SCHOOL OF DISTANCE EDUCATION

B. Sc. DEGREE PROGRAMEE
MATHEMATICS(Core Course)
FIFTH SEMESTER

MM5B08: DIFFERENTIAL EQUATIONS

STUDY NOTES

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Contents

CHAPTER 1

Differential Equations

1.1 Introduction

The laws of physics are generally written down as differential equations. Therefore, all of science and engineering use differential equations to some degree. Understanding differential equations is essential to understanding almost anything you will study in your science and engineering classes. You can think of mathematics as the language of science, and differential equations are one of the most important parts of this language as far as science and engineering are concerned. In this chapter we introduce the following :

1. differential equations
2. order and degree of differential equations
3. solutions of differential equations
4. geometrical and analytical methods for investigating the solutions of first order differential equations
5. different types of differential equations

1.2 Ordinary Differential Equations

An *ordinary differential equation* is an equation that contain one or several derivatives of an unknown function, which we call $y(x)$ and which we want to determine from the equation. For example,

$$\frac{d^2y}{dx^2} + xy\frac{dy}{dx} + y = e^x \sin x$$

is a differential equation. Sometimes one uses shortened notations to write the same equation as

$$y'' + xyy' + y = e^x \sin x$$

or using the differential operator $D = \frac{d}{dx}$

$$D^2y + xyDy + y = e^x \sin x$$

In general, an equation of the form

$$F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}) = 0 \tag{1.1}$$

is called an ordinary differential equation(ODE).

1.2.1 Order and Degree of ODE

The order of a differential equation is the order of the highest derivative that appears in the equation. The degree of a differential equation is the degree of the highest derivative occurring in it, after the equation has been expressed in a form free from radicals and fractions as far as derivatives are concerned. For example,

1. $y'' + (y')^2 + x = 0$ is of order two and degree one.
2. $(y''')^2 + xy'' + y = e^x$ is of order three and degree two.
3. $\left(1 + (y')^2\right)^{2/3} = (y'')$ is of order two and degree three.

Exercise

Find the order and degree of the following differential equations:

1. $(y''')^2 + (y')^3 + y = \sin x$.
2. $x^3 y'' + \tan x y' + y = e^x(1 + x^2)$.
3. $(y'' + y)^{5/2} = y'''$.

1.2.2 Concept of solution

A solution of an n^{th} order differential equation on some open interval $a < x < b$ is a function $y = h(x)$ which is n times differentiable and satisfies the differential equation for all x in that interval. That is a solution of the ODE $F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}) = 0$ is an n times differentiable function $y = h(x)$ such that

$$F(x, y, h'(x), h''(x), \ldots, h^n(x)) = 0$$

for all x in the interval where $h(x)$ is defined.

Example 1. Consider the differential equation

$$\frac{dx}{dt} + x = 2\cos t. \tag{1.2}$$

We claim that $x = x(t) = \cos t + \sin t$ is a solution.How do we check? We simply put x into equation (1.2)! First we need to compute $\frac{dx}{dt}$. We find that

$$\frac{dx}{dt} = -\sin t + \cos t$$

Let us compute the left hand side of equation (1.2).

$$\frac{dx}{dt} + x = -\sin t + \cos t + \cos t + \sin t = 2\cos t.$$

We got precisely the right hand side.

But there is more! We claim $x = cost + sint + e^{-t}$ is also a solution. Let us try,

$$\frac{dx}{dt} = -\sin t + \cos t - e^{-t}$$

Again putting into the left hand side of equation (1.2).

$$\frac{dx}{dt} + x = -\sin t + \cos t - e^{-t} + cost + sint + e^{-t} = 2\cos t.$$

And it works yet again! So there can be many different solutions. In fact, for this equation all solutions can be written in the form

$$x = \cos t + \sin t + Ce^{-t}$$

for some constant C.

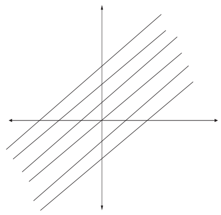


Figure 1.1: Integral curves of $y' = 1$.

Exercise

1. Verify that $y = e^{-x}(a \sin x + b \cos x)$ is a solution of the ODE $y'' + 2y' + 2y = 0$.

2. Verify that the function of y of x implicitly given by $x^2 + y^2 - 1 = 0 (y > 0)$ is a solution of the differential equation $yy' = -x$ on the interval $-1 < x < 1$.

