SCHOOL OF DISTANCE EDUCATION

B. Sc. MATHEMATICS

MM6B010:COMPLEXANALYSIS

(Core Course)

SIXTH SEMESTER

STUDY NOTES

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UNIVERSITY OF CALICUT

B.Sc. MATHEMATICS

MM6B010: COMPLEX ANALYSIS

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This study notes are only an abridged version of the topics given in the syllabus. For further details like proofs of theorems, illustrations etc., students are advised to go through the prescribed Text Books .

Contents

"As are the crests on the heads of peacocks,

As are the gems on the hoods of cobras,

So is Mathematics, at the top of all sciences."

The Yajurveda, circa 600 B.C.

Chapter 1

Analytic Functions

A **complex number** is a number of the form $a + bi$ where a, b are real numbers and i is the square root of -1. They have the algebraic structure of a field. In engineering and physics, complex numbers are used extensively to describe electric circuits and electromagnetic waves. The number i appears explicitly in the Schrödinger wave equation, which is fundamental to the quantum theory of the atom. Complex analysis, which combines complex numbers with ideas from calculus, has been widely applied to various subjects.

Historically, complex numbers arose in the search for solutions to equations such as $x^2 = -1$. Because there is no real number x for which the square is -1, early mathematicians believed this equation had no solution. However, by the middle of the 16^{th} century, Italian mathematician Gerolamo Cardano and his contemporaries were experimenting with solutions to equations that involved the square roots of negative numbers. Swiss mathematician Leonhard Euler introduced the modern symbol i for $\sqrt{-1}$ in 1777 and expressed the famous relationship $e^{i\pi} = -1$, which connects four of the fundamental numbers of mathematics. In his doctoral dissertation in 1799, German mathematician Carl Friedrich Gauss proved the fundamental theorem of algebra, which states that every polynomial with complex coefficients has a complex root. The study of complex functions was continued by French mathematician Augustin Louis Cauchy, who in 1825 generalized the real definite integral of calculus to functions of a complex variable.

We first discuss about complex functions and then define the concepts limit, continuity, differentiability of complex functions. We will study in detail about ***analytic functions***, an important class of complex functions, which plays a central role in complex analysis.

1.1 Regions in the Complex Plane

In this section, we recollect some facts concerned with sets of complex numbers, or points in the z plane, and their closeness to one another.

For any $\varepsilon > 0$, an ε- ***neighborhood*** of a given point z_0 is the set $|z - z_0| < \varepsilon$. It consists of all points z lying inside but not on a circle centered at z_0 and with positive radius ε.

A ***deleted neighborhood*** of z_0, or ***punctured disk***, $0 < |z - z_0| < \varepsilon$ consisting of all points z in an ε- neighborhood of z_0 except for the point z_0 itself.

A point z_0 is said to be an **interior point** of a set S whenever there is some neighborhood of z_0 that contains only points of S. z_0 is called an ***exterior point*** of S when there exists a neighborhood of it containing no points of S. If z_0 is neither of these, it is a ***boundary point*** of S. Thus, a boundary point is a point all of whose neighborhoods contain at least one point in S and at least one point not in S. The totality of all boundary points is called the ***boundary*** of S.

A set is called ***open*** if it contains none of its boundary points. Clearly, a set is open if and only if each of its points is an interior point. A set is ***closed*** if it contains all of its boundary points, and the ***closure*** of a set S is the closed set consisting of all points in S together with the boundary of S.

An open set S is said to be ***connected*** if each pair of points z_1 and z_2 in S can be joined by a polygonal line, consisting of a finite number of line segments joined end to end, that lies entirely in S.

A nonempty open set that is connected is called a ***domain***. A domain together with some, none, or all of its boundary points is said to be a ***region***.

A set S is ***bounded*** if every point of S lies inside some circle $|z| = R$; otherwise, it is ***unbounded***. A point z_0 is said to be an ***accumulation point*** of a set S if each deleted neighborhood of z_0 contains at least one point of S.

1.2 Functions of a Complex Variable

Let S be a set of complex numbers. A ***function*** f defined on S is a rule that assigns to each z in S a complex number w. The number w is called the ***value*** of f at z and is denoted by $f(z)$. i.e., $w = f(z)$. The set S is called the ***domain of definition*** of f.

Let $w = f(z)$ be a complex function of the complex variable z. Let $w = u+iv$ and $z = x + iy$. Then, u and v depends upon the values of the real variables x and y. Therefore, $f(z) = u(x, y)+iv(x, y)$. This shows that any complex function $f(z)$ of a complex variable $z = x + iy$ is equivalent to a pair of two real-valued functions u and v of the real variables x and y.

In polar coordinates, $z = re^{i\theta}$, we have $f(z) = u(r, \theta) + iv(r, \theta)$.

Example 1.

Consider the complex function $f(z) = z^2$, then $f(x+iy) = (x+iy)^2 = x^2-y^2+i2xy$. Hence $u(x,y) = x^2-y^2$ and $v(x,y) = 2xy$. When polar coordinates are used, $f(re^{i\theta}) = (re^{i\theta})^2 = r^2e^{i2\theta} = r^2cos2\theta + ir^2sin2\theta.$. Therefore, $u(r,\theta) = r^2cos2\theta$ and $v(r,\theta) = r^2sin2\theta$.

If n is zero or a positive integer and if $a_0, a_1, a_2, ..., a_n$ are complex constants, where $a_n \neq 0$, the function $P(z) = a_0 + a_1z + a_2z^2 + ... + a_nz^n$ is called a ***polynomial*** of ***degree*** n. Here, the sum has only a finite number of terms and the domain of definition is the entire z plane. A quotient of the form $\dfrac{P(z)}{Q(z)}$ where, $P(z)$ and $Q(z)$ are polynomials is called a ***rational function*** and is defined at each point z where $Q(z) \neq 0$.

Problem 1.

Express the function $f(z) = z^3 + z + 1$ in the form $f(z) = u(x,y) + iv(x,y)$.

Solution.

Let $z = x+iy$. Then, $f(z) = z^3+z+1 = (x+iy)^3+(x+iy)+1$. On simplification, we get $f(z) = (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y)$. ■

1.2.1 Limit of a Function of a Complex Variable

Let a function f be defined at all points z in some deleted neighborhood z_0. Then we say that the ***limit*** of $f(z)$ as z approaches z_0 is a number w_0, or in symbols $lim_{z\to z_0}f(z) = w_0$, if for each positive number ε, there is a positive number δ such that $|f(z) - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$. (i.e., the limit of $f(z)$ as z approaches z_0 is the number w_0, if the point $w = f(z)$ can be made arbitrarily close to w_0 if we choose the point z close enough to z_0 but distinct from it.) Geometrically, this means that for each ε-neighborhood $|w-w_0|<\varepsilon$ of w_0, there exists a deleted neighborhood $0<|z-z_0|<\delta$ of z_0 such that every point z in it has an image w lying in the ε-neighborhood.

Note that limit of a function $f(z)$ at a point z_0 is unique, if it exists.

Theorem 1.2.1.

Suppose that $f(z)=u(x,y)+iv(x,y)$, $(z=x+iy)$ *and* $z_0=x_0+iy_0, w_0=u_0+iv_0$. *Then,* $lim_{z\to z_0}f(z)=w_0$ *if and only if* $lim_{(x,y)\to(x_0,y_0)}u(x,y)=u_0$ *and* $lim_{(x,y)\to(x_0,y_0)}v(x,y)=v_0$.

Similar rules as in the case of the limits of real functions, will hold in the complex case. For instance, if $lim_{z\to z_0}f(z)=w_0$ and if $lim_{z\to z_0}F(z)=W_0$, then

$$lim_{z\to z_0}[f(z)+F(z)]=w_0+W_0,$$

$$lim_{z\to z_0}[f(z)F(z)]=w_0W_0,$$

and, if $W_0\neq 0$,

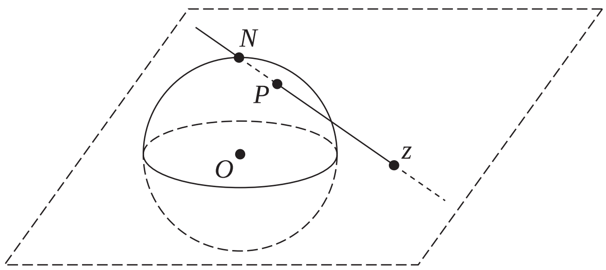
$$lim_{z\to z_0}\frac{f(z)}{F(z)}=\frac{w_0}{W_0}.$$

Also, $lim_{z\to z_0}c=c$ and $lim_{z\to z_0}z=z_0$, where z_0 and c are any complex numbers, and $lim_{z\to z_0}z^n=z_0^n$ $(n=1,2,...)$.

The limit of a polynomial $P(z)=a_0+a_1z+a_2z^2+...+a_nz^n$ as z approaches a point z_0 is the value of the polynomial at that point. i.e., $lim_{z\to z_0}P(z)=P(z_0)$.

1.2.2 Limits involving the Point at Infinity

The complex plane together with the *point at infinity* is called the ***extended complex plane***. To visualize the point at infinity, denoted by ∞ one can think of the complex plane as passing through the equator of a unit sphere centered at the origin. To each point z in the plane there corresponds exactly one point P on the surface of the sphere. The point P is the point where the line through z and the north pole N intersects the sphere. In like manner, to each point P on the surface of the sphere, other than the north pole N, there corresponds exactly one point z in the plane. By letting the point N of the sphere correspond to the point at infinity, we obtain a *one to one correspondence* between the points of the sphere and the points of the extended complex plane. The sphere is known as the ***Riemann sphere***, and the correspondence is called a ***stereographic projection***.



Note that in the above identification, the exterior of the unit circle centered at the origin in the complex plane corresponds to the upper hemisphere with the equator and the point N deleted. Moreover, for each small positive number $\varepsilon > 0$, those points in the complex plane exterior to the circle $|z| = \dfrac{1}{\varepsilon}$ correspond to points on the sphere close to N. Therefore, the set $|z| = \dfrac{1}{\varepsilon}$ is called an ε-neighborhood of ∞. With this definition of ε-neighborhood of ∞, we can define limits involving the point at infinity as in 1.2.1. Also, we have: If z_0 and w_0 are points in the z and w planes, respectively, then $lim_{z\to z_0}f(z) = \infty$ if and only if $lim_{z\to z_0}\frac{1}{f(z)} = 0$ and $lim_{z\to\infty}f(z) = w_0$ if and only if $lim_{z\to 0}f(\frac{1}{z}) = w_0$. Moreover, $lim_{z\to\infty}f(z) = \infty$ if and only if $lim_{z\to 0}\frac{1}{f(1/z)} = 0$.

1.2.3 Continuity

A function f is said to be ***continuous*** at a point z_0 if: (i) $lim_{z\to z_0}f(z)$ exists, (ii) $f(z_0)$ exists, and (iii) $lim_{z\to z_0}f(z) = f(z_0)$.

A function of a complex variable is said to be ***continuous*** in a region R if it is continuous at each point in R.

Note that if two functions are continuous at a point, then their sum and product are also continuous at that point and their quotient is continuous at any such point if the denominator is not zero there. Also, a composition of continuous functions is itself continuous. If $f(z) = u(x,y) + iv(x,y)$, then the function $f(z)$ is continuous at a point $z_0 = (x_0, y_0)$ if and only if its component functions $u(x,y)$ and $v(x,y)$ are continuous at (x_0, y_0).

Theorem 1.2.2.

If a function $f(z)$ is continuous and nonzero at a point z_0, then $f(z) \neq 0$ throughout some neighborhood of that point.

Proof.

Assume that $f(z)$ is continuous and nonzero at z_0. Let $\varepsilon = \frac{|f(z_0)|}{2}$. Then corresponding to this $\varepsilon > 0$, there is a positive number δ such that $|f(z)-f(z_0)| < \frac{|f(z_0)|}{2}$ whenever $|z - z_0| < \delta$. Now, if there is a point z in the neighborhood

$|z - z_0| < \delta$ at which $f(z) = 0$, then we have $|f(z_0)| < \dfrac{|f(z_0)|}{2}$, which is a contradiction. This shows that $f(z) \neq 0$ throughout the neighborhood $|z - z_0| < \delta$ of z_0. □

Theorem 1.2.3.

If a function f is continuous throughout a region R that is both closed and bounded, there exists a nonnegative real number M such that $|f(z)| \leq M$ for all points z in R, where equality holds for at least one such z.

1.2.4 Derivatives

Let f be a function whose domain of definition contains a neighborhood $|z - z_0| < \delta$ of a point z_0. The ***derivative*** of f at z_0 is the limit

$$f'(z_0) = lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

and the function f is said to be ***differentiable*** at z_0 when $f'(z_0)$ exists.

Writing $\triangle z = z - z_0$, where $z \neq z_0$, we have

$$f'(z_0) = lim_{\triangle z \to 0} \frac{f(z_0 + \triangle z) - f(z_0)}{\triangle z}.$$

Since f is defined throughout a neighborhood of z_0, the number $f(z_0 + \triangle z)$ is always defined for $|\triangle z|$ sufficiently small. If we write $w = f(z + \triangle z) - f(z)$, then

$$f'(z) = \frac{dw}{dz} = lim_{\triangle z \to 0} \frac{\triangle w}{\triangle z}.$$

Remark.

Note that the existence of the derivative of a function at a point implies the continuity of the function at that point. To see this, assume that $f'(z_0)$ exists. Then,

$$lim_{z\to z_0}[f(z)-f(z_0)] = lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}lim_{z\to z_0}(z-z_0) = f'(z_0).0 = 0.$$

$\Rightarrow lim_{z\to z_0}f(z) = f(z_0)$. This shows that f is continuous at z_0.

The following example shows that the continuity of a function at a point does not imply the existence of a derivative there.

Example 2.

Consider the real-valued function $f(z) = |z|^2 = u(x,y) + i\ v(x,y)$. Then, $u(x,y) = \sqrt{x^2+y^2}$ and $v(x,y) = 0$. Note that $f(z) = |z|^2$ is continuous at each point in the complex plane since its components are continuous at each point. Here,

$$\frac{\triangle w}{\triangle z} = \frac{|z+\triangle z|^2 - |z|^2}{\triangle z} = \frac{(z+\triangle z)(\overline{z}+\overline{\triangle z}) - z\overline{z}}{\triangle z} = \overline{z} + \overline{\triangle z} + z\frac{\overline{\triangle z}}{\triangle z}.$$

If $\triangle z = (\triangle x, \triangle y) \to (0,0)$ horizontally through the points $(\triangle x, 0)$ on the real axis, then $\frac{\overline{\triangle z}}{\triangle z} \to 1$ and if $\triangle z = (\triangle x, \triangle y) \to (0,0)$ vertically through the points $(0, \triangle y)$ on the imaginary axis, then $\frac{\overline{\triangle z}}{\triangle z} \to -1$, so that we have $\frac{\triangle w}{\triangle z} \to \overline{z} + z$ and $\frac{\triangle w}{\triangle z} \to \overline{z} - z$ in these cases. By the uniqueness of limits, this implies $\overline{z} + z = \overline{z} - z \Rightarrow z = 0$. Therefore, $\frac{dw}{dz}$ cannot exist when $z \neq 0$. Also, when $z = 0$, $\frac{\triangle w}{\triangle z} = \overline{\triangle z}$. Therefore, $\frac{dw}{dz}$ exists only at $z = 0$ and its value is 0. Thus, a function is continuous at a point does not imply that the function is differentiable at that point.

This example also shows that a function $f(z) = u(x,y) + i\ v(x,y)$ can be differentiable at a point $z = (x, y)$ but nowhere else in any neighborhood of that point.

Suppose that f has a derivative at z_0 and that g has a derivative at the point $f(z_0)$. Then the composite function $F(z) = g[f(z)]$ has a derivative at z_0, and $F'(z_0) = g'[f(z_0)]f'(z_0)$.

As in the real case, the following formulas also holds for complex differentiation, and we ask the reader to prove these rules using the definition of derivatives.

(i) $\frac{d}{dz}c = 0$, *for any complex constant* c,
(ii) $\frac{d}{dz}z = 1$
(iii) $\frac{d}{dz}cf(z) = cf'(z)$,
(iv) $\frac{d}{dz}z^n = nz^{n-1}$ *where* n *is a positive integer(valid for negative integers if* $z \neq 0$*)*,
(v) $\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z)$,
(vi) $\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z)$, *and when* $g(z) \neq 0$,
(vi) $\frac{d}{dz}[\frac{f(z)}{g(z)}] = \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}$.

Exercises.

1. Using the definition of limit, prove that

   (a) $lim_{z \to z_0} Rez = Rez_0$

   (b) $lim_{z \to z_0} \bar{z} = \bar{z}_0$,

   (c) $lim_{z \to z_0} \frac{\bar{z}^2}{z} = 0$.

   (d) $lim_{z \to 1-i}[x + i(2x + y)] = 1 + i$, where $(z = x + iy)$.

2. Show that

   (a) $lim_{z \to i} \dfrac{iz^3 - 1}{z + i} = 0.$

   (b) $lim_{z \to z_0} z^n = z_0^n \quad (n = 1, 2, 3, ...)$(Hint: Use mathematical induction.)

   (c) $lim_{z \to \infty} \dfrac{4z^2}{(z-1)^2} = 4.$

   (d) $lim_{z \to \infty} \dfrac{z^2 + 1}{z - 1} = \infty.$

3. Show that the limit of the function $f(z) = (\frac{z}{\bar{z}})^2$ as z tends to 0 does not exist.

4. Show that $lim_{z \to z_0} f(z)g(z) = 0$ if $lim_{z \to z_0} f(z) = 0$ and if there exists a positive number M such that $|g(z)| \leq M$ for all z in some neighborhood of z_0.

5. Show that a set S is unbounded if and only if every neighborhood of the point at infinity contains at least one point in S.

6. Find $f'(z)$, where (a). $f(z) = 3z^2 - 2z + 4$ (b). $f(z) = \dfrac{z+1}{2z-1}$ $(z \neq \frac{1}{2})$

7. Show that $f'(z)$ does not exist at any point z when (a). $f(z) = Rez$, (b). $f(z) = Imz$.

1.3 Cauchy - Riemann Equations

In this section, we obtain a pair of equations that the first-order partial derivatives of the component functions u and v of a function $f(z) = u(x, y) + i\ v(x, y)$ must satisfy at a point $z_0 = (x_0, y_0)$ when the derivative of f exists there. We also derive an expression for $f'(z_0)$ in terms of partial derivatives of u and v.

Theorem 1.3.1.

Suppose that $f(z) = u(x, y) + i\ v(x, y)$ and that $f'(z)$ exists at a point $z_0 = x_0 + i\ y_0$. Then the first-order partial derivatives of u and v must exist at (x_0, y_0), and they must satisfy the **Cauchy - Riemann equations** $u_x = v_y,\ u_y = -v_x$ *there. Also, $f'(z_0)$ can be written $f'(z_0) = u_x + i\ v_x$, where these partial derivatives are to be evaluated at (x_0, y_0).*

Proof.