1.2.3 General Solution and integral curves

The solution of a differential equation in which the number of arbitrary constants occurring is equal to the order of the differential equation is called the general solution. A solution obtained by giving particular values to the arbitrary constants in the general solution is called a particular solution. For example, consider the differential equation $\frac{dy}{dx} = 1$. Observe that

$$y = x + c, \quad c = \text{constant}$$

is the general solution of the given ODE. For each value of c, we get particular solution of the ODE, which is a curve in the xy-plane. Hence the geometrical representation of the general solution is an infinite family of curves in the xy-plane, called integral curves. Integral curves of the differential equation $\frac{dy}{dx} = 1$ is shown in figure 1.1.

1.2.4 Exercise

1. Verify that $y = \sin x + c$, where c is an arbitrary constant is the general solution of the ODE: $y' = \cos x$. Plot some integral curves of this ODE.

2. Verify that $y = a \sin x + b \cos x$ is the general solution of the ODE: $y'' + y = 0$.

1.2.5 Initial and Boundary value problems

If a differential equation is required to satisfy conditions on the dependent variable and its derivatives specified at one point of the independent variable, these conditions are called initial conditions and the problem is called an initial value problem. For example, $y'' + 4y = 0, y(0) = 0, y'(0) = 1$ is an example of an initial value problem. If a differential equation is required to satisfy conditions on the dependent variable and its derivatives specified at two or more values of the independent variable, these conditions are called boundary conditions and the problem is called an boundary value problem.

For example, $y'' + xy = 0, y(0) = 0, y'(1) = 0$ is an example of a boundary value problem.

1.2.6 Direction Fields

Often times it is not easy to solve a differential equation, even for a first-order equation. Yet we may need to know at least the behavior of the solution. First note that the first order equation

$$\frac{dy}{dx} = f(x, y)$$

gives us the slope of the solution curve y at each point (x, y). If we draw the slope at various points on the xy-plane as short line segments, we will get what is called the direction field (often also called slope field). The direction field will give us an idea of how the solution might look like.

Consider, for example, the differential equation

$$\frac{dy}{dx} = y. \tag{1.3}$$

The direction field of (1.3) is shown in Figure 1.2.

Example 2. Draw the direction field for the ODE $y' = 3 - 2y$. Based on the direction field, determine the behavior of y as $t \to \infty$.

For$y > 1.5$, the slopes are negative, and hence the solutions decrease. For $y < 1.5$, the slopes are positive, and hence the solutions increase. The equilibrium solution appears to be $y(t) = 1.5$, to which all other solutions converge.

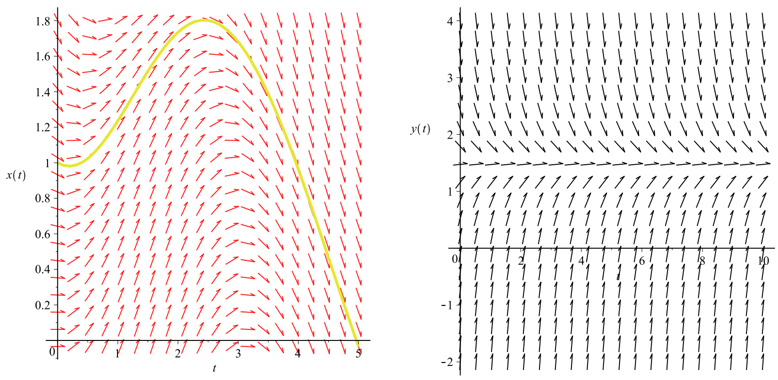


Figure 1.2: Direction field of the differential equations $y' = y$(left) and $y' = 3 - 2y$(right)