Let $\triangle z = \triangle x + i\ \triangle y$ and $\triangle w = f(z_0 + \triangle z) - f(z_0)$.
Then, $\triangle w = [u(x_0 + \triangle x, y_0 + \triangle y) - u(x_0, y_0)] + i\ [v(x_0 + x, y_0 + y) - v(x_0, y_0)]$.
Suppose that $(\triangle x, \triangle y) \to (0, 0)$ horizontally through the points $(\triangle x, 0)$ on the real axis, then we have, $lim_{\triangle z\to 0}\dfrac{\triangle w}{\triangle z} = lim_{\triangle x\to 0}\dfrac{u(x_0 + \triangle x, y_0) - u(x_0, y_0)}{\triangle x} + i\ lim_{\triangle x\to 0}\dfrac{v(x_0 + \triangle x, y_0) - v(x_0, y_0)}{\triangle x}$
$\Rightarrow lim_{\triangle z\to 0}\dfrac{\triangle w}{\triangle z} = u_x(x_0, y_0) + i\ v_x(x_0, y_0)$ where $u_x(x_0, y_0)$ and $v_x(x_0, y_0)$ denote the first–order partial derivatives with respect to x of the functions u and v, respectively, at (x_0, y_0).
If $\triangle z = (\triangle x, \triangle y) \to (0, 0)$ vertically through the points $(0, \triangle y)$ on the imaginary axis, we get $lim_{\triangle z\to 0}\dfrac{\triangle w}{\triangle z} =$
$lim_{\triangle y\to 0}\dfrac{u(x_0, y_0 + \triangle y) - u(x_0, y_0)}{i\ \triangle y} + i\ lim_{\triangle y\to 0}\dfrac{v(x_0, y_0 + \triangle y) - v(x_0, y_0)}{i\ \triangle y} \Rightarrow$
$lim_{\triangle z\to 0}\dfrac{\triangle w}{\triangle z} = lim_{\triangle y\to 0}\dfrac{v(x_0, y_0 + \triangle y) - v(x_0, y_0)}{\triangle y} - i\ lim_{\triangle y\to 0}\dfrac{u(x_0, y_0 + \triangle y) - u(x_0, y_0)}{\triangle y}$
$= v_y(x_0, y_0) - i\ u_y(x_0, y_0)$ where $u_y(x_0, y_0)$ and $v_y(x_0, y_0)$ denote the first–order partial derivatives with respect to y of the functions u and v, respectively, at (x_0, y_0). Since $f'(z)$ exists at the point $z_0 = x_0 + i\ y_0$, both of the above values of $lim_{\triangle z\to 0}\dfrac{\triangle w}{\triangle z}$ must be equal. Therefore, we must have

$$u_x(x_0, y_0) + i\ v_x(x_0, y_0) = v_y(x_0, y_0) - i\ u_y(x_0, y_0).$$

Therefore, we must have $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$. Also, we have $f'(z_0) = lim_{\triangle z \to 0} \dfrac{\triangle w}{\triangle z} = u_x(x_0, y_0) + i\ v_x(x_0, y_0)$. $\square$

Example 3.

Let $f(z) = |z|^2 = u(x, y) + i\ v(x, y)$. Then, we have $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$.

Then $u_x = 2x$, $u_y = 2y$, and $v_x = v_y = 0$. So the Cauchy - Riemann equations holds at a point (x, y) only if $2x = 0$ and $2y = 0$, i.e., only if $x = y = 0$. Therefore, $f'(z)$ does not exist at any nonzero point.

From, Exercise 1 given below, we know that satisfaction of the Cauchy-Riemann equations at a point is not sufficient to ensure the existence of the derivative of a function at that point.

Now we derive a sufficient condition for differentiability of $f(z)$ at a point $z_0 = x_0 + i\ y_0$.

For proving this theorem we make use of the following result from your *Vector Calculus* (V- Semester) Course.

The increment Theorem for Functions of Two Variables:

Suppose that the first order partial derivatives of $f(x, y)$ are defined throughout an open region R containing the point (x_0, y_0) and that f_x and f_y are continuous at (x_0, y_0) . Then the change, $\triangle z = f(x_0+\triangle x, y_0+\triangle y)-f(x_0, y_0)$ in the value of f that results from moving from (x_0, y_0) to another point $x_0 + \triangle x, y_0 + \triangle y)$ in R satisfies an equation of the form $\triangle z = f_x(x_0, y_0)\triangle x+f_y(x_0, y_0)\triangle y+\varepsilon_1\triangle x+\varepsilon_2\triangle y$ in which $\varepsilon_1, \varepsilon_2 \to 0$ as $\triangle x, \triangle y \to 0$.

Theorem 1.3.2.

Let the function $f(z) = u(x, y) + i\ v(x, y)$ be defined throughout some ε-

neighborhood of a point $z_0 = x_0 + i\ y_0$*, and suppose that*

(a) the first–order partial derivatives of the functions u *and* v *with respect to* x *and* y *exist everywhere in the neighborhood;*

(b) those partial derivatives are continuous at (x_0, y_0) *and satisfy the Cauchy-Riemann equations* $u_x = v_y, u_y = -v_x$ *at* (x_0, y_0)*. Then* $f'(z_0)$ *exists, its value being* $f'(z_0) = u_x + i\ v_x$ *where the right–hand side is to be evaluated at* (x_0, y_0)*.*

Proof.

Let $\triangle z = \triangle x + i\ \triangle y$, where $0 < |\triangle z| < \varepsilon$. Then we write $\triangle w = f(z_0 + \triangle z) - f(z_0) = \triangle u + i\ \triangle v$, where $\triangle u = u(x_0 + \triangle x, y_0 + \triangle y) - u(x_0, y_0)$ and $\triangle v = v(x_0 + x, y_0 + y) - v(x_0, y_0)$. Since the first–order partial derivatives of u and v are continuous at the point (x_0, y_0), by the increment theorem for functions of two variables, we have

$$\triangle u = u_x(x_0, y_0)\triangle x + u_y(x_0, y_0)\triangle y + \varepsilon_1 \triangle x + \varepsilon_2 \triangle y$$

and

$$\triangle v = v_x(x_0, y_0)\triangle x + v_y(x_0, y_0)\triangle y + \varepsilon_3 \triangle x + \varepsilon_4 \triangle y,$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and ε_4 tend to zero as (x, y) approaches $(0, 0)$ in the z-plane. Now, $\triangle w = \triangle u + i\ \triangle v \Rightarrow \triangle w = u_x(x_0, y_0)\triangle x + u_y(x_0, y_0)\triangle y + \varepsilon_1 \triangle x + \varepsilon_2 \triangle y + i\ [v_x(x_0, y_0)\triangle x + v_y(x_0, y_0)\triangle y + \varepsilon_3 \triangle x + \varepsilon_4 \triangle y]$. Since Cauchy – Riemann equations are satisfied at (x_0, y_0), this implies that

$$\frac{\triangle w}{\triangle z} = u_x(x_0, y_0) + i\ v_x(x_0, y_0) + (\varepsilon_1 + i\ \varepsilon_3)\frac{\triangle x}{\triangle z} + (\varepsilon_2 + i\ \varepsilon_4)\frac{\triangle y}{\triangle z}.$$

But, $|\triangle x| \leq |\triangle z|$ and $|\triangle y| \leq |\triangle z|$, we have $|\dfrac{\triangle x}{\triangle z}| \leq 1$ and $|\dfrac{\triangle y}{\triangle z}| \leq 1$, so that

$$|(\varepsilon_1 + i\ \varepsilon_3)\frac{\triangle x}{\triangle z}| \leq |\varepsilon_1 + i\ \varepsilon_3| \leq |\varepsilon_1| + |\varepsilon_3|$$

and

$$|(\varepsilon_2 + i\ \varepsilon_4)\frac{\triangle y}{\triangle z}| \leq |\varepsilon_2 + i\ \varepsilon_4| \leq |\varepsilon_2| + |\varepsilon_4|.$$

Thus, as $\triangle z \to 0$, we get $f'(z_0) = lim_{\triangle z \to 0}\dfrac{\triangle w}{\triangle z} = u_x(x_0, y_0) + i\ v_x(x_0, y_0)$. $\square$

Problem 2.

Show that $f(z) = e^z$ is differentiable everywhere in the complex plane.

Solution.

We have $f(z) = e^z = e^{x+i\ y} = e^x e^{iy} = e^x(cos\ y + i\ sin\ y) = u + i\ v$. Then $u_x = e^x cos\ y$, $u_y = -e^x sin\ y$, $v_x = e^x sin\ y$ and $v_y = e^x cos\ y$, all are continuous and satisfies the Cauchy– Riemann equations everywhere. Therefore $f'(z)$ exists everywhere and $f'(z) = u_x + i\ v_x = e^x cos\ y + i\ e^x sin\ y = e^z = f(z)$, for all z. $\blacksquare$

The following theorem gives the polar form of Cauchy– Riemann equations.

Theorem 1.3.3.

Let the function $f(z) = u(r, \theta) + i\ v(r, \theta)$ be defined throughout some $\varepsilon-$ neighborhood of a nonzero point $z_0 = r_0 exp(i\ \theta_0)$, and suppose that (a) the first–order partial derivatives of the functions u and v with respect to r and θ exist everywhere in the neighborhood; (b) those partial derivatives are continuous at (r_0, θ_0) and satisfy the polar form $ru_r = v_\theta, u_\theta = -rv_r$ of the Cauchy-Riemann equations at (r_0, θ_0). Then $f'(z_0)$ exists, its value being $f'(z_0) = e^{-i\ \theta}(u_r + i\ v_r)$, where the right–hand side is to be evaluated at (r_0, θ_0).

Problem 3.

Show that $f(z) = \frac{1}{z}$ is differentiable at all $z \neq 0$ and find its derivative.

Solution.

Writing in the polar form, we have for $z \neq 0$, $f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}(cos\ \theta - i\ sin\ \theta) = u + i\ v$. Thus, $u(r, \theta) = \frac{cos\ \theta}{r}$ and $v(r, \theta) = -\frac{sin\ \theta}{r}$. Therefore,$ru_r = -\frac{cos\ \theta}{r} = v_\theta$ and $u_\theta = -\frac{sin\ \theta}{r} = -rv_r$. $\Rightarrow$ The partial derivatives are continuous and the Cauchy– Riemann equations are satisfied at all $z \neq 0$. Therefore the derivative of f exists at all $z \neq 0$ and $f'(z) = e^{-i\theta}(-\frac{cos\ \theta}{r^2} + i\ \frac{sin\ \theta}{r^2}) = -\frac{1}{(re^{i\theta})^2} = -\frac{1}{z^2}$ when $z \neq 0$. ∎

Exercises.

1. Let u and v denote the real and imaginary components of the function f defined by means of the equations $f(z) = \frac{\overline{z}^2}{z}$ when $z \neq 0$, and $f(0) = 0$. Verify that the Cauchy-Riemann equations are satisfied at the origin. Show that $f'(0)$ does not exists.

2. If $f(z)$ is analytic, use chain rule to show that $\frac{\partial f}{\partial \overline{z}} = 0$.

3. Show that $f(z) = z^3$ is differentiable at all z and find its derivative.

4. Show that $f(z) = e^{\overline{z}}$ is nowhere differentiable.

5. Show that $f'(z)$ does not exists at any point if $f(z) = 2x + i\ xy^2$.

6. Show that $f(z) = \frac{1}{z^4}$ is differentiable at all $z \neq 0$ and find its derivative.

1.4 Analytic Functions

A function f of the complex variable z is said to be ***analytic*** at a point z_0 if it has a derivative at each point in some neighborhood of z_0.

Note that if f is analytic at a point z_0, it must be analytic at each point in some neighborhood of z_0. A function f is analytic in an open set if it has a derivative everywhere in that set.

Note that the function $f(z) = \dfrac{1}{z}$ is analytic at each nonzero point in the finite complex plane. But the function $f(z) = |z|^2$ is not analytic at any point since its derivative exists only at $z = 0$.
An ***entire function*** is a function that is analytic at each point in the entire finite complex plane.

Since the derivative of a polynomial exists everywhere, it follows that every polynomial is an entire function. Similarly, $e^z, sin\ z, cos\ z$ are entire functions.

If a function f fails to be analytic at a point z_0 but is analytic at some point in every neighborhood of z_0, then z_0 is called a ***singular point***, or ***singularity***, of f.

For example, the point $z = 0$ is a singular point of the function $f(z) = \dfrac{1}{z}$, whereas the function $f(z) = |z|^2$, has no singular points since it is nowhere analytic.

A necessary, but not sufficient condition for a function f to be analytic in a domain D is clearly the continuity of f throughout D. Satisfaction of the Cauchy-Riemann equations is also necessary, but not sufficient. Sufficient conditions for analyticity in D are provided by the theorems 1.3.2 and 1.3.3.

Note that a constant multiple of an analytic function is analytic. If two functions are analytic in a domain D, their sum and their product are both analytic in D. Similarly, their quotient is analytic in D provided the function in the denominator does not vanish at any point in D. In particular, the quotient $P(z)/Q(z)$ of two polynomials is analytic in any domain throughout which $Q(z) \neq 0$. From the chain rule for the derivative of a composite function, we see that a composition of two analytic functions is analytic.

Problem 4.

If $f'(z) = 0$ everywhere in a domain D, then show that $f(z)$ must be constant throughout D.

Solution.

Since $f'(z)$ exists everywhere in D, $f(z) = u + i\ v$ is analytic in D. Hence u and v satisfies the Cauchy–Riemann equations and $f'(z) = u_x + i\ v_x$. But, $f'(z) = 0$, $\forall z \in D$, we have $u_x = 0, v_x = 0$. By Cauchy –Riemann equations, we get $u_y = 0, v_y = 0$. This shows that both u and v are independent of x and y. i.e. u and v are constants in $D \Rightarrow f(z = u + i\ v)$ is constant in D. ■

Problem 5.

If $f(z)$ is a real–valued analytic function in a domain D, then show that $f(z)$ must be constant in D.

Solution.

Let $f(z) = u+i\ v$. Since $f(z)$ is real–valued in D, we have $v = 0$. $\Rightarrow v_x = v_y = 0$. Since $f(z)$ is analytic in D, u and v satisfies the Cauchy–Riemann equations, so we get $u_x = 0, u_y = 0$. This shows that u is a constant in D. Thus, any real–valued analytic function in a domain D is constant in D. ■

Problem 6.

If both $f(z)$ and $\overline{f(z)}$ are analytic functions in a domain D, then show that $f(z)$ must be constant in D.

Solution.

Let $f(z) = u + i\ v$. Then, $\overline{f(z)} = u - i\ v = U + i\ V$. $\Rightarrow U = u$ and $V = -v$. Since $f(z)$ is analytic in D, we have the Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$. Since $\overline{f(z)} = U + i\ V$ is analytic in D, we have $U_x = V_y$ and $U_y = -V_x$. Since $U = u$ and $V = -v$, this implies that $u_x = -v_y$ and $u_y = v_x$. $\Rightarrow u_x = 0, v_x = 0$. This shows that $f'(z) = u_x + i\ v_x = 0$. Therefore by Problem 4, $f(z)$ is a constant in D. ■

Definition 1.4.1.

A real–valued function H of two real variables x and y is said to be ***harmonic*** in a given domain of the $xy-$plane if, throughout that domain, it has continuous partial derivatives of the first and second order and satisfies the partial differential equation $H_{xx}(x, y) + H_{yy}(x, y) = 0$, known as *Laplace's equation.*

For example, $f(x, y) = x^2 - y^2$ is clearly a harmonic function in any domain of the xy-plane.

Note that, if a function $f(z) = u(x, y) + i\ v(x, y)$ is analytic in a domain D, then its component functions u and v are harmonic in D. To see this, note that by Cauchy – Riemann equations, we have $u_x = v_y$ and $u_y = -v_x$. Differentiating these again w.r.t x and y, we get $u_{xx} = v_{yx}$ and $u_{yy} = -v_{xy}$. The continuity of the partial derivatives of u and v implies $v_{xy} = v_{yx}$. Hence, we get $u_{xx} + u_{yy} = 0$. Similarly, $v_{xx} + v_{yy} = 0$. This shows that both u and v are harmonic in D.

If two given functions u and v are harmonic in a domain D and their first–

order partial derivatives satisfy the Cauchy-Riemann equations throughout D, then v is said to be a ***harmonic conjugate*** of u.

It can be proved that if a function $f(z) = u(x, y) + i\ v(x, y)$ is analytic in a domain D if and only if v is a harmonic conjugate of u.

For example, since $u = x^2 - y^2$ and $v = 2xy$ are the real and imaginary parts of the entire function $f(z) = z^2$, v is a harmonic conjugate of u throughout the plane. But u cannot be a harmonic conjugate of v since, the function $2xy + i\ (x^2 - y^2)$ is not analytic anywhere (Verify this!).

Problem 7.

Show that $u(x, y) = y^3 - 3x^2y$ is harmonic throughout the plane and find a harmonic conjugate $v(x, y)$.

Solution.

We have $\dfrac{\partial u}{\partial x} = -6xy$ and $\dfrac{\partial u}{\partial y} = 3y^2 - 3x^2$. $\Rightarrow \dfrac{\partial^2 u}{\partial x^2} = -6y$, and $\dfrac{\partial^2 u}{\partial y^2} = 6y$. Thus, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. $\Rightarrow u$ is harmonic.

Let $v(x, y)$ be the harmonic conjugate of u so that $f(z) = u + i\ v$ is analytic. By Cauchy–Riemann equations, $u_x = v_y$ and $u_y = -v_x$. But $v_y = u_x = -6xy$ Integrating this equation w.r.t y, we get $v(x, y) = -3xy^2 + k(x)$. Differentiating this w.r.t x, we get $v_x = -3y^2 + k'(x)$. But $v_x = -u_y = -3y^2 + 3x^2 \Rightarrow k'(x) = 3x^2$. Integrating, we get $k(x) = x^3 + C$ where C is an arbitrary real number, so that $v(x, y) = -3xy^2 + x^3 + C$. ■

Exercises.

1. If $f(z)$ is an analytic function in a domain D with $Imf(z)$ =constant, then show that $f(z)$ is a constant in D.

2. If $f(z)$ is an analytic function in a domain D with $|f(z)|$ is a constant, then show that $f(z)$ is a constant in D.

3. If $f(z)$ is an analytic function in a domain D with $arg f(z)$ =constant, then show that $f(z)$ is a constant in D.

4. Show that a linear combination of two entire functions is entire.

5. Show that $f(z) = e^y e^{ix}$ is nowhere analytic.

6. Show that $f(z) = xy + i\ y$ is nowhere analytic.

7. Determine the singular points of

   (a) $f(z) = \dfrac{2z+1}{z(z^2+1)}$

   (b) $f(z) = tan\ z$

   (c) $f(z) = \dfrac{z^3+i}{z^2-3z+2}$.

8. Show that if v and V are harmonic conjugates of $u(x, y)$ in a domain D, then $v(x, y)$ and $V(x, y)$ can differ at most by an additive constant.

9. Show that $u(x, y) = \dfrac{y}{x^2+y^2}$ is harmonic throughout the plane and find a harmonic conjugate $v(x, y)$.

10. Suppose that v is a harmonic conjugate of u in a domain D and also that u is a harmonic conjugate of v in D. Show that both $u(x, y)$ and $v(x, y)$ must be constant throughout D.

11. Show that v is a harmonic conjugate of u in a domain D if and only if $-u$ is a harmonic conjugate of v in D.

12. Let the function $f(z) = u(x, y) + i\ v(x, y)$ be analytic in a domain D, and consider the families of level curves $u(x, y) = c_1$ and $v(x, y) = c_2$,

where c_1 and c_2 are arbitrary real constants. Prove that these families are orthogonal.

13. Show that $u(x, y) = 2x - x^3 + 3xy^2$ is harmonic throughout the plane and find a harmonic conjugate $v(x, y)$.

14. Let the function $f(z) = u(r, \theta) + i\ v(r, \theta)$ be analytic in a domain D that does not include the origin. Using the polar form of Cauchy-Riemann equations and assuming continuity of partial derivatives, show that throughout D the function $u(r, \theta)$ satisfies the partial differential equation $r^2 u_{rr}(r, \theta) + r u_r(r, \theta) + u_{\theta\theta}(r, \theta) = 0$, which is the polar form of Laplace's equation. Show that the same is true for the function $v(r, \theta)$.

1.5 Elementary Functions

In this section, we consider various elementary functions that studied in calculus and define corresponding functions of a complex variable. We begin by defining the complex exponential function.

Let $z = x + i\ y$. Then we define the ***complex exponential function*** as $e^z = e^x e^{iy}$.

But, by Euler's formula $e^{iy} = cos\ y + i\ sin\ y$ where y is to be taken in radians. Thus, we have

$$exp(z) = e^z = e^x(cos\ y + i\ sin\ y) = e^x cos\ y + i\ e^x sin\ y = u + i\ v$$

Then u and v have continuous first order partial derivatives that satisfy the Cauchy–Riemann equations, showing that e^z is an entire function and $(e^z)' = e^z$.

Also, note that e^z is periodic with pure imaginary period $2\pi i$. We have, $|e^z| = e^x \neq 0$, so that $e^z \neq 0$ for any complex number z. Also, $arg\ e^z = y + 2n\pi$ $(n = 0, \pm 1, \pm 2, ...)$.

If z is any nonzero complex number, we define the ***complex logarithmic function*** $log\ z$ (or, $ln\ z$)as the inverse of the complex exponential function. i.e., we have $log\ z = w = u + i\ v$ if $e^w = z$.

Writing $z = re^{i\ \theta}$ with $-\pi < \theta \leq \pi$ and $w = u + i\ v$, we obtain: $e^{u+i\ v} = re^{i\ \theta} \Rightarrow e^u e^{i\ v} = re^{i\ \theta}$ so that $e^u = r$ and $v = \theta + 2n\pi$ (since $e^{i2n\pi} = 1$ for any integer n.)

Therefore, $u = ln\ r$ and $v = \theta + 2n\pi$ for $n = 0, \pm 1 \pm 2, ...$

Thus, $log\ z = ln\ r + i(\theta + 2n\pi)$ where $\theta = arg\ z$, $n = 0, \pm 1 \pm 2, ...$

Since $arg\ z$ is infinitely many valued, the complex logarithmic function is also a multiple–valued function.

If we let α denote any real number and restrict the value of $\theta = arg\ z$ in the above definition of $log\ z$ so that $\alpha < \theta < \alpha + 2\pi$, the function $log\ z = ln\ r + i\theta$, $(r > 0, \alpha < \theta < \alpha + 2\pi)$, with components $u(r, \theta) = ln\ r$ and $v(r, \theta) = \theta$, is single–valued and continuous in the stated domain.

The function $log\ z = ln\ r + i\ \theta$ defined above is continuous and also analytic throughout the domain $r > 0, \alpha < \theta < \alpha + 2\pi$, since the first–order partial derivatives of u and v are continuous there and satisfy the polar form of the Cauchy–Riemann equations.

Also, $\dfrac{d}{dz} log\ z = e^{-i\ \theta}(u_r + i\ v_r) = e^{-i\ \theta}(\dfrac{1}{r} + i\ 0) = \dfrac{1}{re^{i\theta}} = \dfrac{1}{z}$, where $|z| > 0, \alpha < arg\ z < \alpha + 2\pi$.

The ***principal value*** of $log\ z$, denoted by $Log\ z$ is the value corresponding to the principal value of $arg\ z$. i.e., $Log\ z = ln\ |z| + i\ (Arg\ z)$, where $Arg\ z$ is that value of $arg\ z$ lies between $-\pi < arg\ z \leq \pi$.

A ***branch*** of a multiple–valued function f is any single–valued function F that is analytic in some domain at each point z of which the value $F(z)$ is one of the values of f.

The requirement of analyticity, prevents F from taking on a random selection of the values of f. Observe that for each fixed α, the single–valued function $log\ z = ln\ r + i\theta$, $(r > 0, \alpha < \theta < \alpha + 2\pi)$ is a branch of the multiple–valued function $log\ z = ln\ r + i\ arg\ z$.

Note that the function $Log\ z = ln\ r + i\theta(r > 0, -\pi < \theta < \pi)$ is the ***principal branch*** of $log\ z$.

A ***branch cut*** is a portion of a line or curve that is introduced in order to define a branch F of a multiple–valued function f.

Points on the branch cut for F are singular points of F, and any point that is common to all branch cuts of f is called a ***branch point***.

The origin and the ray $\theta = \alpha$ make up the branch cut for the branch $log\ z = ln\ r + i\theta$, $(r > 0, \alpha < \theta < \alpha + 2\pi)$ of the logarithmic function. The branch cut for the principal branch of $log\ z$ consists of the origin and the ray $\theta = \pi$. The origin is evidently a branch point for branches of the multiple-valued logarithmic function.

Now we define ***complex exponents***. When $z \neq 0$ and the exponent c is any complex number, the function z^c is defined by means of the equation $z^c = e^{c\ log\ z}$, where $log\ z$ denotes the multiple–valued logarithmic function. So, in general, z^c is multiple–valued. To get a single value for z^c, we replace the multiple–valued logarithmic function by a particular branch of $log\ z$.

The ***principal value*** of z^c is that value of z^c when $log\ z$ is replaced by $Log\ z$ in the definition of z^c. i.e., the principal value of $z^c = e^{cLog\ z}$.

Problem 8.

Solve for z, the equation $e^z = 1 + \ i$.

Solution.

In polar form, we have $1 + \ i = \sqrt{2}e^{i\ \frac{\pi}{4}}$. Let $z = x + i\ y$.
Therefore, $e^z = e^x e^{i\ y} = \sqrt{2}e^{i\ (\frac{\pi}{4}+2n\pi)}$ $(n = 0, \pm1, \pm2, ...)$.
$\Rightarrow e^x = \sqrt{2}$ and $y = \dfrac{\pi}{4} + 2n\pi$, $(n = 0, \pm1, \pm2, ...)$.
$\Rightarrow x = ln\ \sqrt{2} = \dfrac{1}{2}\ ln\ 2$ and $y = (2n + \dfrac{1}{4})\pi$ $(n = 0, \pm1, \pm2, ...)$.

Hence $z = x + i\ y = \dfrac{1}{2}\ ln\ 2 + i\ (2n + \dfrac{1}{4})\pi$ $(n = 0, \pm1, \pm2, ...)$. ■

Problem 9.

Show that $exp\ (z + \pi\ i) = -\ exp\ z$.

Solution.

We have $exp\ (z + \pi\ i) = e^{z+\pi\ i} = e^z e^{\ i\pi} = e^z(cos\ \pi + i\ sin\ \pi) = e^z(-1 + i\ 0) = -e^z = -\ exp\ z$. ■

Problem 10.

Find the value of $log\ (-1 - \sqrt{3}\ i)$.

Solution.

Writing in polar form, $-1 - \sqrt{3}\ i = 2e^{i(-\frac{2\pi}{3})}$. Therefore, $log\ (-1 - \sqrt{3}\ i) = ln\ 2 + i\ (-\frac{2\pi}{3} + 2n\pi)$, $(n = 0, \pm1, \pm2, ...)$. $\Rightarrow log\ (-1 - \sqrt{3}\ i) = ln\ 2 + 2\ (n - \frac{1}{3})\pi\ i$, $(n = 0, \pm1, \pm2, ...)$. ■

Problem 11.

Show that $Log(-e\ i) = 1 - \frac{\pi}{2}\ i$.

Solution.

We have , $Log(-e\ i) = ln|-e\ i| + i\ (Arg(-e\ i))$. $\Rightarrow Log(-e\ i) = ln\ e + i\ (-\frac{\pi}{2}$
$\Rightarrow Log(-e\ i) = 1 + i\ (-\frac{\pi}{2} = 1 - \frac{\pi}{2}\ i$. ■

Problem 12.

Find the principal value of i^i.

Solution.

We have, the principal value of $i^i = e^{i\ Log\ i}$. But, $Log\ i = ln\ |i| + i\ Arg\ (i) = 0 + \ i\ \frac{\pi}{2} = \frac{\pi}{2}i$. Therefore, the principal value of $i^i = e^{i\ Log\ i} = e^{i(\ i\ \frac{\pi}{2})} = e^{-\frac{\pi}{2}}$.

■

Now we define the trigonometric and hyperbolic functions of a complex variable z.

We define the *sine* and *cosine* functions of a complex variable z as follows:

$$sin\ z = \frac{e^{iz} - e^{-iz}}{2\ i}\ and\ cos\ z = \frac{e^{iz} + e^{-iz}}{2}.$$

These functions are entire since they are linear combinations of the entire functions e^{iz} and e^{-iz}.

From the definitions, it follows that $\frac{d}{dz}sin\ z = cos\ z$ and $\frac{d}{dz}cos\ z = -sin\ z$.

The other four trigonometric functions are defined in terms of the sine and cosine functions as in the real case.

The *hyperbolic sine* and the *hyperbolic cosine* of a complex variable z are defined as :

$$sinh\ z = \frac{e^z - e^{-z}}{2}\ and\ cosh\ z = \frac{e^z + e^{-z}}{2}.$$

Since e^z and e^{-z} are entire, $sinh\ z$ and $cosh\ z$ are entire functions. Furthermore, $\frac{d}{dz} sinh\ z = cosh\ z,\ \frac{d}{dz} cosh\ z = sinh\ z.$

The hyperbolic tangent of z is defined by means of the equation $tanh\ z = \frac{sinh\ z}{cosh\ z}$ and is analytic in every domain in which $cosh\ z \neq 0$.
The functions $coth\ z,\ sech\ z$, and $csch\ z$ are the reciprocals of $tanh\ z,\ cosh\ z$, and $sinh\ z$, respectively.

The hyperbolic sine and cosine functions are closely related to the trigonometric functions as follows:

$$-i\ sinh\ (iz) = sin\ z,\ cosh\ (iz) = cos\ z,$$

$$-isin\ (iz) = sinh\ z,\ \ cos\ (iz) = cosh\ z.$$

In order to define the inverse sine function $sin^{-1}z$, we write $w = sin^{-1}z$ when $z = sin\ w$.
i.e., $w = sin^{-1}z$ when $z = \frac{e^{iw} - e^{-iw}}{2i}$.
$\Rightarrow (e^{iw})^2 - 2\ i\ z(e^{iw}) - 1 = 0,$
which is quadratic equation in e^{iw}, and solving for e^{iw}, we get
$e^{iw} = i\ z + (1 - z^2)^{1/2}$ where $(1 - z^2)^{1/2}$ is a double–valued function of z.
Taking logarithms on both sides we get $w = sin^{-1}z = -i\ log[i\ z + (1 - z^2)^{1/2}]$.
Since the logarithmic function is multiple-valued, we see that $sin^{-1}z$ is a multiple-valued function, with infinitely many values at each point z.

Similarly, one can find that $cos^{-1}z = -i\ log\ [z + i\ (1 - z^2)^{1/2}]$ and that $tan^{-1}z = \frac{i}{2}\ log\ \frac{i+z}{i-z}$. The functions $cos^{-1}z$ and $tan^{-1}z$ are also multiple-valued.

In a similar way, inverse hyperbolic functions can be found in a corresponding manner.

It turns out that $sinh^{-1}z = log\ [z + (z^2+1)^{1/2}]$, $cosh^{-1}z = log\ [z + (z^2-1)^{1/2}]$, and $tanh^{-1}z = \dfrac{1}{2}\ log\ \dfrac{1+z}{1-z}$.

Exercises.

1. Find all values of z such that

$$\text{(a) } e^z = -2; \text{ (b) } e^z = 1+\sqrt{3}i; \;\text{ (c) } exp(2z-1) = 1; \;\text{ (d) } log\ z = i\pi/2;$$
$$\text{(e) } sinh\ z = i \;\text{ (f) } cosh\ z = \frac{1}{2}; \;\text{ (g) } sin\ z = 2; \;\text{ (h) } cos\ z = \sqrt{2}.$$

2. Show that if e^z is real, then $Im\ z = n\pi, (n = 0, \pm1, \pm2, ...)$.

3. Show that

   (a) $exp(\dfrac{2+\pi i}{4}) == \sqrt{\dfrac{e}{2}}(1+i)$.

   (b) $Log(1-i) = \dfrac{1}{2}\ ln2 - \dfrac{\pi}{4}\ i$.

   (c) $log\ e = 1 + 2n\pi i\ (n = 0, \pm1, \pm2, ...)$

   (d) $log\ i = (2n + \dfrac{1}{2})\pi i\ (n = 0, \pm1, \pm2, ...)$

   (e) $log\ (-1+\sqrt{3}i) = ln\ 2 + 2(n + \dfrac{1}{3})\pi i\ (n = 0, \pm1, \pm2, ...)$.

   (f) $(-1)^{1/\pi} = e^{(2n+1)i}\ (n = 0, \pm1, \pm2, ...)$

4. Find the principal value of $(1-i)^{4i}$.

5. Show that $|sin\ z|^2 = sin^2x + sinh^2y$ and $|cos\ z|^2 = cos^2x + sinh^2y$.

6. Show that $sin^{-1}(-i) = n\pi + i(-1)^{n+1}ln(1+\sqrt{2})\ (n = 0, \pm1, \pm2, ...)$

7. Find all the values of (a) $tan^{-1}(2\ i)$, (b) i^{-2i}, (c) $(1+i)^i$.

8. Solve the equation $cos\ z = \sqrt{2}$ for z.

9. Show that $\overline{cos\ (iz)} = cos(i\overline{z})$ for all z.

10. Show that neither $sin\ z$ nor $cos\ z$ is an analytic function of z anywhere.

11. Show that the function $f(z) = Log\ (z - i)$ is analytic everywhere except on the portion $x \leq 0$ of the line $y = 1$.

12. Show that the function $ln\ (x^2 + y^2)$ is harmonic in every domain that does not contain the origin.

Chapter 2

Complex Integration

In the first chapter, we have studied about derivatives of complex functions. We now turn to the problem of integrating complex functions. The theory that we will learn is elegant, powerful, and a useful tool for physicists and engineers. It also connects widely with other branches of mathematics. For example, even though the ideas presented here belong to the general area of mathematics known as analysis, we will see as an application of them gives us one of the simplest proofs of the *fundamental theorem of algebra*.

The complex integration helps us in the evaluation of certain real definite integrals and improper integrals, by changing them as the integration of a suitable complex function around a special *simple closed path* or *contour* of integration, where the usual methods of real integration fails. Also, complex integration theory is useful in establishing some basic properties of analytic functions.

We will find that integrals of analytic functions are well behaved and that many properties from calculus carry over to the complex case. We introduce the integral of a complex function by defining the integral of a complex-valued function of a real variable.

2.1 Definite Integrals

Let w be a complex function of a real variable t, then we can write $w(t) = u(t) + i\ v(t)$, where u and v are real valued functions of t.

Then, the *derivative* $w'(t)$ at a point t is defined as $w'(t) = u'(t) + i\ v'(t)$, provided each of the derivatives u' and v' exists at t.

The *definite integral* of $w(t)$ over an interval $a \le t \le b$ is defined as $\int_a^b w(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt$, provided each of the integrals on the right exists.

Thus, $Re[\int_a^b w(t)dt] = \int_a^b Re[w(t)]dt$ and $Im[\int_a^b w(t)dt] = \int_a^b Im[w(t)]dt$.

Remark.

The existence of the integrals of u and v in the above definition is ensured if those functions are piece wise continuous on the interval $a \le t \le b$.

The following properties will holds for complex definite integrals.

(i) $\int_a^b kw(t)dt = k\int_a^b w(t)dt$, *for any complex constant* k,

(ii) $\int_a^b [w_1(t) + w_2(t)]dt = \int_a^b w_1(t)dt + \int_a^b w_2(t)dt$, *for any complex functions* w_1 *and* w_2,

(iii) $\int_a^b w(t)dt = -\int_b^a w(t)dt$,

(iv) $\int_a^b w(t)dt = \int_a^c w(t)dt + \int_c^b w(t)dt$, *where* $a < c < b$, *and*

(v) if $w(t) = u(t) + i\ v(t)$ *and* $W(t) = U(t) + i\ V(t)$ *are continuous on the interval* $a \le t \le b$, *and if* $W'(t) = w(t)$ *on* $a \le t \le b$, *then* $U'(t) = u(t)$ *and* $V'(t) = v(t)$ *and hence* $\int_a^b w(t)dt = W(b) - W(a)$. *(*Fundamental Theorem of Calculus*)* .

Problem 13.

Evaluate the following integrals.

(a). $\int_0^{\frac{\pi}{4}} e^{it}dt.$

(b). $\int_0^1 (1+i\ t)^2 dt$

Solution.

(a). We have $\int_0^{\frac{\pi}{4}} e^{it}dt = \left[\ \frac{e^{it}}{i}\ \right]_0^{\frac{\pi}{4}} = -i\left[\ \ e^{i\frac{\pi}{4}} - 1\ \right] = \dfrac{1}{\sqrt{2}} + i\ (1 - \dfrac{1}{\sqrt{2}}).$

(b). $\int_0^1 (1+i\ t)^2 dt = \int_0^1 (1-t^2)dt +\ i\int_0^1 2\ t\ dt = \dfrac{2}{3} +\ i.$ ■

Problem 14.

If m and n are integers, show that $\int_0^{2\pi} e^{im\theta}e^{-in\theta}d\theta = \left\{ \begin{array}{ccc} 0 & when & m \neq n \\ 2\pi & when & m = n \end{array} \right\}.$

Solution.

First, assume that $m \neq n$, then

$$\int_0^{2\pi} e^{im\theta}e^{-in\theta}d\theta\ =\ \int_0^{2\pi} e^{i(m-n)\theta}d\theta\ =\ \left[\ \frac{e^{i(m-n)\theta}}{i(m-n)}\ \right]_0^{2\pi}\ =\ \left[\ \frac{e^{i(m-n)2\pi} - e^0}{i(m-n)}\ \right]\ =\ 0$$

Now, let $m = n$, then

$$\int_0^{2\pi} e^{im\theta}e^{-im\theta}d\theta = \int_0^{2\pi} d\theta = \left[\ \theta\ \right]_0^{2\pi} = 2\pi$$

■

2.1.1 Contours

Definition 2.1.1.