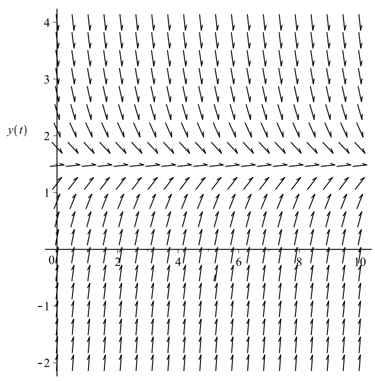


Figure 1.3: Direction field of $y' = 3 - 2y$

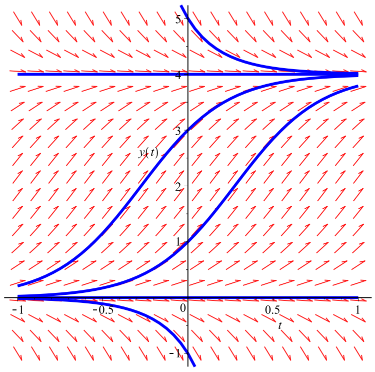


Figure 1.4: Direction field of $y' = y(4-y)$

Example 3. Draw the direction field for the ODE $y' = y(4-y)$. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency.

Example 4. Draw the direction field for the ODE $y' = y(y-2)^2$. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency.

Observe that $y' = 0$ for $y = 0$ and $y = 2$. The two equilibrium solutions are $y(t) = 0$ and $y(t) = 2$. Based on the direction field, $y' > 0$ for $y > 2$; thus solutions with initial values greater than 2 diverge from $y(t) = 2$. For $0 < y < 2$, the slopes are also positive, and hence solutions with initial values between 0 and 2 all increase toward the solution $y(t) = 2$. For $y < 0$, the slopes are all negative; thus solutions with initial values less than 0 diverge from the solution $y(t) = 0$.

Example 5. Draw the direction field for the ODE $y' = -2 + t - y$. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency.

All solutions appear to approach a linear asymptote (with slope equal to 1). It is easy to verify that $y(t) = t - 3$ is a solution.

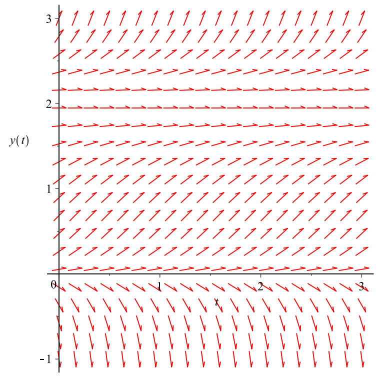


Figure 1.5: Direction field of $y' = y(y-2)^2$

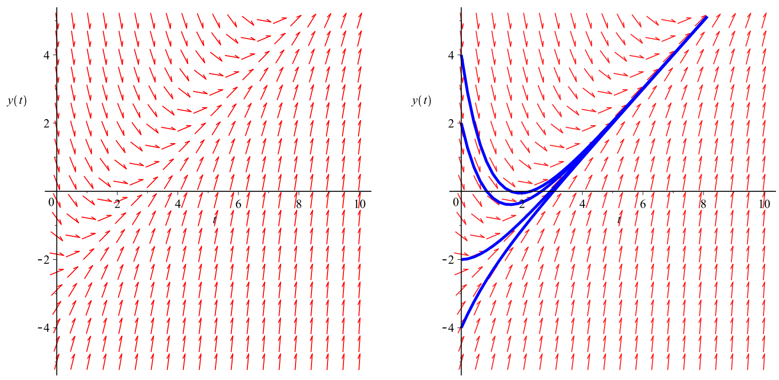


Figure 1.6: Direction field of $y' = -2 + t - y$

1.2.7 The differential

If $y = f(x)$ is a differentiable function of x, the differential dy of $f(x)$ is defined to be

$$df = f'(x)dx \tag{1.4}$$

and sometimes written simply as $dy = f'(x)dx$. Notice that this leads to the familiar derivative or differential coefficient $y' = dy/dx = f'(x)$. Observe that dy is a function of both x and dx. Hence equation (1.4) can be written as

$$dy(x, dx) = f'(x)dx$$

If we know the differential of f, $df = f'(x)dx$, it is immediate that $f(x) = \int f'(x)dx + c$, where c is a constant of integration.