A set of points $z = (x, y)$ in the complex plane is said to be an ***arc*** if $x = x(t),\ \ y = y(t),\ (a \leq t \leq b)$, where $x(t)$ and $y(t)$ are continuous functions of the real parameter t.

Thus, an arc C in the complex plane is a continuous function from $[a, b]$ to the complex plane and we describe C by the equation $z = z(t) = x(t) + \ i\ y(t)$, $(a \le t \le b)$. The sense of increasing values of t induces an orientation to C.

The arc C is said to be a ***simple arc***, or a ***Jordan arc***, if it does not cross itself. Thus, C is simple if $z(t_1) \ne z(t_2)$ when $t_1 \ne t_2$. When the arc C is simple except for the fact that $z(b) = z(a)$, we say that C is a ***simple closed curve***, or a ***Jordan curve***. For example, a circle is a simple closed curve, whereas an 8- shaped curve is not. A curve is ***positively oriented*** when it is in the counterclockwise direction.

Example 4.

The unit circle $z = e^{i\theta}$, $(0 \le \theta \le 2\pi)$ about the origin is a simple closed curve, oriented in the counterclockwise direction. The circle $z = z_0 + Re^{i\theta}$, $(0 \le \theta \le 2\pi)$, centered at the point z_0 and with radius R is also a simple closed curve oriented in the counterclockwise direction. The arc given by $z(\theta) = e^{-i\theta}$, $(0 \le \theta \le 2\pi)$ represents the unit circle traversed in the clockwise direction. The equation $z(\theta) = e^{i2\theta}$, $(0 \le \theta \le 2\pi)$ represents the unit circle traversed twice in the counterclockwise direction.

Definition 2.1.2.

Consider an arc C represented by the equation $z = z(t) = x(t) + \ i\ y(t)$, $(a \le t \le b)$. If the components $x'(t)$ and $y'(t)$ of the derivative $z'(t) = x'(t) + i\ y'(t)$ of $z(t)$ are continuous on the entire interval $[a, b]$, then the arc C is said to be a ***differentiable arc***. In this case, the real-valued function $|z'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$ is integrable over the interval $[a, b]$, and $\int_a^b |z'(t)| dt$ gives the length L of C.

An arc C represented by the equation $z = z(t) = x(t) + \ i\ y(t)$, $(a \le t \le b)$ is said to be a ***smooth arc*** if the derivative $z'(t)$ is continuous on the closed interval $[a, b]$ and nonzero throughout the open interval (a, b). An arc consisting of a finite number of smooth arcs joined end to end is called a ***contour***, or a ***piecewise smooth arc***. Hence, if $z = z(t)$ represents a contour, then $z(t)$ is continuous, and its derivative $z'(t)$ is piecewise continuous.

When only the initial and final values of $z(t)$ are the same, a contour C is called a ***simple closed contour***. Circles, the boundary of a triangle or a rectangle taken in a specific direction are examples of simple closed contours. The length of a contour or a simple closed contour is the sum of the lengths of the smooth arcs that make up the contour.

The points on any simple closed curve or simple closed contour C are boundary points of two distinct domains, one of which is the interior of C and is bounded. The other, which is the exterior of C, is unbounded. This is known as the ***Jordan curve theorem***.

Problem 15.

Express the following curves in parametric form.

(a). The polygonal line consisting of a line segment from 0 to $1 + i$ followed by one from $1 + i$ to $2 + i$.

(b). The curve $y = \frac{1}{x}$ from $(1, 1)$ to $(4, \frac{1}{4})$.

(c). The upper half of the circle $|z + 3 - i| = 5$ in the counterclockwise direction.

Solution.

(a). The equation $z(t) = t + i\ t$, when $0 \leq t \leq 1$, $z = t + i$, when $1 \leq t \leq 2$ represents the arc consisting of a line segment from 0 to $1 + i$ followed by one from $1 + i$ to $2 + i$, and is a simple arc.

(b). The equation $z(t) = t + \frac{1}{t} i$, $1 \leq t \leq 4$ represents the curve $y = \frac{1}{x}$ from $(1, 1)$

to $(4, \frac{1}{4})$.

(c). The equation $z(\theta) = (-3 + \ i) + 5e^{i\theta}$, $0 \leq \theta \leq \pi$ represents the upper half of the circle $|z + 3 - i| = 5$ in the counterclockwise direction. ■

2.1.2 Contour Integrals

We now discuss integrals of complex-valued functions f of the complex variable z. Such an integral is defined in terms of the values $f(z)$ along a given contour C, starting from a point $z = z_1$ to a point $z = z_2$ in the complex plane. Such an integral is called a ***line integral***, and is denoted by $\int_C f(z)dz$ and its value depends, in general, on the contour C as well as on the function f.

Let $z = z(t) = x(t) + \ i\ y(t)$, $(a \leq t \leq b)$ represents a contour C and assume that $f(z)$ is defined on C. Also, we assume that $f[z(t)]$ is piecewise continuous on the interval $[a, b]$ and in this case we say the function $f(z)$ is piecewise continuous on C. Then, the ***line integral***, or ***contour integral***, of f along C is defined as

$$\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt$$

. Since C is a contour, $z'(t)$ is also piecewise continuous on $[a, b]$; and so the existence of the above integral is ensured.

From the definition of line integrals, it is clear that

$\int_C kf(z)dz = k \int_C f(z)dz$, for any complex constant k, and

$$\int_C [f_1(z) + f_2(z)]dz = \int_C f_1(z)dz + \int_C f_2(z)dz,$$

for any complex functions f_1 and f_2 defined on C.

Also, if $-C$ denotes the same set of points of C, but traversed in the reverse direction, then

$$\int_{-C} f(z)dz = -\int_C f(z)dz.$$

Now, consider a path C consisting of a contour C_1 from z_1 to z_2 followed by a contour C_2 from z_2 to z_3 such that the initial point of C_2 is the final point of C_1. Then,

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz.$$

Here, the contour C is called the sum of its ***legs*** C_1 and C_2 and is denoted by $C_1 + C_2$.

Problem 16.

Evaluate the integral $\int_C \bar{z}dz$, where C is the right hand half of the circle $|z| = 2$, from $-2i$ to $2i$.

Solution.

The equation for C is given by $z(\theta) = 2e^{i\theta}$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Therefore,

$$\begin{aligned}\int_C \bar{z}dz =& \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \overline{2e^{i\theta}}(2e^{i\theta})'d\theta\\ =&4\ i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i\theta}e^{i\theta}d\theta\\ =&4\ i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = 4\pi i\end{aligned}$$

■

Remark.

In the above problem, note that $z\bar{z} = |z|^2 = 4$ on C, so that $\int_C \frac{1}{z}dz = \frac{1}{4}\int_C \bar{z}dz = \pi i$.

Problem 17.

Integrate $\frac{1}{z}$, around C, the unit circle $|z| = 1$, taken in the counterclockwise direction.

Solution.

The equation for C is given by $z(\theta) = e^{i\theta}$, $0 \leq \theta \leq 2\pi$. Therefore,

$$\begin{aligned}
\int_C \frac{1}{z} dz &= \int_0^{2\pi} \frac{1}{e^{i\theta}} (e^{i\theta})' d\theta \\
&= i \int_0^{2\pi} e^{-i\theta} e^{i\theta} d\theta \\
&= i \int_0^{2\pi} d\theta = 2\pi i
\end{aligned}$$

■

2.1.3 Upper Bounds for Absolute Value of Contour Integrals

We now discuss a theorem which helps us to obtain an upper bound for the modulus of a contour integral $\int_C f(z)dz$, without evaluating the interval.

Lemma 2.1.3.

If $w(t)$ is a piecewise continuous complex-valued function defined on an interval $a \leq t \leq b$, then

$$|\int_a^b w(t)dt| \leq \int_a^b |w(t)|dt.$$

Theorem 2.1.4.

Let C denotes a contour of length L, and suppose that a function $f(z)$ is piece wise continuous on C. If M is a nonnegative constant such that $|f(z)| \leq M$ for *all points z on C, then $|\int_C f(z)dz| \leq ML$.*

Problem 18.

Find an upper bound $|\int_C \dfrac{z+4}{z^3-1}dz|$, where C is the arc of the circle $|z|=2$ from $z=2$ to $z=2i$, without evaluating the integral.

Solution.

Here, the length of C is π. Also along C, $|z+4| \leq |z|+4 = 6$, and $|z^3-1| \geq ||z|^3-1| = 7$. Therefore, $|\dfrac{z+4}{z^3-1}| = \dfrac{|z+4|}{|z^3-1|} \leq \frac{6}{7}$. Thus, by Theorem 2.1.4, $|\int_C \dfrac{z+4}{z^3-1}dz| \leq \frac{6\pi}{7}$. ■

Problem 19.

If C is the boundary of the triangle with vertices at the points $0, 3i$ and -4, oriented in the counterclockwise direction, then show that $|\int_C (e^z - \bar{z})dz| \leq 60$, without evaluating the integral.

Solution.

Here, the length L of C is the sum of the lengths of the sides of the triangle, and therefore $L = 3+4+5 = 12$. We have, $|e^z - \bar{z}| \leq |e^z| + |\bar{z}|$. Along C, $|e^z| = e^x \leq 1$ and $|\bar{z}| \leq 4$. Therefore, $|e^z - \bar{z}| \leq 1+4 = 5$. Thus, by theorem 2.1.4, $|\int_C (e^z - \bar{z})dz| \leq 5.12 = 60$. ■

Exercises.

1. Evaluate the following integrals.

   (a) $\int_1^2 (\frac{1}{t} - i)^2 dt$.

   (b) $\int_0^{\frac{\pi}{6}} e^{i2t} dt$

(c) $\int_0^\infty e^{-zt}dt$, $(Re\ z > 0)$

2. Find a parametric representation $z = z(t)$ for

   (a) The straight-line segment from 0 to $4 - 7i$.

   (b) The upper half of $|z - 4 + 3i| = 5$ in the counterclockwise direction.

   (c) The arc $y = x^3$ from $(-2, -8)$ to $(3, 27)$.

   (d) The lower half of the circle with center at $z = 1$ and radius 1 in the clockwise direction.

3. Evaluate the contour integral $\int_C f(z)dz$, where,

   (a) $f(z) = \frac{z+2}{z}$ and C is the semi circle $z = 2e^{i\theta}, (0 \leq \theta \leq \pi)$.

   (b) $f(z) = \bar{z}$ and C is parabola $y = x^2$ from 0 to $1 + i$.

   (c) $f(z) = cosz$ and C is the semi circle $|z| = \pi$ from $-\pi i$ to π.

   (d) $f(z) = Rez^2$ and C is the unit circle $|z| = 1$ counterclockwise.

   (e) $f(z) = Rez^2$ and C is the boundary of the square with vertices $0,\ i,\ 1+i,\ 1$, oriented in the clockwise direction.

   (f) $f(z) = \pi\ \exp\ (\pi\bar{z})$ and C is the boundary of the square with vertices $0,\ 1,\ 1+i,\ i$, oriented in the counterclockwise direction.

4. Find an upper bound $|\int_C \dfrac{dz}{z^2 - 1}|$, where C is the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$, without evaluating the integral.

5. Let C denote the line segment joining the points $z = i$ to $z = 1$. Show that $|\int_C \dfrac{dz}{z^4}| \leq 4\sqrt{2}$, without evaluating the integral.

2.2 Theorems on Complex Integration

We now, state a theorem that contains an extension of the fundamental theorem of calculus that simplifies the evaluation of many contour integrals. This extension involves the concept on an ***antiderivative*** of a continuous function $f(z)$ on a domain D, i.e., a function $F(z)$ such that $F'(z) = f(z)$ for all z in D.

Note that an antiderivative is an analytic function and that an antiderivative of a given function $f(z)$ is unique except for an additive constant. (This is because the derivative of the difference $F(z) - G(z)$ of any two such antiderivatives is zero, and since an analytic function whose derivative is zero throughout in a domain D, is a constant in$D$.)

Theorem 2.2.1.

Suppose that a function $f(z)$ is continuous on a domain D. If any one of the following statements is true, then so are the others:
(a) $f(z)$ has an antiderivative $F(z)$ throughout D,
(b) the integrals of $f(z)$ along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value, namely $\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$, where $F(z)$ is the antiderivative of $f(z)$,
(c) the integrals of $f(z)$ around closed contours lying entirely in D all have value zero.

Remark.

Note that the theorem 2.2.1 does not claim that any of these statements is true for a given function $f(z)$. It says only that all of them are true or that none of them is true.

Problem 20.

By finding an antiderivative, find the value of $\int \frac{1}{z^2}dz$, where C is the circle $|z| = 2$ oriented in counterclockwise sense.

Solution.

The function $f(z) = \frac{1}{z^2}$, which is continuous everywhere except at the origin, has an antiderivative $F(z) = \frac{-1}{z}$ in the domain $|z| > 0$, consisting of the entire plane with the origin deleted. Therefore, by theorem 2.2.1, $\int \frac{1}{z^2}dz = 0$. ■

2.2.1 Cauchy - Goursat Theorem

In theorem 2.2.1, we noted that when a continuous function f has an antiderivative in a domain D, then the integral of $f(z)$ around any given closed contour C lying entirely in D has value zero.

Now, we present a theorem giving other conditions on a function f which ensure that the value of the integral of $f(z)$ around a simple closed contour is zero.

Theorem 2.2.2. (Cauchy - Goursat Theorem)

If a function f is analytic at all points interior to and on a simple closed contour C, then $\int_C f(z)dz = 0$.

Definition 2.2.3.

A ***simply connected domain*** D is a domain such that every simple closed contour within it encloses only points of D.

A domain that is not simply connected is said to be ***multiply connected***.

Example 5.

The set of points interior to a simple closed contour is a simply connected domain. The annular domain between two concentric circles is, however, not simply connected, it is multiply connected.

We have the following generalization of Cauchy - Goursat Theorem.

Theorem 2.2.4.

If a function f is analytic throughout a simply connected domain D, then $\int_C f(z)dz = 0$ for every closed contour C lying in D.

From theorem 2.2.1, we get the following result.

Corollary 2.2.5.

A function f that is analytic throughout a simply connected domain D must have an antiderivative everywhere in D.

Remark.

Note that the finite complex plane is simply connected. Therefore, by the above corollary, we see that entire functions always possess antiderivatives.

The following theorem generalizes the Cauchy - Goursat theorem to multiply connected domains.

Theorem 2.2.6.

Suppose that

(a) C is a simple closed contour, described in the counterclockwise direction,

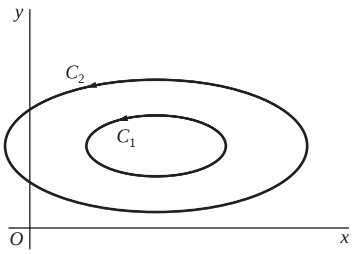
(b) C_k $(k = 1, 2, ..., n)$ are simple closed contours interior to C, all described in the clockwise direction, that are disjoint and whose interiors have no points in common.

If a function f is analytic on all of these contours and throughout the multiply connected domain consisting of the points inside C and exterior to each C_k, then $\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0$.

The following corollary shows that if C_1 is continuously deformed into C_2, always passing through points at which f is analytic, then the value of the integral of f over C_1 never changes.

Corollary 2.2.7. *(***Principle of Deformation of Paths** *)*

Let C_1 and C_2 denote positively oriented simple closed contours, where C_1 is interior to C_2. If a function f is analytic in the closed region consisting of those contours and all points between them, then $\int_{C_2} f(z)dz = \int_{C_1} f(z)dz$.



Example 6.

Let C be any positively oriented simple closed contour surrounding the origin. Also, let C_0 be a positively oriented circle with center at the origin and radius so small that C_0 lies entirely inside C. Since, $\int_{C_0} \frac{dz}{z} = 2\pi i$ and since $\frac{1}{z}$ is analytic everywhere except at $z = 0$, then by the above Corollary 2.2.7, it follows that $\int_C \frac{dz}{z} = 2\pi i$.

2.2.2 Cauchy's Integral Formula

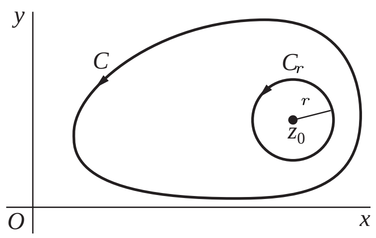
Now we state the ***Cauchy's Integral Formula***, which asserts that if a function f is analytic within and on a simple closed contour C, then the values of f interior to C are completely determined by the values of f on C.

Theorem 2.2.8.

Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then $f(z_0) = \dfrac{1}{2\pi i}\int_C \dfrac{f(z)dz}{z - z_0}$.

Proof.

Let C_r denote a positively oriented circle $|z - z_0| = r$, where r is small enough that C_r is interior to C. Then, the quotient $\dfrac{f(z)}{(z - z_0)}$ is analytic between and on the contours C_r and C.



Therefore, by Corollary 2.2.7, we have

$$\int_C \frac{f(z)}{(z - z_0)}dz = \int_{C_r} \frac{f(z)}{(z - z_0)}dz.$$

This implies that,

$$\int_C \frac{f(z)}{(z - z_0)}dz - f(z_0)\int_{C_r} \frac{1}{(z - z_0)}dz. = \int_{C_r} \frac{f(z) - f(z_0)}{(z - z_0)}dz.$$

As in Problem 17, we obtain $\int_{C_r} \dfrac{1}{(z - z_0)}dz = 2\pi i$, so that

$$\int_C \frac{f(z)}{(z - z_0)}dz - 2\pi i\ f(z_0) = \int_{C_r} \frac{f(z) - f(z_0)}{(z - z_0)}dz.$$

Since f is analytic, and therefore continuous, at z_0 ensures that corresponding to each positive number ε, there is a positive number δ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$. Let the radius r of the circle C_r be smaller than the number δ. Then $|z - z_0| = r < \delta$ when z is on C_r, so that $|f(z) - f(z_0)| < \varepsilon$ when z is such a point. Therefore, by Theorem 2.1.4, we obtain

$$\int_C \frac{f(z) - f(z_0)}{z - z_0} dz < \frac{\varepsilon}{r} 2\pi r = 2\pi\varepsilon.$$

Thus,

$$|\int_C \frac{f(z)}{(z - z_0)} dz - 2\pi i f(z_0)| < 2\pi\varepsilon.$$

Since, $\varepsilon > 0$ is arbitrary, it follows that

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{z - z_0}.$$

□

2.2.3 Cauchy's integral formula for Derivatives

The Cauchy's integral formula can be extended to provide an integral representation for derivatives of f at z_0. We assume that the function f is analytic everywhere inside and on a simple closed contour C, taken in the positive sense and z_0 is any point interior to C. Then, for $n = 1, 2, 3, ...,$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}}.$$

This is known as ***Cauchy's integral formula for derivatives***.

Remark.

From Cauchy's integral formula for derivatives, it follows that if a function f is analytic at a given point, then its derivatives of all orders are analytic there too.

Problem 21.