For example, if $dy = 2xdx$, then $y = x^2 + c$, which is a family of parabolas all opening upwards and with y-axis as their axis. There is one such parabola through every point on the y-axis. If c = 0, it goes through the origin. If $c = 1$, we get the parabola $y = x^2 + 1$ which goes through $(0, 1)$. Given a differential equation such as

$$y' = \frac{xy}{x^2 + y^2},$$

we can easily convert it into the differential form as follows:

$$(x^2 + y^2)dy = xydx.$$

Thus the form of a first order differential equation can be written as:

$$M(x, y)dx + N(x, y)dy = 0$$

As an example,$(x^2 + y^2)y' = (x^2 - y^2)$ can also be written as

$$(x^2 + y^2)dy - (x^2 - y^2)dx = 0.$$

1.2.8 Linear and nonlinear ODE

The ordinary differential equations may be divided into two large classes, namely, linear equations and nonlinear equations.
The ODE:

$$F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}) = 0 \tag{1.5}$$

is said to be linear if F is a linear function of the variables $y, y', \ldots, y^{(n)}$. That is, a ODE is called linear if the dependent variable (y) and its derivatives occur only in the first degree and they are not multiplied together. Thus the general form of a linear differential equation of order n is

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = g(x) \tag{1.6}$$

An equation that is not of the form (1.6) is a nonlinear equation. An example of a linear equation is

$$x^2y'' + 4xy' + y = \sin x$$

Examples of nonlinear equations are:

$$yy'' + xy' + y = e^x$$

$$y''' + (\sin x)y' + xy = y^2$$

$$(y'')^2 + (\cos x)y' + y = \tan x$$

Exercise

State whether the following equations are linear or nonlinear.

1. $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + 2y = \sin x$
2. $(1 + y^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = e^y$
3. $\frac{d^4y}{dx^4} + \frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + 2y = 1$
4. $\frac{d^2y}{dx^2} + \sin(x + y) = \sin x$
5. $\frac{dy}{dx} + xy^2$
6. $\frac{d^3y}{dx^3} + x\frac{dy}{dx} + (\cos^2 x)y = x^3$

1.2.9 First order differential equations

This subsection deals with differential equations of first order,

$$y' = f(x, y), \tag{1.7}$$

where where f is a given function of two variables. Any differentiable function $y = \varphi(x)$ that satisfies this equation for all x in some interval is called a solution. Our object is to develop methods for finding solutions. Unfortunately, for an arbitrary function f, there is no general method for solving equation (1.7) in terms of elementary functions. Instead, we will describe several methods, each of which is applicable to a certain subclass of first order equations. The most important of these are

1. separable equations
2. linear equations and
3. exact equations

Separable equations

A differential equation of first order,

$$y' = f(x, y), \tag{1.8}$$

is said to be separable if, $f(x, y)$ can be factored as $f(x, y) = g(x)h(y)$, where g and h are known functions. If this factoring is not possible, the equation is not separable. Hence separable equations can be written as:

$$y' = g(x)h(y). \tag{1.9}$$

Case 1: If $h(y) \neq 0$, equation (1.9) can be written as:

$$\underbrace{\varphi(y)dy}_{\text{function of } y \text{ only}} = \underbrace{g(x)dx}_{\text{function of } x \text{ only}}, \quad \varphi(y) = \frac{1}{h(y)} \tag{1.10}$$

Integrating both sides of the above equation, we get:

$$\int \varphi(y)dy = \int g(x)dx + C, \tag{1.11}$$

where C is an arbitrary constant. This is the general solution of equation (1.9).

Case 2: If $h(y) = 0$, solve for the roots of this equation. Let $y_0, y_1, \ldots, y_r$ be the roots of this equation. Then $y(x) = y_0, y(x) = y_1, \ldots, y(x) = y_r$ are the equilibrium solutions of equation (1.9).

Example 6. Show that the equation

$$\frac{dy}{dx} = \frac{x^2}{1 - y^2} \tag{1.12}$$

is separable, and then find an equation for its integral curves.

The given equation can be written as:

$$\frac{dy}{dx} = (x^2)\left(\frac{1}{1 - y^2}\right)$$

The above equation is of the form

$$\frac{dy}{dx} = g(x)h(y)$$

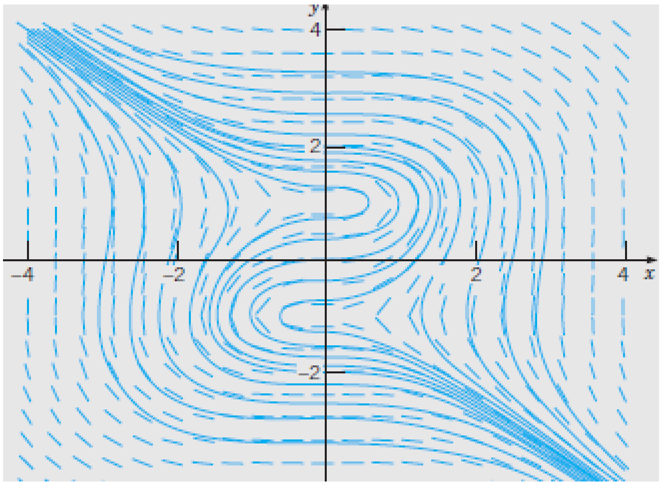


Figure 1.7: Direction field and integral curves of $y' = x^2/(1-y^2)$.

Hence the equation is separable.

Separating variables we get:

$$(1-y^2)dy = (x^2)dx$$

Integrating, we get:

$$y - y^3/3 = x^3/3 + C, \tag{1.13}$$

where C is an arbitrary constant. Equation (1.13) is an equation for the integral curves of (1.12). Integral curves are shown in figure 1.7.

Example 1. Solve the initial value problem

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}, y(0) = -1,$$

and determine the interval in which the solution exists.

The differential equation can be written as

$$2(y-1)dy = (3x^2+4x+2)dx.$$

Integrating the left side with respect to y and the right side with respect to x gives

$$y^2 - 2y = x^3 + 2x^2 + 2x + c, \tag{1.14}$$

where c is an arbitrary constant. To determine the solution satisfying the prescribed initial condition, we substitute $x = 0$ and $y = -1$ in Eq. (1.14), obtaining c = 3.

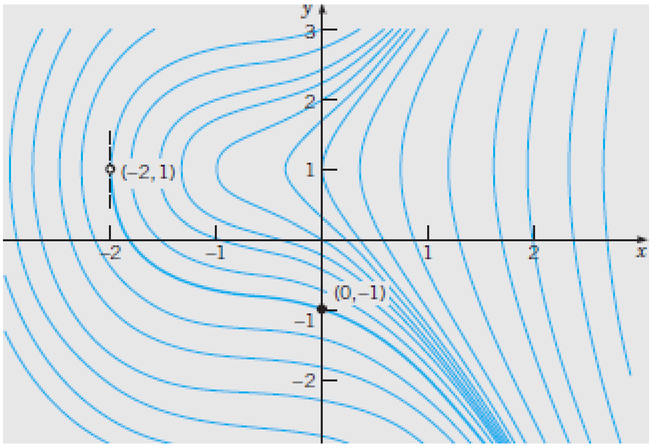


Figure 1.8: Integral curves of $y' = \frac{3x^2+4x+2}{2(y-1)}$

Hence the solution of the initial value problem is given implicitly by

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3. \tag{1.15}$$

To obtain the solution explicitly, we must solve Eq. (1.15) for y in terms of x. Thus we obtain

$$\begin{aligned} y &= 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4} \\ &= 1 \pm \sqrt{(x+2)(x^2+2)} \end{aligned} \tag{1.16}$$

Thus $y = 1 + \sqrt{(x+2)(x^2+2)}$ and $1 - \sqrt{(x+2)(x^2+2)}$ are solutions of the given differential equation. Putting $x = 0$ in the equation $y = 1 + \sqrt{(x+2)(x^2+4)}$, we get $y = 3$. So $y = 1 + \sqrt{(x+2)(x^2+2)}$ is not an integral curve passing through the point (0, -1). Again, putting $x = 0$ in the equation $y = 1 - \sqrt{(x+2)(x^2+2)}$, we get $y = -1$. Hence the integral curve passing through the point (0, -1) is given by:

$$y = 1 - \sqrt{(x+2)(x^2+2)} \tag{1.17}$$

Finally, to determine the interval in which the solution (1.17) is valid, we must find the interval in which the quantity under the radical is positive. Note that $x^2 + 2$ is always positive. Therefore $(x+2)(x^2+2)$ is positive if and only if $x + 2 > 0$. That is, $x > -2$. Thus the required interval is $(-2, \infty)$. The integral curves of the given differential equation is shown in figure 1.8

1.2.10 Linear equations

A first-order differential equation is said to be linear if and only if it can be written as

$$\frac{dy}{dt} + p(t)y = g(t) \tag{1.18}$$

where $p(t)$ and $g(t)$ are known functions of t only. Equation (1.18) is normally considered to be the "standard" form for first-order linear equations. Note that the only appearance of y in a linear equation (other than in the derivative) is in a term where y alone is multiplied by some formula of t .