Evaluate $\int_c \frac{z^2-z+1}{z-1}dz$, where C is (a). $|z| = \frac{1}{2}$, and (b). $|z| = 2$, oriented in the positive sense.

Solution.

(a). Here, $f(z) = \dfrac{z^2-z+1}{z-1}$ is analytic at all points except the point $z = 1$ and $z = 1$ lies outside C.
Therefore by Cauchy – Goursat theorem, $\int_C \frac{z^2-z+1}{z-1}dz = 0$.
(b). Here $f(z) = z^2 - z + 1$ is analytic everywhere and C encloses the point $z = 1$. Therefore, by Cauchy's integral formula, we get

$$\int_C \frac{z^2-z+1}{z-1}dz = 2\pi i f(1) = 2\pi i.$$

■

Problem 22. Evaluate $\int_c \frac{z}{(9-z^2)(z+i)}dz$, where C is the positively oriented circle $|z| = 2$.

Solution.

Here, $f(z) = \dfrac{z}{(9-z^2)}$ is analytic within and on C, and $z_0 = -i$ lies inside C. Therefore by Cauchy's integral formula, we get

$$\int_c \frac{z}{(9-z^2)(z+i)}dz = \int_c \frac{\frac{z}{(9-z^2)}}{(z-(-i))}dz = 2\pi i f(-i) = 2\pi i(\frac{-i}{10}) = \frac{\pi}{5}.$$

■

Problem 23. Evaluate $\int_c \frac{e^{2z}}{z^4}dz$, where C is the positively oriented unit circle.

Solution.

Here, $f(z) = e^{2z}$ is analytic within and on C, and $z_0 = 0$ lies inside C. Therefore by Cauchy's integral formula for derivatives, we get

$$\int_c \frac{e^{2z}}{z^4} dz = \int_c \frac{e^{2z}}{(z-0)^{3+1}} dz = \frac{2\pi i}{3!} f^{(3)}(0) = \frac{8\pi i}{3}.$$

■

Problem 24. Evaluate $\int_c \frac{e^{2z}}{(z-1)(z-2)} dz$, where C is the positively oriented circle $|z| = 3$.

Solution.

Here, $\frac{e^{2z}}{(z-1)(z-2)}$ is analytic everywhere, except the points $z = 1$ and $z = 2$, and both of these points lies inside C. Using partial fractions, we have

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}.$$

Therefore by Cauchy's integral formula, we get

$$\int_C \frac{e^{2z}}{(z-1)(z-2)} dz = \int_C \frac{e^{2z}}{z-2} dz - \int_C \frac{e^{2z}}{z-1} dz = 2\pi i(e^4 - e^2).$$

■

Problem 25. Evaluate $\int_c \frac{z^4 - 3z^2 + 6}{(z+i)^3} dz$, where C is any positively oriented contour enclosing the point $z_0 = -i$.

Solution.

Here, $f(z) = z^4 - 3z^2 + 6$ is analytic everywhere and the point $z_0 = -i$ lies inside C. Therefore by Cauchy's integral formula for derivatives, we get

$$\int_c \frac{z^4 - 3z^2 + 6}{(z+i)^3} dz = \int_c \frac{z^4 - 3z^2 + 6}{(z+i)^{2+1}} dz = \frac{2\pi i}{2!} f^{(2)}(-i) = -18\pi i..$$

■

2.2.4 Consequences of Cauchy's Integral Formula

The following theorem, known as ***Morera's theorem***, gives a partial converse to the Cauchy – Goursat theorem.

Theorem 2.2.9.

Let f be continuous on a domain D. If $\int_C f(z)dz = 0$ for every closed contour C in D, then f is analytic throughout D.

Proof.

By Theorem 2.2.1, f has an antiderivative in D, i.e., there exists an analytic function F such that $F'(z) = f(z)$ at each point in D. Since f is the derivative of an analytic function F, it follows that (*See* the Remark above) f is analytic in D. □

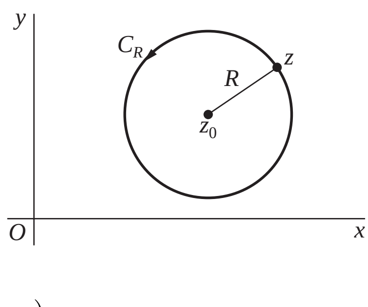
Theorem 2.2.10. *(***Cauchy's inequality***)*

Suppose that a function f is analytic inside and on a positively oriented circle C_R, centered at z_0 and with radius R. If M_R denotes the maximum value of $|f(z)|$ on C_R, then

$$|f^{(n)}(z_0)| \leq \frac{n!M_R}{R^n}, \quad (n = 1, 2, ...).$$

Proof.

By Cauchy's integral formula for derivatives, we have



$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}} \quad (n = 1, 2, ...).$$

Applying Theorem 2.1.4, we see that

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \frac{M_R 2\pi R}{R^{n+1}} \quad (n = 1, 2, ...).$$

Therefore,

$$|f^{(n)}(z_0)| \leq \frac{n!M_R}{R^n}, \quad (n = 1, 2, ...).$$

□

Now we make use of Cauchy's inequality to prove that no entire function except a constant is bounded in the complex plane. This is known as ***Liouville's theorem***.

Theorem 2.2.11.

If a function f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the complex plane.

Proof.

Since f is bounded in the complex plane, there exists a nonnegative constant M such that $|f(z)| \leq M$ for all z.

Therefore, since f is entire, for any choice of z_0 and R, for the value $n = 1$, the Cauchy's inequality implies that, $|f'(z_0)| \leq \dfrac{M}{R}$.

Letting $R \to \infty$, we get $f'(z_0) = 0$. Since the choice of z_0 was arbitrary, this means that $f'(z) = 0$ everywhere in the complex plane. This shows that f is a constant function. □

By using Liouville's theorem, we now give a simple proof for the ***fundamental theorem of algebra***.

Theorem 2.2.12.

Any polynomial $P(z) = a_0 + a_1z + a_2z^2 + ... + a_nz^n$, $(a_n \neq 0)$ of degree n $(n \geq 1)$ has at least one zero. i.e., there exists at least one point z_0 such that $P(z_0) = 0$.

Proof.

Suppose that $P(z)$ is not zero for any value of z. Then the reciprocal $f(z) = \dfrac{1}{P(z)}$ is an entire function, and it is also bounded in the complex plane. Therefore, by Liouville's theorem $f(z)$, and hence $P(z)$ is a constant. But $P(z)$ is not a constant. This contradiction shows that there exists at least one point z_0 such that $P(z_0) = 0$. $\square$

Lemma 2.2.13.

Suppose that $|f(z)| \leq |f(z_0)|$ at each point z in some neighborhood $|z - z_0| < \varepsilon$ in which f is analytic. Then $f(z)$ has the constant value $f(z_0)$ throughout that neighborhood.

Theorem 2.2.14. (Maximum Modulus Principle)

If a function f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D. That is, there is no point z_0 in the domain such that $|f(z)| \leq |f(z_0)|$ for all points z in D.

Corollary 2.2.15.

Suppose that a function f is continuous on a closed bounded region R and that it is analytic and not constant in the interior of R. Then the maximum value of $|f(z)|$ in R, which is always reached, occurs somewhere on the boundary of R and never in the interior.

Exercises.

1. By finding an antiderivative, evaluate each of these integrals, where the path is any contour between the indicated limits of integration: (a) $\int_i^{i/2} e^{\pi z} dz$, (b) $\int_0^{\pi+2i} cos(\frac{z}{2}) dz$, (c) $\int_1^3 (z-2)^3 dz$.

2. Apply the Cauchy - Goursat theorem to show that $\int_C f(z)dz = 0$, when the contour C is the unit circle $|z| = 1$, in either direction, and when (a) $f(z) = \dfrac{z^2}{z-3}$, (b) $f(z) = ze^{-z}$, (c) $f(z) = \dfrac{1}{z^2+2z+2}$, (d) $f(z) = tanz$.

3. Evaluate $\int_c \dfrac{dz}{z^2+2}$, where C is the positively oriented unit circle.

4. Evaluate $\int_c \dfrac{dz}{\bar{z}}$, where C is the positively oriented unit circle.

5. Verify Cauchy – Goursat theorem for the function $f(z) = z^2$, where C is the boundary of the rectangle with vertices $-1, 1, 1+i, -1+i$, taken in the counterclockwise sense.

6. Evaluate $\int_c \dfrac{e^{2z}}{(z+1)^4}dz$, where C is the positively oriented circle $|z| = 2$.

7. Evaluate $\int_c \dfrac{e^{z^2}}{z(z-2i)^2}dz$, where C is the boundary of the square with vertices $\pm 3 \pm 3i$, oriented in the counterclockwise direction.

8. Suppose that $f(z)$ is entire and that the harmonic function $u(x, y) = Re[f(z)]$ has an upper bound u_0 i.e., $u(x, y) \leq u_0$ for all points (x, y) in the xy-plane. Show that $u(x, y)$ must be constant throughout the plane.

Chapter 3

Series of Complex Numbers

In this chapter we study about the convergence and divergence of complex sequences and complex infinite series. The basic definitions for complex sequences and series are essentially the same as for the real case.

3.1 Convergence of Sequences and Series of Complex Numbers

A complex sequence is a function of the form $f : N \to \mathbb{C}$, where for every $n \in \mathbb{N}$, we write $f(n) = z_n$. An ***infinite sequence*** $z_1, z_2, ..., z_n, ...$ of complex numbers has a limit z if, for each positive number ε, there exists a positive integer n_0 such that $|z_n - z| < \varepsilon$ whenever $n > n_0$.

The quantity $|z_n - z|$ measures the difference between z_n and its intended limit z. The definition thus says that this difference can be made as small as we like, provided that n is large enough. It follows that the convergence is not affected by the initial terms. Observe that the inequality $|z_n - z| < \varepsilon$ is equivalent to saying that the point z_n lies inside a circle of radius ε and centered at z. In the case when $z_n = x_n$ and $z = x$ are real, the inequality $|x_n - x| < \varepsilon$ is equivalent to the inequalities $x - \varepsilon < xn < x + \varepsilon$, so that x_n lies in the open interval $(x - \varepsilon, x + \varepsilon)$.

The limit of the above sequence is unique if it exists. If that limit exists, the sequence is said to ***converge*** to z, and we write $lim_{n\to\infty} z_n = z$. If the sequence has no limit, we say that it ***diverges***. The proof of the following theorem is left as an exercise.

Theorem 3.1.1.

Suppose that $z_n = x_n + iy_n$ $(n = 1, 2, ...)$ *and* $z = x + iy$. *Then* $lim_{n\to\infty} z_n = z$ *if and only if* $lim_{n\to\infty} x_n = x$ *and* $lim_{n\to\infty} y_n = y$.

An infinite series $\Sigma_{n=1}^{\infty} z_n = z_1 + z_2 + ... + z_n + ...$ of complex numbers ***converges*** to the ***sum*** S if the sequence $S_N = \Sigma_{n=1}^{N} z_n = z_1 + z_2 + ... + z_N$, $(N = 1, 2, ...)$ of partial sums converges to S, we then write $\Sigma_{n=1}^{\infty} z_n = S$. Since a sequence can have at most one limit, a series can have at most one sum. When a series does not converge, we say that it ***diverges***.

Theorem 3.1.2.

Suppose that $z_n = x_n + iy_n$, $(n = 1, 2, ...)$ *and* $S = X + iY$. *Then* $\Sigma_{n=1}^{\infty} z_n = S$ *if and only if* $\Sigma_{n=1}^{\infty} x_n = X$ *and* $\Sigma_{n=1}^{\infty} y_n = Y$.

Proof.

Let S_N denote the sequence of partial sums of the series $\Sigma_{n=1}^{\infty} z_n$. Then we can write $S_N = X_N + iY_N$, where $X_N = \Sigma_{n=1}^{N} x_n$ and $Y_N = \Sigma_{n=1}^{N} y_n$. Therefore, $\Sigma_{n=1}^{\infty} z_n = S$ if and only if $lim_{n\to\infty} S_N = S$. By Theorem 3.1.1, $lim_{N\to\infty} S_N = S$ if and only if $lim_{N\to\infty} X_N = X$ and $lim_{N\to\infty} Y_N = Y$. Thus, $\Sigma_{n=1}^{\infty} z_n = S$ if and only if $\Sigma_{n=1}^{\infty} x_n = X$ and $\Sigma_{n=1}^{\infty} y_n = Y$. $\square$

Corollary 3.1.3.

If a series of complex numbers converges, the n^{th} term converges to zero as n tends to infinity.

This corollary shows that the terms of convergent series are bounded. That is, when series $\Sigma_{n=1}^{\infty} z_n$ converges, there exists a positive constant M such that $|z_n| \leq M$ for each positive integer n.

A series of complex numbers $\Sigma_{n=1}^{\infty} z_n$ is said to be ***absolutely convergent*** if the series $\Sigma_{n=1}^{\infty} |z_n| = \sqrt{x_n^2 + y_n^2}$, $(z_n = x_n + iy_n)$ of real numbers $\sqrt{x_n^2 + y_n^2}$ converges.

Corollary 3.1.4.

The absolute convergence of a series of complex numbers implies the convergence of that series.

In complex analysis, we define power series in a formally identical way to the real case: namely, a power series centered at a complex number z_0 is an expression of the form $\Sigma_{n=0}^{\infty} a_n (z - z_0)^n$; where a_n be complex numbers. Notice that for any complex number z, this infinite series is a series of complex numbers, which either converges or diverges.

A power series can be thought of as a generalization of a polynomial, but unlike polynomials power series do not necessarily converge at all points z. Power series will provide a large source of analytic functions, and we will see that power series play a key role in understanding properties of analytic functions.

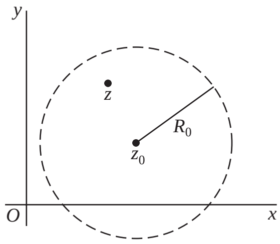
3.2 Taylor Series

We begin with the ***Taylor's theorem***.

Theorem 3.2.1.

Suppose that a function f is analytic throughout a disk $|z-z_0| < R_0$, centered at z_0 and with radius R_0. Then $f(z)$ has the power series representation

$$f(z) = \Sigma_{n=1}^{\infty} a_n (z-z_0)^n \qquad (|z-z_0| < R_0), \text{ where } a_n = \frac{f^{(n)}(z_0)}{n!} \quad (n = 0, 1, 2, ...).$$



i.e., $\Sigma_{n=1}^{\infty} a_n (z-z_0)^n$ converges to $f(z)$ when z lies in the open disk $|z-z_0| < R_0$.

(This expansion of $f(z)$ is called the **Taylor series** of $f(z)$ about the point z_0.)

Since $f^{(0)}(z_0) = f(z_0)$ and $0! = 1$, the Taylor series of $f(z)$ about the point z_0 can be written as $f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + ...,\ (|z-z_0| < R_0)$.

Remark.

A Taylor's series about the point $z_0 = 0$, $f(z) = \Sigma_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \quad (|z| < R_0)$ is called a ***Maclaurin series***.

Any function which is analytic at a point z_0 must have a Taylor series about z_0. For, if f is analytic at z_0, it is analytic throughout some neighborhood $|z-z_0| < \varepsilon$ of z_0. Therefore by Taylor's theorem, $f(z)$ have a Taylor series about z_0 valid in $|z-z_0| < \varepsilon$. Also, if f is entire, R_0 can be chosen arbitrarily large, and the condition of validity becomes $|z-z_0| < \infty$ and the Taylor series then converges to $f(z)$ at each point z in the finite plane. If f is analytic everywhere inside a circle centered at z_0, then the Taylor series of $f(z)$ about z_0 converges to $f(z)$ for each point z within that circle and in fact, according to Taylor's theorem, the series converges to $f(z)$ within the circle about z_0 whose radius is the distance from z_0 to the nearest point z_1 at which f fails to be analytic.

Example 7.

Consider the function $f(z) = e^z$. Since $f(z) = e^z$ is an entire function, it has a Maclaurin series representation which is valid for all z. Here, $f^{(n)}(z) = e^z$ $(n = 0, 1, 2, ...) \Rightarrow f^{(n)}(0) = 1$, $(n = 0, 1, 2, ...)$

Therefore, $e^z = \Sigma_{n=0}^{\infty} \frac{z^n}{n!}$, $\quad (|z| < \infty)$.

Example 8.

Let $f(z) = \frac{1}{1-z}$. Then, the derivatives of the function $f(z) = \frac{1}{1-z}$, which fails to be analytic at $z = 1$, are $f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}$ $(n = 0, 1, 2, ...)$. $\Rightarrow f^{(n)}(0) = n!$ $(n = 0, 1, 2, ...)$.. Therefore, $f(z) = \frac{1}{1-z} = 1 + z + z^2 + z^3 + ...$ $(|z| < 1)$.

Problem 26. Determine the Taylor's series for $f(z) = \frac{1}{z+z^4}$ valid for $|z| < 1$.

Solution.

We have,
$$\frac{1}{z+z^4} = \frac{1}{z}\frac{1}{1+z^3} = \frac{1}{z}\frac{1}{1-(-z^3)} = \frac{1}{z}\Sigma_{n=0}^{\infty}(-z^3)^n$$
$$= \frac{1}{z}\Sigma_{n=0}^{\infty}(-1)^n(z)^{3n} = \Sigma_{n=0}^{\infty}(-1)^n z^{3n-1}. \qquad \blacksquare$$

Exercises.

1. Obtain the Maclaurin series representation $f(z) = z\ cosh\ (z^2)$.
2. Obtain the Taylor series for the function $f(z) = e^z$ in powers of $(z-1)$.
3. Find the Maclaurin series expansion of the function $f(z) = \frac{z}{z^4+9}$.
4. Derive the Taylor series representation for $\frac{1}{1-z}$ valid in $|z - i| < \sqrt{2}$ in powers of $(z - i)$.

5. Expand $cos\ z$ into a Taylor series about the point $z_0 = \pi/2$.

6. Expand the function $f(z) = \dfrac{1+2z^2}{z^3+z^5}$ into a series involving powers of z.

7. Obtain the Maclaurin series for the function $f(z) = sin(z^2)$.

3.3 Laurent Series

If a function f fails to be analytic at a point z_0, but it is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered at z_0, then the power series representation for $f(z)$ involves both positive and negative powers of $z - z_0$. Such a series representation for $f(z)$ is called a ***Laurent's series***.

Theorem 3.3.1.

Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered at z_0 , and let C denote any positively oriented simple closed contour around z_0 and lying in that domain. Then, at each point in the domain, $f(z)$ has the series representation

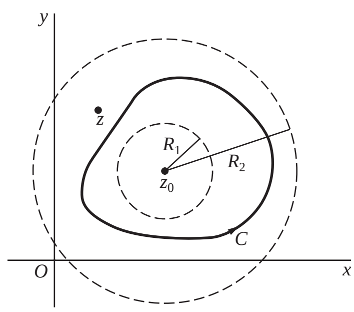
$$f(z) = \Sigma_{n=0}^{\infty} a_n (z - z_0)^n + \Sigma_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad (R_1 < |z - z_0| < R_2),$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}} \quad (n = 0, 1, 2, ...)$$

and

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}} \quad (n = 1, 2, ...).$$



Remark.

Replacing n by $-n$ in the second series in the above Laurent's series enables us to write that series as

$$\Sigma_{n=-\infty}^{-1} \frac{b_{-n}}{(z-z_0)^{-n}},$$

where

$$b_{-n} = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{n+1}} \quad (n = -1, -2, ...).$$

Thus, we have

$$f(z) = \Sigma_{n=-\infty}^{-1} b_{-n}(z-z_0)^n + \Sigma_{n=1}^{\infty} a_n(z-z_0)^n \quad (R_1 < |z-z_0| < R_2).$$

Or, we can write

$$f(z) = \Sigma_{n=-\infty}^{\infty} c_n(z-z_0)^n \quad (R_1 < |z-z_0| < R_2),$$

where

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{n+1}} \quad (n = 0, \pm 1, \pm 2, ...).$$

When the annular domain is specified, it can be proved that a Laurent's series for a given function is unique. This fact helps us to found the coefficients in a Laurent's series by means other than appealing directly to their integral representations. We illustrate this through the following examples.

Example 9.

We have the Maclaurin's series expansion of e^z as

$$e^z = \Sigma_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + ... \ (|z| < \infty).$$

Replacing z by $\frac{1}{z}$, in this series representation, we get the Laurent's series for $e^{\frac{1}{z}}$

as:

$$e^{\frac{1}{z}} = \Sigma_{n=0}^{\infty} \frac{1}{n!\ z^n} = 1 + \frac{1}{1!\ z} + \frac{1}{2!\ z^2} + \frac{1}{3!\ z^3} + ... \ (0 < |z| < \infty).$$

Note that no positive powers of z appear in this Laurent's series, the coefficients of the positive powers being zero.

Example 10.

Consider $f(z) = \dfrac{1}{(z+5)}$. We now find the Laurent's series expansions of $f(z)$ that are valid in the regions (i) $|z| < 5$, and (ii) $|z| > 5$. Here, the region in (i) is an open disk inside a circle of radius 5, centered on $z = 0$. We write $f(z) = \dfrac{1}{(z+5)} = \dfrac{1}{5(1+\frac{z}{5})} = \dfrac{1}{5(1-(-\frac{z}{5}))}$.

Therefore, by using geometric series expansion, we get $f(z) = \dfrac{1}{5}\Sigma_{n=0}^{\infty}(-\dfrac{z}{5})^n = \Sigma_{n=0}^{\infty}\dfrac{(-1)^n\ z^n}{5^{n+1}}$, valid in the region $|z| < 5$, which involves only non negative powers of z.

Now, the region in (ii) is an open annulus outside a circle of radius 5, centered on $z = 0$. Here, $|z| > 5 \Rightarrow |\frac{5}{z}| < 1$.

So, we have $f(z) = \dfrac{1}{(z+5)} = \dfrac{1}{z(1+\frac{5}{z})} = \dfrac{1}{z(1-(-\frac{5}{z}))}$. By using geometric series expansion, we get $f(z) = \dfrac{1}{z}\Sigma_{n=0}^{\infty}(-\dfrac{5}{z})^n = \Sigma_{n=0}^{\infty}\dfrac{(-1)^n\ 5^n}{z^{n+1}}$, valid in the region $|z| > 5$, which involves only negative powers of z.

Problem 27. Determine the Laurent's series for $f(z) = \dfrac{1}{z(z+5)}$ valid in the region $|z| < 5$.

Solution.

We know from the above example that $\dfrac{1}{z+5} = \Sigma_{n=0}^{\infty}\dfrac{(-1)^n\ z^n}{5^{n+1}}$, valid in the region $|z| < 5$. Therefore,

$$\frac{1}{z(z+5)} = \frac{1}{z}\,\frac{1}{z+5} = \frac{1}{z}\,\Sigma_{n=0}^{\infty}\frac{(-1)^n\ z^n}{5^{n+1}} = \Sigma_{n=0}^{\infty}\frac{(-1)^n\ z^{n-1}}{5^{n+1}} \qquad (|z| < 5).$$

■

Exercises.

1. Obtain the Laurent's series expansion for $f(z) = \dfrac{1}{z^2+4}$ valid in the region $|z-2i| > 4$.

2. For the function $f(z) = \dfrac{1}{z(z+2)}$, determine the Laurent's series that is valid within the region $1 < |z-1| < 3$.

3. Expand $f(z) = \dfrac{1}{z-1} - \dfrac{1}{z-2}$ in the following regions (a) $|z| < 1$ (b) $1 < |z| < 2$ (c) $|z| > 2$.

4. Find the Laurent series that represents the function $f(z) = z^2 sin\dfrac{1}{z^2}$ in the domain $0 < |z| < \infty$.

5. Derive the Laurent series representation for $\dfrac{e^z}{(z+1)^2}$ valid for $0 < |z+1| < \infty$.

6. Give two Laurent's series expansions in powers of z for the function $f(z) = \dfrac{1}{z^2(1-z)}$, and specify the regions in which those expansions are valid.

7. Find the Laurent series expansion of $f(z) = \dfrac{z}{(z-1)(z-3)}$ valid in the region $0 < |z-1| < 2$.

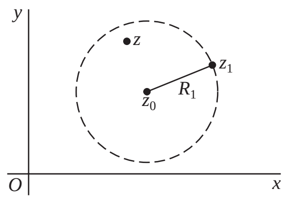
3.4 Absolute and Uniform Convergence of Power Series

We will now discuss basic properties of power series.

A natural question is to determine the set of complex numbers z for which a given power series converges. We have the following theorem.

Theorem 3.4.1.

If a power series $\Sigma_{n=0}^{\infty} a_n(z-z_0)^n$ converges when $z = z_1$ $(z_1 \neq z_0)$, then it is absolutely convergent at each point z in the open disk $|z - z_0| < R_1$ where $R_1 = |z_1 - z_0|$.



Analogous to the concept of an interval of convergence in real calculus, a complex power series $\Sigma_{n=0}^{\infty} a_n(z-z_0)^n$ has a circle of convergence defined by $|z - z_0| = R$ for some $R \geq 0$.

The above theorem implies that the set of all points inside some circle centered at z_0 is a region of convergence for the above power series $\Sigma_{n=0}^{\infty} a_n(z-z_0)^n$, provided it converges at some point other than z_0.

The greatest circle centered at z_0 such that series $\Sigma_{n=0}^{\infty} a_n(z-z_0)^n$ converges at each point inside is called the ***circle of convergence*** of the series.

The series cannot converge at any point z_2 outside that circle, according to the theorem ; for if it did, it would converge everywhere inside the circle centered at z_0 and passing through z_2. The first circle could not, then, be the circle of convergence.

The power series converges absolutely for all z satisfying $|z - z_0| < R$ and diverges for $|z - z_0| > R$. Here R is called the radius of convergence of the power

series. The radius R of convergence can be (a) zero (in which case $\Sigma_{n=0}^{\infty} a_n(z-z_0)^n$ converges only at $z = z_0$), (b) a finite number (in which case the given power series converges at all interior points of the circle $|z - z_0| = R$), (c) ∞ (in which case the given power series converges for all z).
A power series may converge at some, all, or none of the points on the actual circle of convergence.

Suppose that the power series $\Sigma_{n=0}^{\infty} a_n(z - z_0)^n$ has circle of convergence $|z - z_0| = R$, and let $S(z)$ and $S_N(z)$ represent the sum and partial sums, respectively, of that series:

$$S(z) = \Sigma_{n=0}^{\infty} a_n(z - z_0)^n, \quad S_N(z) = \Sigma_{n=0}^{N-1} a_n(z - z_0)^n \ (|z - z_0| < R).$$

Then, the ***remainder function*** $\rho_N(z)$ is given by $\rho_N(z) = S(z) - S_N(z)$ $(|z - z_0| < R)$. Since the power series converges for any fixed value of z when $|z - z_0| < R$, we know that the remainder $\rho_N(z)$ approaches zero for any such z as N tends to infinity.

This means that corresponding to each positive number ε, there is a positive integer N_ε such that $|\rho_N(z)| < \varepsilon$ whenever $N > N_\varepsilon$.

When the choice of N_ε depends only on the value of ε and is independent of the point z taken in a specified region within the circle of convergence, the convergence is said to be ***uniform*** in that region.

It can be shown that if z_1 is a point inside the circle of convergence $|z-z_0| = R$ of a power series $\Sigma_{n=0}^{\infty} a_n(z - z_0)^n$, then that series must be uniformly convergent in the closed disk $|z - z_0| < R_1$, where $R_1 = |z_1 - z_0|$.

Note that a power series $\Sigma_{n=0}^{\infty} a_n(z - z_0)^n$ represents a continuous function

$S(z)$ at each point inside its circle of convergence $|z - z_0| = R$. Furthermore, the sum $S(z)$ of the power series $\Sigma_{n=0}^{\infty} a_n(z - z_0)^n$ is actually analytic within the circle of convergence.

Theorem 3.4.2.

Let C denote any contour interior to the circle of convergence of the power series $\Sigma_{n=0}^{\infty} a_n(z - z_0)^n$, and let $g(z)$ be any function that is continuous on C. The series formed by multiplying each term of the power series by $g(z)$ can be integrated term by term over C; i.e., $\int_C g(z)S(z)dz = \Sigma_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n dz$.

If a series $\Sigma_{n=0}^{\infty} a_n(z - z_0)^n$ converges to $f(z)$ at all points interior to some circle $|z - z_0| = R$, then it is the Taylor series expansion for f in powers of $z - z_0$.

If a series $\Sigma_{n=-\infty}^{\infty} c_n(z - z_0)^n = \Sigma_{n=0}^{\infty} a_n(z - z_0)^n + \Sigma_{n=1}^{\infty} \dfrac{b_n}{(z - z_0)^n}$ converges to $f(z)$ at all points in some annular domain about z_0, then it is the Laurent series expansion for f in powers of $z - z_0$ for that domain.

An important result in real calculus states that, within a power series's radius of convergence, a power series is differentiable, and its derivative can be obtained by differentiating the individual terms of the power series term–by–term. The same holds true for complex power series:

Theorem 3.4.3.

The power series $S(z) = \Sigma_{n=0}^{\infty} a_n(z - z_0)^n$ can be differentiated term by term. i.e., at each point z interior to the circle of convergence of that series, $S'(z) = \Sigma_{n=1}^{\infty} n a_n(z - z_0)^{n-1}$. Also, $S'(z)$ has the same radius of convergence as $S(z)$.

Suppose that each of the power series $\Sigma_{n=0}^{\infty} a_n(z - z_0)^n$ and $\Sigma_{n=0}^{\infty} b_n(z - z_0)^n$ converges within some circle $|z-z_0| = R$. Their sums $f(z)$ and $g(z)$, respectively, are then analytic functions in the disk $|z-z_0| < R$, and the product of those sums has a Taylor series expansion which is valid there: $\Sigma_{n=0}^{\infty} c_n(z-z_0)^n$ $(|z-z_0| < R)$, where c_n are given by

$$c_n = \Sigma_{k=0}^{n} a_k b_{n-k}.$$

The series $\Sigma_{n=0}^{\infty} c_n(z-z_0)^n$ with $c_n = \Sigma_{k=0}^{n} a_k b_{n-k}$ is the same as the series obtained by formally multiplying the two series $\Sigma_{n=0}^{\infty} a_n(z-z_0)^n$ and $\Sigma_{n=0}^{\infty} b_n(z-z_0)^n$ term by term and collecting the resulting terms in like powers of $z - z_0$; it is called the ***Cauchy product*** of the two given series.

Now, let $f(z)$ and $g(z)$ denote the sums of series $\Sigma_{n=0}^{\infty} a_n(z-z_0)^n$ and $\Sigma_{n=0}^{\infty} b_n(z-z_0)^n$, in the region $|z-z_0| < R$ respectively. Suppose that $g(z) \neq 0$ when $|z-z_0| < R$. Since the quotient $f(z)/g(z)$ is analytic throughout the disk $|z-z_0| < R$, it has a Taylor series representation $\dfrac{f(z)}{g(z)} = \Sigma_{n=0}^{\infty} d_n(z-z_0)^n$, where the coefficients d_n can be found by differentiating $f(z)/g(z)$ successively and evaluating the derivatives at $z = z_0$.

Problem 28.

Determine the power series for $f(z) = \dfrac{1}{z^2}$ valid in the region $|z-1| < 1$, by differentiating the power series representation of $\dfrac{1}{z}$ in the region $|z-1| < 1$.

Solution.

By geometric series expansion, we have $\dfrac{1}{z} = \dfrac{1}{1+(z-1)} = \dfrac{1}{1-(-(z-1))} = \Sigma_{n=0}^{\infty}(-1)^n(z-1)^n$, $|z-1| < 1$.

Differentiating each sides of this equation gives, $\dfrac{-1}{z^2} = \Sigma_{n=1}^{\infty}(-1)^n n(z-1)^{n-1}$

$\Rightarrow \dfrac{1}{z^2} = \Sigma_{n=0}^{\infty}(-1)^n(n+1)(z-1)^n, \;\; |z-1| < 1.$ ■

Problem 29.

Use the power series expansions for e^z and $\dfrac{1}{1+z}$, to obtain the power series representation of $\frac{e^z}{1+z}$ in the region $|z| < 1$.

Solution.

By geometric series expansion, we have $\frac{1}{1+z} = 1 - z + z^2 - z^3 + ...,\ \ |z| < 1$.
Also, $e^z = \Sigma_{n=0}^{\infty} \frac{z^n}{n!}, \quad (|z| < \infty)$.

Therefore, $\frac{e^z}{1+z} = (1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + ...)(1 - z + z^2 - z^3 + ...)$.

Multiplying these two series term by term, we get $\frac{e^z}{1+z} = 1 + \frac{1}{2}z^2 - \frac{1}{3}z^3 + ...$ $(|z| < \infty)$. ■

Exercises.

1. By differentiating the Maclaurin series representation $\frac{1}{1-z} = \Sigma_{n=0}^{\infty} z^n$ $(|z| < 1)$, obtain the expansions for $\frac{1}{(1-z)^2}$ and $\frac{2}{(1-z)^3}$

2. Find the Taylor series for the function $\frac{1}{z}$ about the point $z_0 = 2$. By differentiating that series term by term, obtain the Taylor series for the function $\frac{1}{z^2}$ valid in the region $|z - 2| < 2$.

3. Use multiplication of series to show that $\frac{e^z}{z(z^2+1)} = \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + ...$ $(0 < |z| < 1)$.

4. Use division to obtain the Laurent series representation
$\frac{1}{e^z - 1)} = \frac{1}{z} - \frac{1}{2} + \frac{1}{12}z - \frac{1}{720}z^3 + ...\ (0 < |z| < 2\pi)$.

5. Show that $\frac{1}{z^2 sinh\ z} = \frac{1}{z^3} - \frac{1}{6} \cdot \frac{1}{z} + \frac{7}{360}z + ...\ (0 < |z| < \pi)$.

Chapter 4

Residue Integration

In this chapter, we will discuss the Cauchy's residue theorem, which is a powerful tool to evaluate line integrals of analytic functions over closed curves; it can often be used to compute real integrals as well. It generalizes the Cauchy - Goursat theorem and Cauchy's integral formula. We begin with a detailed study of isolated singular points.

4.1 Singular Points and Residues

Recall that a point z_0 is called a ***singular point*** of a function f if f fails to be analytic at z_0 but is analytic at some point in every neighborhood of z_0.

A singular point z_0 is said to be ***isolated*** if there is a deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0 throughout which f is analytic.

If there is a positive number R_1 such that f is analytic for $R_1 < |z| < \infty$, then f is said to have an isolated singular point at $z_0 = \infty$.

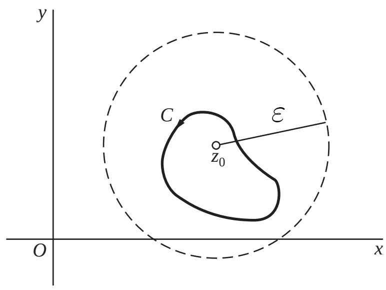
For example, the function $f(z) = \dfrac{z+1}{z^3(z^2+1)}$ has the three isolated singular points $z = 0$ and $z = \pm i$.

The function $f(z) = \dfrac{1}{sin(\pi/z)}$ has the singular points $z = 0$ and $z = 1/n$, $(n = \pm 1, \pm 2, ...)$, all lying on the segment of the real axis from $z = -1$ to $z = 1$. Each singular point except $z = 0$ is isolated. The singular point $z = 0$ is not isolated because every deleted ε-neighborhood of the origin contains other singular points of the function (since $1/n \to 0$ as $n \to \infty$).

Now, suppose that z_0 is an isolated singular point of a function f. Then there exists $\varepsilon > 0$ such that f is analytic in the annulus $0 < |z - z_0| < \varepsilon$. Hence $f(z)$ has a Laurent series representation

$$f(z) = \Sigma_{n=0}^{\infty} a_n (z - z_0)^n + \Sigma_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad (0 < |z - z_0| < \varepsilon).$$

Let C is any positively oriented simple closed contour around z_0 that lies in the punctured disk $0 < |z - z_0| < \varepsilon$.



Since $\int_C (z-z_0)^n dz = 0$ when $z \neq -1$, and $\int_C \dfrac{1}{(z - z_0)} dz = 2\pi i$, by integrating the above Laurent series, term by term around C, we obtain:

$$\int_C f(z)dz = 2\pi i \ b_1.$$

The complex number b_1, which is the coefficient of $1/(z-z_0)$ in the above Laurent series expansion of $f(z)$, is called the ***residue*** of f at the isolated singular point z_0, and we denote it as $\text{Res}_{z=z_0} f(z)$.

Therefore, we have $\int_C f(z)dz = 2\pi i \ \text{Res}_{z=z_0} f(z)$.

This provides a powerful method for evaluating certain integrals around simple closed contours.

Example 11.

Consider the integral $\int_C z^2 \ sin \ \frac{1}{z} \ dz$ where C is the positively oriented unit circle $|z| = 1$. Since the integrand is analytic everywhere in the finite complex plane except at $z = 0$, it has a Laurent series representation that is valid in the region $0 < |z| < \infty$. Therefore by the equation $\int_C f(z)dz = 2\pi i \ \text{Res}_{z=z_0} f(z)$, the value of integral $\int_C z^2 \ sin \ \frac{1}{z} \ dz$ is $2\pi \ i$ times the residue of its integrand at $z = 0$.