Example 2. Consider the equation

$$t^2\frac{dy}{dt} + t^3[y - sin(t))] = 0.$$

Dividing through by t^2 and doing a little multiplication and addition converts the equation to

$$\frac{dy}{dt} + ty = tsin(t),$$

which is the standard form for a linear equation. So this differential equation is linear.

1.2.11 Solution of linear equations

Suppose we want to solve some first-order linear equation

$$\frac{dy}{dt} + p(t)y = g(t) \tag{1.19}$$

(for brevity, $p = p(t)$ and $g = g(t)$). To avoid triviality, let's assume $p(t)$ is not always 0 . Whether $g(t)$ vanishes or not will not be relevant. The small trick to solving equation (1.19) comes from the product rule for derivatives: If μ and y are two functions of t , then

$$\frac{d}{dt}[\mu y] = \frac{d\mu}{dt}y + \mu\frac{dy}{dt}. \tag{1.20}$$

Rearranging the terms on the right side, we get

$$\frac{d}{dt}[\mu y] = \mu\frac{dy}{dt} + \frac{d\mu}{dt}y \tag{1.21}$$

, and the right side of this equation looks a little like the left side of equation (1.19). To get a better match, let's multiply equation (1.19)) by μ,

$$\mu\frac{dy}{dt} + \mu p(t)y = \mu g(t) \tag{1.22}$$

With luck, the left side of this equation will match the right side of the last equation for the product rule, and we will have

$$\frac{d}{dt}[\mu y] = \frac{d\mu}{dt}y + \mu\frac{dy}{dt} \tag{1.23}$$

$$= \mu\frac{dy}{dt} + \mu p(t)y = \mu g(t) \tag{1.24}$$

This, of course, requires that

$$\frac{d}{dt}[\mu] = \mu p(t) \tag{1.25}$$

Assuming this requirement is met, the equations in (1.19) hold. Cutting out the middle of that (and recalling that g and μ are functions of t only), we see that the differential equation reduces to

$$\frac{d}{dt}[\mu y] = \mu(t)g(t) \tag{1.26}$$

The advantage of having our differential equation in this form is that we can actually integrate both sides with respect to t , with the left side being especially easy since it is just a derivative with respect to t . The function μ is called an integrating factor for the differential equation. As noted in the derivation, it must satisfy

$$\frac{d\mu}{dt} = \mu p(t) \tag{1.27}$$

If we assume temporarily that μ is positive, then the above equation can be written as:

$$\frac{1}{\mu}\frac{d\mu}{dt} = p(t)$$

Integrating,

$$\int \frac{1}{\mu}\frac{d\mu}{dt}dt = \int p(t)dt$$

This implies that

$$\ln \mu = \int p(t)dt$$

That is,

$$\mu = e^{\int p(t)dt} \tag{1.28}$$

Returning to equation (1.26), we have

$$\frac{d}{dt}[\mu y] = \mu(t)g(t) \tag{1.29}$$

Hence

$$\mu y = \int \mu(t)g(t) + c, \tag{1.30}$$

where c is an arbitrary constant.

Example 3. Solve the initial value problem

$$ty' + 2y = 4t^2 \tag{1.31}$$

$$y(1) = 2 \tag{1.32}$$

1. Get the equation into the standard form for first-order linear differential equations,

$$\frac{dy}{dt} + p(t)y = g(t) \tag{1.33}$$

   For our example, we just divide through by t , obtaining

$$y' + (2/t)y = 4t$$

   Note that $p(t) = 2/t$ and $g(t) = 4t$.