Note that

$$z^2 \ sin \ \frac{1}{z} = z^2(\frac{1}{z} - \frac{1}{3!} \cdot \frac{1}{z^3} + \frac{1}{5!} \cdot \frac{1}{z^5} - ....) = z - \frac{1}{3!} \cdot \frac{1}{z} + \frac{1}{5!} \cdot \frac{1}{z^3} - .... \ \ 0 < |z| < \infty.$$

Here, the coefficient of $\frac{1}{z}$ is $\frac{-1}{3!}$. $\Rightarrow \text{Res}_{z=z_0} z^2 \ sin \ \frac{1}{z} = \frac{-1}{3!}$. Therefore,

$$\int_C z^2 \ sin \ \frac{1}{z} \ dz = 2\pi i \cdot \frac{-1}{3!} = \frac{-\pi \ i}{3}.$$

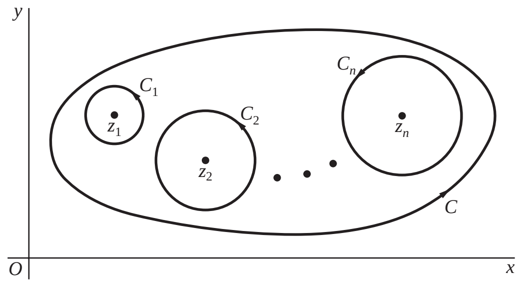
Theorem 4.1.1. *(***Cauchy's Residue Theorem***)*

Let C be a simple closed contour, described in the positive sense. If a function f is analytic inside and on C except for a finite number of singular points z_k $(k = 1, 2, ..., n)$ inside C, then

$$\int_C f(z)dz = 2\pi i \ \Sigma_{k=1}^{n} Res_{z=z_k} f(z)$$

Proof.

Let the points z_k $(k = 1, 2, ..., n)$ be centers of positively oriented circles C_k which are interior to C and are so small that no two of them have points in common. The circles C_k, together with the simple closed contour C, form the boundary of a closed region throughout which f is analytic and whose interior is a multiply connected domain consisting of the points inside C and exterior to each C_k.



Hence, by the Cauchy–Goursat theorem for multiply connected domains,

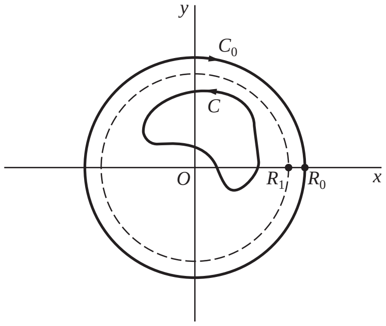
$$\int_C f(z)dz - \Sigma_{k=1}^{n} \int_{C_k} f(z)dz = 0$$

. But, $\int_{C_k} f(z)dz = 2\pi i \ \text{Res}_{z=z_k} f(z) \ (k = 1, 2, ..., n)$.

Therefore, $\int_C f(z)dz = 2\pi i \ \Sigma_{k=1}^{n} \text{Res}_{z=z_k} f(z)$. □

Residue at Infinity

Suppose that a function f is analytic throughout the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C. Let R_1 denote a positive number which is large enough that C lies inside the circle $|z| = R_1$. Then, the function f is clearly analytic throughout the domain $R_1 < |z| < \infty$ and in this case, the point at infinity is said to be an isolated singular point of f. Now, let C_0 denote a circle $|z| = R_0$, oriented in the clockwise direction, where $R_0 > R_1$.



The residue of f at infinity is defined by means of the equation

$$\int_{C_0} f(z)dz = 2\pi\ i\ Res_{z=\infty} f(z) \quad - - - - - (1)$$

Since f is analytic throughout the closed region bounded by C and C_0, the principle of deformation of paths implies that

$$\int_C f(z)dz = \int_{-C_0} f(z)dz = -\int_{C_0} f(z)dz.$$

Therefore, $\int_C f(z)dz = -2\pi\ i\ Res_{z=\infty} f(z) \quad - - - - - (2)$.

Now to find this residue,we write the Laurent series

$$f(z) = \Sigma_{n=-\infty}^{\infty} c_n\ z^n \quad (R_1 < |z| < \infty),$$

where

$$c_n = \frac{1}{2\pi i}\int_{-C_0} \frac{f(z)dz}{z^{n+1}} \quad (n = 0, \pm 1, \pm 2, ...).$$

Replacing z by $1/z$ in the above Laurent series and then multiplying by $1/z^2$, we see that

$$\frac{1}{z^2} f(\frac{1}{z}) = \Sigma_{n=-\infty}^{\infty} c_n\ z^{n+2} = \Sigma_{n=-\infty}^{\infty} c_{n-2}\ z^n \quad (0 < |z| < \frac{1}{R_1}).$$

Therefore, by definition of residues, $c_{-1} = Res_{z=0}[\frac{1}{z^2} f(\frac{1}{z})]$. But, by the above formula to compute the coefficients of Laurent series,

$$c_{-1} = \frac{1}{2\pi i}\int_{-C_0} f(z)dz$$

$$\Rightarrow \int_{C_0} f(z)dz = -2\pi\ i\ Res_{z=0}[\frac{1}{z^2} f(\frac{1}{z})] \quad - - - - - (3).$$

Now from equations (1) and (3), it follows that

$$Res_{z=\infty}f(z) = -Res_{z=0}[\frac{1}{z^2}f(\frac{1}{z})] \quad -----(4)$$

From equations (2) and (4), we obtain the following theorem, which is sometimes more efficient to use than Cauchy's residue theorem since it involves only one residue.

Theorem 4.1.2.

If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then

$$\int_C f(z)dz = 2\pi\ i\ Res_{z=0}[\frac{1}{z^2}f(\frac{1}{z})].$$

Problem 30.

Evaluate the integral $\int_C \frac{5z-2}{z(z-1)}dz$, where C is the positively oriented circle $|z| = 2$.

Solution.

Here, the integrand $f(z) = \frac{5z-2}{z(z-1)}$ has the two isolated singularities $z = 0$ and $z = 1$, both of which are interior to C. We first expand $f(z) = \frac{5z-2}{z(z-1)}$ as a Laurent series about $z = 0$ as follows:

$$\frac{5z-2}{z(z-1)} = \frac{5z-2}{z} \cdot \frac{-1}{1-z} = (5 - \frac{2}{z})(-1 - z - z^2 - ....) \ \ (0 < |z| < 1).$$

Therefore, the $\text{Res}_{z=0}f(z)$ is the coefficient of $1/z$ in this Laurent series expansion, i.e., $\text{Res}_{z=0}f(z) = 2$.

Now we expand $f(z) = \dfrac{5z-2}{z(z-1)}$ as a Laurent series about $z = 1$ as follows:

$$\frac{5z-2}{z(z-1)} = \frac{5(z-1)+3}{(z-1)} \cdot \frac{1}{1+(z-1)} = (5 + \frac{3}{(z-1)})[1-(z-1)+(z-1)^2 - ....],$$

when $0 < |z-1| < 1$. From this expansion, we get $\text{Res}_{z=1} f(z) = 3$, the coefficient of $i/(z-1)$. Therefore, by Cauchy's residue theorem,

$$\int_C \frac{5z-2}{z(z-1)} dz = 2\pi i \ [Res_{z=0} f(z) + Res_{z=1} f(z)] = 2\pi i \ [2+3] = 10\pi \ i.$$

$\blacksquare$

Remark.

The above problem can also be solved by using Theorem 4.1.2. Here,

$$\frac{1}{z^2} f(\frac{1}{z}) = \frac{5-2z}{z(1-z)} = \frac{5-2z}{z} \cdot \frac{1}{1-z} = (\frac{5}{z} - 2) \ (1 + z + z^2 + ...) \ \ (0 < |z| < 1).$$

Therefore, $Res_{z=0}[\frac{1}{z^2} f(\frac{1}{z})]$ is the coefficient of $1/z$ in the above Laurent series expansion. i.e., $Res_{z=0}[\frac{1}{z^2} f(\frac{1}{z})] = 5$. Therefore,

$$\int_C f(z) dz = 2\pi \ i \ Res_{z=0}[\frac{1}{z^2} f(\frac{1}{z})] = 2\pi \ i \ \cdot 5 = 10\pi \ i \ .$$

Exercises.

1. Find the residue at $z = 0$ of the following functions.
   (a) $\dfrac{1}{z+z^2}$ (b) $z \ cos(\frac{1}{z})$ (c) $\dfrac{z - sin \ z}{z}$ (d) $\dfrac{exp(-z)}{z^2}$.

2. Evaluate the integral $\int_C \dfrac{exp(-z)}{(z-1)^2} dz$, where C is the positively oriented circle $|z| = 3$.

3. Evaluate the integral $\int_C \ z^2 \ exp(\frac{1}{z})dz$, where C is the positively oriented unit circle.

4. Suppose that a function f is analytic throughout the finite complex plane except for a finite number of singular points $z_1, z_2, ..., z_n$. Show that

$$Res_{z=z_1}f(z) + Res_{z=z_2}f(z) + ... + Res_{z=z_n}f(z) + Res_{z=\infty}f(z) = 0.$$

5. Use Cauchy's residue theorem, to compute $\int_C \frac{z+1}{z^2-2z}dz$, where C is the positively oriented circle $|z| = 3$.

6. Use Theorem 4.1.2, to compute $\int_C \frac{z+1}{z^2-2z}dz$, where C is the positively oriented circle $|z| = 3$.

4.2 Types of Isolated Singular Points

Let $z = z_0$ be an isolated singular point of a function f. Then there exists $\varepsilon > 0$ such that f is analytic in the annulus $0 < |z - z_0| < \varepsilon$. Hence $f(z)$ has a Laurent series representation

$$f(z) = \Sigma_{n=0}^{\infty} a_n(z - z_0)^n + \Sigma_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad (0 < |z - z_0| < \varepsilon).$$

Here, the portion of the Laurent series of $f(z)$ about the isolated singular point $z = z_0$ consisting of negative powers of $z - z_0$, i.e, $\Sigma_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$ is called the ***principal part*** of f at z_0. We now use the principal part to identify the isolated singular point z0 as one of three special types.

If the principal part of f at z_0 contains at least one nonzero term but the number of such terms is only finite, then there exists a positive integer$m$ $(m \geq 1)$ such that $b_m \neq 0$ and $b_{m+1} = b_{m+2} = ... = 0$. Then the Laurent series expansion of $f(z)$ becomes

$$f(z) = \Sigma_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + ... + \frac{b_m}{(z - z_0)^m} \quad (0 < |z - z_0| < \varepsilon),$$

where $b_m \neq 0$. In this case, the isolated singular point z_0 is called ***a pole of order*** m. A pole of order $m = 1$ is called a ***simple pole***.

If every b_n in the Laurent series expansion of $f(z)$ about the isolated singular point $z = z_0$ is zero, so that

$$f(z) = \Sigma_{n=0}^{\infty} a_n (z - z_0)^n \quad (0 < |z - z_0| < \varepsilon).$$

In this case, the isolated singular point z_0 is called a ***removable singular point***.

Note that the residue at a removable singular point is always zero. If we redefine, f at z_0 so that $f(z_0) = a_0$, the Laurent expansion becomes valid throughout the entire disk $|z - z_0| < \varepsilon$. Since a power series always represents an analytic function interior to its circle of convergence, it follows that f is now analytic at z_0. The singularity z_0 of f is, therefore, removed.

If an infinite number of the coefficients b_n in the principal part of the Laurent series expansion of $f(z)$ about the isolated singular point z_0 are nonzero, then z_0 is said to be an ***essential singular point*** of f.

In each neighborhood of an essential singular point, a function assumes every finite value, with one possible exception, an infinite number of times. This is known as ***Picard's theorem***.

Example 12. (Simple Pole)

Consider the function $f(z) = \dfrac{5z - 2}{z - 1}$.

Note that $f(z) = \dfrac{5z-2}{z-1} = \dfrac{5(z-1)+3}{(z-1)} = 5 + \dfrac{3}{(z-1)}$, which is the Laurent series expansion of $f(z) = \dfrac{5z-2}{z-1}$ about the isolated singular point $z = 1$. Here, the principal part contains only one nonzero term namely $\dfrac{3}{(z-1)}$, and hence $z = 1$ is a simple pole of $f(z)$ and $\text{Res}_{z=1} \dfrac{5z-2}{z-1} = 3$.

Example 13. (Pole of Order 2)

Consider the function $f(z) = \dfrac{1}{z^2(z+1)}$. Note that

$f(z) = \dfrac{1}{z^2(z+1)} = \dfrac{1}{z^2} \dfrac{1}{1-(-z)} = \dfrac{1}{z^2} - \dfrac{1}{z} + 1 - z + z^2 - \ldots\ldots\ (0 < |z| < 1)$.

Thus, the principal part of the Laurent series expansion of $f(z) = \dfrac{1}{z^2(z+1)}$ about the isolated singular point $z = 0$ shows that $z = 0$ is a pole of order 2, and $\text{Res}_{z=0} \dfrac{1}{z^2(z+1)} = -1$.

Example 14. (Removable Singular Point)

Consider the function $f(z) = \dfrac{sin\ z}{z}$.

Note that $\dfrac{sin\ z}{z} = \dfrac{1}{z}\ [z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \ldots.] = 1 - \dfrac{z^2}{3!} + \dfrac{z^4}{3!} - \ldots.\ (0 < |z| < \infty)$.

Thus, the principal part of the Laurent series expansion of $f(z) = \dfrac{sin\ z}{z}$ has no terms. $\Rightarrow z = 0$ is a removable singular point of $f(z)$, $\text{Res}_{z=0} \dfrac{sin\ z}{z} = 0$, and if we set $f(0) = 1$, $f(z)$ becomes an entire function.

Example 15. (Essential Singular Point)

We have

$$e^{\frac{1}{z}} = \Sigma_{n=0}^{\infty} \frac{1}{n!\ z^n} = 1 + \frac{1}{1!\ z} + \frac{1}{2!\ z^2} + \frac{1}{3!\ z^3} + \ldots.\ (0 < |z| < \infty).$$

Thus, the principal part of the Laurent series expansion of $f(z) = e^{\frac{1}{z}}$ contains infinitely many terms. $\Rightarrow z = 0$ is an essential singular point of $f(z)$ and $\text{Res}_{z=0}\ e^{\frac{1}{z}} = 1$.

Problem 31.

Show that $z = 0$ is a removable singularity of the function $f(z) = \dfrac{1 - cos\ z}{z^2}$.

Solution.

We have the Macalurin' series expansion

$$cos\ z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ... \ (|z| < \infty).$$

Therefore, for $0 < |z| < \infty$, we have

$$f(z) = \frac{1 - cos\ z}{z^2} = \frac{1}{z^2}[1 - (1 + \frac{1}{1!\ z} + \frac{1}{2!\ z^2} + \frac{1}{3!\ z^3} + ...)] = \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} + ...$$

The principal part of the Laurent series expansion has no terms. $\Rightarrow z = 0$ is a removable singular point of $\dfrac{1 - cos\ z}{z^2}$. If we set $f(0) = 1/2$, $f(z)$ becomes an entire function. ■

Problem 32.

Evaluate $\int_C e^{-1/z} sin\ (\frac{1}{z})\ dz$ where C is the positively oriented unit circle.

Solution.

We have the series expansions

$$e^{-1/z} = 1 - \frac{1}{z} + \frac{1}{2!} \cdot \frac{1}{z^2} - \frac{1}{3!} \cdot \frac{1}{z^3} + ... \ (0 < |z| < \infty)$$

and

$$sin\ (\frac{1}{z}) = \frac{1}{z} - \frac{1}{3!} \cdot \frac{1}{z^3} + \frac{1}{5!} \cdot \frac{1}{z^5} - .... \ (0 < |z| < \infty)$$

Therefore, for $0 < |z| < \infty$, we have

$$e^{-1/z} sin\ (\frac{1}{z}) = \frac{1}{z} - \frac{1}{z^2} + ...$$

$\Rightarrow$ The principal part of the Laurent series expansion of $e^{-1/z} sin\ (\frac{1}{z})$ has infinitely many terms. $\Rightarrow z = 0$ is an essential singular point with residue 1. Hence $\int_C e^{-1/z} sin\ (\frac{1}{z})\ dz = 2\pi\ i \cdot 1 = 2\pi\ i.$ ■

Exercises.

1. Write the principal part of the following functions at the isolated singular points and determine whether that point is a pole, a removable singular point, or an essential singular point. Also, find the corresponding residue in each case.

   (a) $z\ exp(\frac{1}{z})$

   (b) $\frac{cos\ z}{z}$

   (c) $\frac{z^2}{1+z}$

   (d) $\frac{1}{(2-z)^3}$

   (e) $\frac{sinh\ z}{z^4}$

   (f) $\frac{z^2 - 2z + 3}{z-2}$

   (g) $\frac{1 - exp(2z)}{z^4}$

   (h) $\frac{exp(2z)}{(z-1)^2}$.

2. Suppose that a function f is analytic at z_0, and write $g(z) = \frac{f(z)}{(z-z_0)}$. Show that

(a) if $f(z_0) \neq 0$, then z_0 is a simple pole of g, with residue $f(z_0)$;

(b) if $f(z_0) = 0$, then z_0 is a removable singular point of g.

4.3 Computation of Residues at Poles

If a function $f(z)$ has an isolated singularity at a point z_0, then, the basic method for identifying z_0 as a pole and finding the residue there is to write the appropriate Laurent series and to note the coefficient of $1/(z - z_0)$. But, the computation of a Laurent series expansion is tedious in most circumstances. The following theorem provides an alternative characterization of poles and a way of finding residues at poles.

Theorem 4.3.1. *An isolated singular point z_0 of a function f is a pole of order m if and only if $f(z)$ can be written in the form $f(z) = \dfrac{\phi(z)}{(z - z_0)^m}$, where $\phi(z)$ is analytic and nonzero at z_0.*
Moreover, $Res_{z=z_0} f(z) = \phi(z_0)$ if $m = 1$ and $Res_{\ z=z_0} f(z) = \dfrac{\phi^{(m-1)}(z_0)}{(m-1)!}$ if $m \geq 2$.

Problem 33.

Find the residues of $f(z) = \dfrac{z+1}{z^2+9}$ at singular points.

Solution.

Here, $f(z) = \dfrac{z+1}{z^2+9}$ is analytic at all points except $z = \pm 3\ i$. We can write $f(z) = \dfrac{\phi(z)}{z - 3\imath}$ where $\phi(z) = \dfrac{z+1}{z+3\ i}$. Since $\phi(z)$ is analytic at $3\ i$ and $\phi(3\ i) \neq 0$, $z = 3\ i$ is a simple pole of f, and $\text{Res}_{z=3\ i} f(z) = \phi(3\ i) = \dfrac{3-i}{6}$.

Similarly, $z = -3\ i$ is also a simple pole of f, and $\text{Res}_{z=-3\ i} f(z) = \dfrac{3+i}{6}$.

■

Problem 34.

Find the value of the integral $\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz$, taken counterclockwise around the circle $|z-2| = 2$.

Solution.

Here, $f(z) = \frac{3z^3+2}{(z-1)(z^2+9)}$ has the singular points $z = 1$ and $z = \pm 3\ i$ and all these are simple poles.

Here, C is $|z-2| = 2$. The simple poles $z = 1$ lies inside C , whereas $z = -3\imath$ and $z = 3\ i$ lies out side C.

As in above problem, we find that $\text{Res}_{z=1} f(z) = 1/2$. By Cauchy's residue theorem,

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz = 2\pi i\ Res_{z=1} f(z) = \pi\ i.$$

■

Problem 35.

Evaluate $\int_C \frac{e^z}{(z+1)^2}dz$, along the circle $|z-1| = 3$ taken in counterclockwise direction.

Solution.

Note that $f(z) = \frac{e^z}{(z+1)^2}$ is analytic at all points except $z = -1$.
Here, C is $|z-1| = 3$. $\Rightarrow$ The singular point $z = -1$ lies inside C .
Also, we can write $f(z) = \frac{\phi(z)}{(z+1)^2}$ where $\phi(z) = e^z$.

Since $\phi(z)$ is analytic and non zero at $z = -1$, it is a double pole of f, and $\text{Res}_{z=-1} f(z) = \frac{\phi'(-1)}{1!} = e^{-1}$.

Hence, $\int_C \frac{e^z}{(z+1)^2}dz = 2\pi i\ Res_{z=-1} f(z) = 2\pi\ i \cdot e^{-1} = \frac{2\pi\ i}{e}$.

■

Exercises.

1. Find the residues of $f(z) = \dfrac{sinh\ z}{z^4}$ at $z = 0$.

2. Find the residues of $f(z) = \dfrac{(log\ z)^3}{z^2+1}$ at singular points.

3. Show that $z = i$ is a pole of order 3 of the function $f(z) = \dfrac{z^3+2z}{(z-i)^3}$ and find the residue at $z = i$.

4. Find the value of the integral $\int_C \dfrac{3z^3+2}{(z-1)(z^2+9)}dz$, taken counterclockwise around the circle $|z| = 4$.

5. Find the value of the integral $\int_C \dfrac{dz}{(z-1)(z^3(z+4))}dz$, taken counterclockwise around the circle $|z| = 2$.

6. Evaluate the integral of $f(z)$ around the positively oriented circle $|z| = 3$ where (a) $f(z) = \dfrac{z^3 e^{\frac{1}{z}}}{1+z^3}$ (b) $f(z) = \dfrac{(3z+2)^2}{z(z-1)(2z+5)}$.

4.4 Zeros of Analytic Functions

Suppose that a function f is analytic at a point z_0. Since $f(z)$ is analytic at z_0, all of the derivatives $f^{(n)}(z)$ $(n = 1, 2, ...)$ exist at z_0. If $f(z_0) = 0$ and if there is a positive integer m such that $f^{(m)}(z+0) \neq 0$ and each derivative of lower order vanishes at z_0 , then f is said to have a ***zero of order*** m at z_0. The following theorem provides an alternative characterization of zeros of order m.

Theorem 4.4.1.

Let a function f be analytic at a point z_0. It has a zero of order m at z_0 if and only if there is a function g, which is analytic and nonzero at z_0 , such that $f(z) = (z - z_0)^m g(z)$.

Example 16.

Consider the polynomial $f(z) = z^3 - 8$. Note that f is entire and that $f(2) = 0$ and $f'(2) = 12 \neq 0$. $\Rightarrow z_0 = 2$ is a zero of order 1.

This can also be seen by using the above theorem. We can write $z^3 - 8 = (z-2)(z^2+2z+4)$. $\Rightarrow f(z)$ has a zero of order 1 at $z_0 = 2$, since $f(z) = (z-2)g(z)$, where $g(z) = z^2 + 2z + 4$, and because f and g are entire and $g(2) \neq 0$.

The next theorem shows that the zeros of an analytic function are isolated when the function is not identically equal to zero.

Theorem 4.4.2.

Given a function f and a point z_0 , suppose that (a) f is analytic at z_0 ; (b) $f(z_0) = 0$ but $f(z)$ is not identically equal to zero in any neighborhood of z_0 . Then $f(z) \neq 0$ throughout some deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0 .

Theorem 4.4.3.

Given a function f and a point z_0, suppose that

(a) f is analytic throughout a neighborhood N_0 of z_0 ;

(b) $f(z) = 0$ at each point z of a domain D or line segment L containing z_0. Then $f(z) = 0$ in N_0; that is, $f(z)$ is identically equal to zero throughout N_0.

Theorem 4.4.4.

Suppose that

(a) two functions p and q are analytic at a point z_0 ;

(b) $p(z_0) \neq 0$ and q has a zero of order m at z_0. Then the quotient $p(z)/q(z)$ has a pole of order m at z_0.

Theorem 4.4.5.

Let two functions p and q be analytic at a point z_0 . If $p(z_0) \neq 0$, $q(z_0) = 0$, and $q'(z_0) \neq 0$, then z_0 is a simple pole of the quotient $p(z)/q(z)$ and

$$Res_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

Example 17.

Consider the function $f(z) = cot z = \dfrac{cos\ z}{sin\ z}$, which is a quotient of the entire functions $p(z) = cos\ z$ and $q(z) = sin\ z$. Its singularities occur at the zeros of q. i.e., at the points $z = n\pi$ $(n = 0, \pm 1, \pm 2, ...)$. Since $p(n\pi) = (-1)^n \neq 0$, $q(n\pi) = 0$, and $q'(n\pi) = (-1)^n \neq 0$, each singular point $z = n\pi$ of f is a simple pole, with residue $= \dfrac{p(n\pi)}{q'(n\pi)} = \dfrac{(-1)^n}{(-1)^n} = 1$.

Problem 36.

Find the value of the integral $\int_C \dfrac{9z + i}{z(z^2 + 1)} dz$, taken counterclockwise around the circle $|z| = 2$.

Solution.

Here, $f(z) = \dfrac{9z + i}{z(z^2 + 1)} = \dfrac{p(z)}{q(z)}$ has the singular points $z = 0$ and $z = \pm i$ and all these lies inside C.

Since $p(0) = i \neq 0$, $q(0) = 0$, and $q'(0) = 1 \neq 0$, the singular point $z = 0$ of f is a simple pole, with residue $= \dfrac{p(0)}{q'(0)} = \dfrac{i}{1} = i$.

Similarly, since $p(i) = 10i \neq 0$, $q(i) = 0$, and $q'(i) = -2 \neq 0$, the singular point $z = i$ of f is a simple pole, with residue $= \dfrac{p(i)}{q'(i)} = \dfrac{10\ i}{-2} = -5\ i$ and, since $p(-i) = -8\ i \neq 0$, $q(-i) = 0$, and $q'(-i) = -2 \neq 0$, the singular point $z = -i$ of f is a simple pole, with residue $= \dfrac{p(-i)}{q'(-i)} = \dfrac{-8\ i}{-2} = 4\ i$.

By Cauchy's residue theorem,
$\int_C \frac{9z+i}{z(z^2+1)}dz = 2\pi i\,[Res_{z=0}f(z)+Res_{z=i}f(z)+Res_{z=-i}f(z)] = 2\pi\, i[i+(-5i)+(4i)] = 0.$ ■

Exercises.

1. Let C denote the positively oriented circle $|z| = 2$ and evaluate the integral (a) $\int_C tan\ zdz$; (b) $\int_C \frac{dz}{sinh\ 2z}$.

2. Show that $\int_C \frac{dz}{(z^2-1)^2+3} = \frac{\pi}{2\sqrt{2}}$, where C is the positively oriented boundary of the rectangle whose sides lie along the lines $x = \pm 2,\ y = 0$, and $y = 1$.

3. Let p and q denote functions that are analytic at a point z_0, where $p(z_0) \neq 0$ and $q(z_0) = 0$. Show that if the quotient $p(z)/q(z)$ has a pole of order m at z_0, then z_0 is a zero of order m of q.

4. Show that $\text{Res}_{z=\pi i}\frac{z - sinh\ z}{z^2 sinh\ z} = \frac{i}{\pi}$.

5. Show that Res $_{z=z_n}(tanh\ z) = 1$ where $z_n = (\frac{\pi}{2}+n\pi)i\ \ (n = 0, \pm1, \pm2, ...)$.

6. Show that the point $z = 0$ is a simple pole of the function $f(z) = csc\ z$ and that the residue of f at $z = 0$ is 1.

4.5 Evaluation of Improper Integrals

We have seen that the Cauchy's residue theorem allows us to evaluate integrals without actually physically integrating i.e., it allows us to evaluate an integral just by knowing the residues contained inside a closed contour. In this section we shall see how to use the residue theorem to to evaluate certain real integrals which were not possible (or difficult) using real integration techniques from single variable calculus.

In calculus, the improper integral of a continuous function $f(x)$ over the semi-infinite interval $0 \leq x < \infty$ is defined by means of the equation

$$\int_0^\infty f(x)dx = lim_{R\to\infty} \int_0^R f(x)dx.$$

When the limit on the right exists, the improper integral is said to ***converge*** to that limit.

If $f(x)$ is continuous for all x, its improper integral over the infinite interval $-\infty < x < \infty$ is defined by writing

$$\int_{-\infty}^\infty f(x)dx = lim_{R_1\to\infty} \int_{-R_1}^0 f(x)dx + lim_{R_2\to\infty} \int_0^{R_2} f(x)dx \quad ----(1)$$

and when both of the limits on right side exist, we say that the integral $\int_{-\infty}^\infty f(x)dx$ converges to their sum.

Another value that is assigned to the integral $\int_{-\infty}^\infty f(x)dx$ is the ***Cauchy principal value (P.V.)*** . The Cauchy principal value (P.V.) of integral $\int_{-\infty}^\infty f(x)\ dx$ is defined as:

$$\text{P.V. } \int_{-\infty}^\infty f(x)\ dx = lim_{R\to\infty} \int_{-R}^R f(x)\ dx \quad ----(2)$$

provided this single limit exists.

If integral (1) converges its Cauchy principal value (2) exists. But it is not always true that integral (1) converges when its Cauchy P.V. exists.

For example, consider the function $f(x) = x$. Here, both of the limits $lim_{R_1\to\infty} \int_{-R_1}^0 x\ dx$ and $lim_{R_2\to\infty} \int_0^{R_2} x\ dx$ does not exists.
But, P.V. $\int_{-\infty}^\infty f(x)\ dx = lim_{R\to\infty} \int_{-R}^R x\ dx = lim_{R\to\infty}[\frac{x^2}{2}]_{-R}^R = 0.$

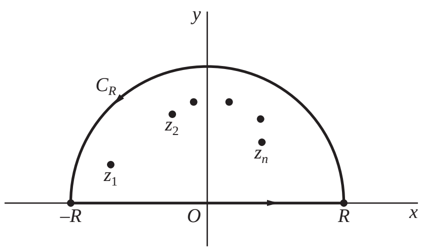
If $f(x)$ $(-\infty < x < \infty)$ is an even function and if the Cauchy principal value (2) exists, then

$$\int_0^\infty f(x)\ dx = \frac{1}{2}[P.V. \int_{-\infty}^\infty f(x)\ dx].$$

We now describe a method involving sums of residues, that is often used to evaluate improper integrals of rational functions $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials with real coefficients and no factors in common. We agree that $q(z)$ has no real zeros but has at least one zero above the real axis.

The method begins with the identification of all the distinct zeros of the polynomial $q(z)$ that lie above the real axis. They are, of course, finite in number and may be labelled as $z_1,\ z_2, ...,\ z_n$, where n is less than or equal to the degree of $q(z)$.

We then integrate the quotient $f(z) = \dfrac{p(z)}{q(z)}$ around the positively oriented boundary of the semicircular region consisting of the portion C_R in the upper half plane, of the circle with center at $z = 0$ and radius R, and the line segment from $-R$ to R on the real axis.



We take the positive number R as so large such that all the points $z_1,\ z_2,\ ...,\ z_n$ lie inside of the above described simple closed path.

The parametric representation $z = x$ $(-R \leq x \leq R)$ of the segment of the real axis and the Cauchy's residue theorem allows us to write that

$$\int_{-R}^{R} f(x)\ dx + \int_{C_R} f(z)\ dz = 2\pi\ i\Sigma_{k=0}^{n} Res_{z=z_k} f(z).$$

If

$$lim_{R\to\infty} \int_{C_R} f(z)\ dz = 0,$$

it follows that

$$P.V. \int_{-\infty}^{\infty} f(x)\ dx = 2\pi\ i\ \Sigma_{k=0}^{n} Res_{z=z_k} f(z).$$

Therefore, if f is even, then

$$\int_{0}^{\infty} f(x)\ dx = \pi\ i\ \Sigma_{k=0}^{n} Res_{z=z_k} f(z).$$

Problem 37.

Find the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{x^2+x}{(x^2+1)(x^2+4)} dx$

Solution.

Consider $f(z) = \frac{z^2+z}{(z^2+1)(z^2+4)}$. Then $f(z)$ is not analytic at $z = \pm\ i$ and $z = \pm 2i$ and the singular points of $f(z)$ that lie in the upper half plane are $z = i$ and $z = 2i$.

Consider the simple closed contour enclosing the semicircular region bounded by the segment $z = x$ $(-R \leq x \leq R)$ of the real axis and the upper half C_R of the circle $|z| = R$. (Here R is large enough such that both the singular points $z = i$ and $z = 2i$ of f that lie in the upper half plane belongs to the interior of the simple closed contour mentioned). Integrating $f(z)$ counterclockwise around the boundary of this semicircular region, we see that

$$\int_{-R}^{R} \frac{x^2+x}{(x^2+1)(x^2+4)}\ dx + \int_{C_R} f(z)\ dz = 2\pi\ i\ [Res_{z=i} f(z) + Res_{z=2i} f(z)].$$

On computation, we get $Res_{z=i} f(z) = \frac{i-1}{6i}$ and $Res_{z=2i} f(z) = \frac{2-i}{6i}$.

Therefore,

$$\int_{-R}^{R} \frac{x^2+x}{(x^2+1)(x^2+4)}\ dx = 2\pi\ i[\frac{i-1}{6i} + \frac{2-i}{6i}] - \int_{C_R} f(z)\ dz = \frac{\pi}{3} - \int_{C_R} f(z)\ dz.$$

This is valid for all values of R greater than 2.

Now, we show that $\int_{C_R} f(z)\ dz \to 0$ as $R \to \infty$.
On $|z| = R$, we have $|z^2+z| \leq R^2+R$, $|z^2+1| \geq ||z|^2-1| = R^2-1$ and $|z^2+4| \geq ||z|^2-4| = R^2-4$. $\Rightarrow |f(z)| \leq \dfrac{R^2+R}{(R^2-1)(R^2-4)}$

This implies that $|\int_{C_R} f(z)\ dz| \leq \int_{C_R} |f(z)|\ dz \leq \dfrac{R^2+R}{(R^2-1)(R^2-4)}\ \pi\ R$ (Using 2.1.4) Thus, as $R \to \infty$, $\int_{C_R} f(z)\ dz \to 0$. Hence

$$P.V. \int_{-\infty}^{\infty} \frac{x^2+x}{(x^2+1)(x^2+4)}\ dx = \frac{\pi}{3}.$$

■

Exercises.

1. Use residues to evaluate $\int_0^\infty \dfrac{x^2}{x^6+1}dx$
2. Use residues to find the Cauchy principal value of $\int_{-\infty}^\infty \dfrac{dx}{x^2+2x+2}$.
3. Use residues to evaluate $\int_0^\infty \dfrac{dx}{x^2+1}$
4. Find the value of $\int_0^\infty \dfrac{dx}{x^4+1}$
5. Use residues to show that $\int_0^\infty \dfrac{dx}{(x^2+1)^2}dx = \dfrac{\pi}{4}$
6. Use residues to find the Cauchy principal value of $\int_{-\infty}^\infty \dfrac{x\ dx}{(x^2+1)(x^2+2x+2)}$.

4.6 Definite Integrals involving Sines and Cosines

The method of residues is also useful in evaluating certain definite integrals of the type

$$\int_0^{2\pi} F(sin\ \theta, cos\ \theta)\ d\theta \quad ----(1)$$

Given the form of an integrand in (1) one can reasonably hope that the integral results from the usual parametrization of the unit circle $z = e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$.

Let $z = e^{i\theta}$, so that $sin\ \theta = \dfrac{z - z^{-1}}{2i}$, $cos\ \theta = \dfrac{z + z^{-1}}{2}$, and $d\theta = \dfrac{dz}{iz}$.

Putting all of this into (1) yields

$$\int_0^{2\pi} F(sin\ \theta, cos\ \theta)\ d\theta = \int_C F(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2})\frac{dz}{iz} \quad ----(2)$$

where C is the unit circle. When the integrand in integral (2) reduces to a rational function of z , we can evaluate that integral by means of Cauchy's residue theorem once the zeros in the denominator have been located and provided that none of them lie on C.

Problem 38.

Use residues to show that if $-1 < a < 1$, $\int_0^{2\ \pi} \dfrac{d\ \theta}{1 + a\ sin\ \theta} = \dfrac{2\pi}{\sqrt{1 - a^2}}$.

Solution.

This integration formula is clearly valid when $a = 0$, so we now assume that $a \neq 0$. With substitutions $sin\ \theta = \dfrac{z - z^{-1}}{2i}$, and $d\theta = \dfrac{dz}{iz}$. the integral takes the form $\int_C \dfrac{2/a}{z^2 + (2i/a)z - 1}dz$, where C is the positively oriented circle $|z| = 1$. Using quadratic formula, we find that the denominator of the integrand has the pure imaginary zeros $z_1 = (\dfrac{-1 + \sqrt{1 - a^2}}{a})i$ and $z_2 = (\dfrac{-1 - \sqrt{1 - a^2}}{a})i$. Let

$f(z)$ denotes the integrand in the above integral, then $f(z) = \dfrac{2/a}{(z-z_1)(z-z_2)}$. Since $|a| < 1$, $|z_2| = \dfrac{1+\sqrt{1-a^2}}{|a|} > 1$. Also, since $|z_1 z_2| = 1$, it follows that $|z_1| < 1$. Hence there are no singular points on C, and the only singular point interior to C is the point z_1. The corresponding residue can be found by writing $f(z) = \dfrac{\phi(z)}{z-z_1}$ where $\phi(z) = \dfrac{2/a}{z-z_2}$. $\Rightarrow z_1$ is a simple pole and $Res_{z=z_1} f(z) = \dfrac{2/a}{z_1-z_2} = \dfrac{1}{i\sqrt{1-a^2}}$. Thus, $\int_0^{2\,\pi} \dfrac{d\,\theta}{1+a\ sin\ \theta} = 2\pi\ iRes_{z=z_1} f(z) = \dfrac{2\pi}{\sqrt{1-a^2}}$.

■

Exercises.

1. Use residues to show that $\int_0^{2\pi} \dfrac{cos\ 3t}{5-4\ cos\ t} dt = \dfrac{\pi}{12}$.

2. Find the value of $\int_0^{2\pi} \dfrac{d\,\theta}{5+4\ sin\ \theta}$.

3. Show that $\int_{-\pi}^{\pi} \dfrac{d\,\theta}{1+sin^2\ \theta} = \sqrt{2}\pi$.

4. Use residues to show that if $-1 < a < 1$, $\int_0^{2\,\pi} \dfrac{d\,\theta}{1+a\ cos\ \theta} = \dfrac{2\pi}{\sqrt{1-a^2}}$.

5. Use residues to find the value of $\int_0^{\pi} \dfrac{d\,\theta}{(a+\ cos\ \theta)^2}$, if $a > 1$.

Text Books (As per Syllabus)

J. W. Brown and R. V. Churchill: ***Complex Variables and Applications***, Mc Graw-Hill, Inc, New York, (8th Edn.)

Further Reading

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