2. Compute an integrating factor

$$\mu = e^{\int p(t)dt}$$

   For our example,

$$\mu = e^{\int p(t)dt} = e^{\int (2/t)dt} = e^{2\ln t} = t^2$$

3. Multiply the differential equation (in standard form) by the integrating factor,

$$t^2[y' + (2/t)y] = 4t^3$$

$$\underbrace{t^2[y' + (2/t)y]}_{d/dt[t^2 y]} = 4t^3$$

4. Integrate with respect to t both sides of the last equation obtained,

$$t^2 y = t^4 + c$$

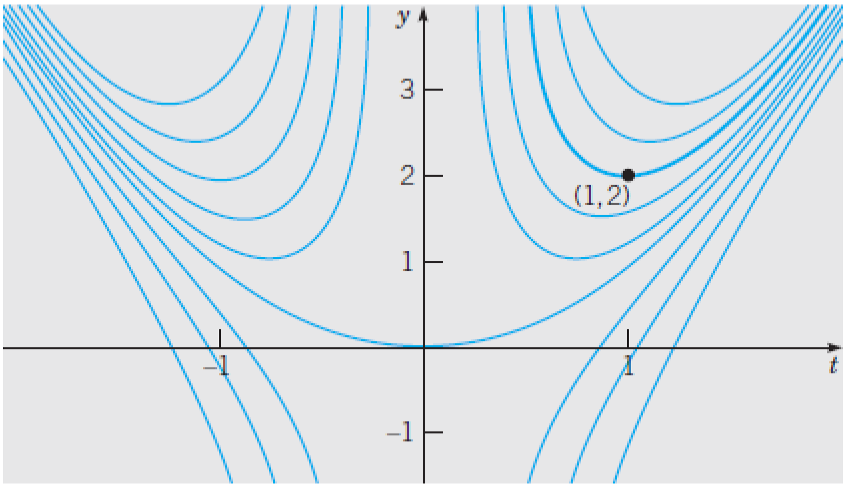


Figure 1.9: The integral curves of $ty' + 2y = 4t^2$

5. Finally, solve for y by dividing through by μ,

$$y = t^2 + c/t^2$$

Thus the general solution of the given differential equation is given by:

$$y = t^2 + c/t^2 \tag{1.34}$$

The integral curves of the given equation is shown in figure [?]. Applying the initial condition $y(1) = 2$ in equation (1.34), we get $c = 1$. Thus the solution to the initial value problem:$ty' + 2y = 4t^2, y(1) = 2$ is

$$y = t^2 + 1/t^2, t > 0.$$

This solution is shown by the heavy curve in Figure 1.9. Note that it becomes unbounded and is asymptotic to the positive y-axis as $t \to 0$ from the right. This is the effect of the infinite discontinuity in the coefficient $p(t)$ at the origin. The function $y = t^2 + (1/t^2)$ for $t < 0$ is not part of the solution of this initial value problem.

1.2.12 On Using Definite Integrals with Linear Equations

Consider the first-order linear equation

$$\frac{dy}{dt} + p(t)y = g(t) \tag{1.35}$$

The integrating factor of this differential equation is:

$$\mu(t) = e^{\int p(t)dt}$$

Multiplying both sides of equation (1.35) by $\mu(t)$, we get:

$$\underbrace{\mu[\frac{dy}{dt} + p(t)y]}_{d/dt[\mu y]} = \mu g(t) \tag{1.36}$$

That is,

$$\frac{d}{dt}[\mu y] = \mu g(t) \tag{1.37}$$

As before, to avoid having t represent two different entities, we replace the t's with another variable, say, s , and rewrite our current differential equation as

$$\frac{d}{ds}[\mu(s)y(s)] = \mu(s)g(s) \tag{1.38}$$

Then we pick a convenient lower limit a for our integration and integrate each side of the above with respect to s from $s = a$ to $s = t$,

$$\int_a^t \frac{d}{ds}[\mu(s)y(s)]ds = \int_a^t \mu(s)g(s) \tag{1.39}$$

But

$$\int_a^t \frac{d}{ds}[\mu(s)y(s)]ds = \mu(s)y(s)|_a^t = \mu(t)y(t) - \mu(a)y(a) \tag{1.40}$$

So equation (1.39) reduces to

$$\mu(t)y(t) - \mu(a)y(a) = \int_a^t \mu(s)g(s) \tag{1.41}$$

Solving for $y(t)$ yield

$$y(t) = \frac{1}{\mu(t)}\left[\mu(a)y(a) + \int_a^t \mu(s)g(s)\right] \tag{1.42}$$

Example 4. Solve the initial value problem

$$2y' + ty = 2,$$
$$y(0) = 1.$$

The given differential equation in the standard form is:

$$y' + (t/2)y = 1$$

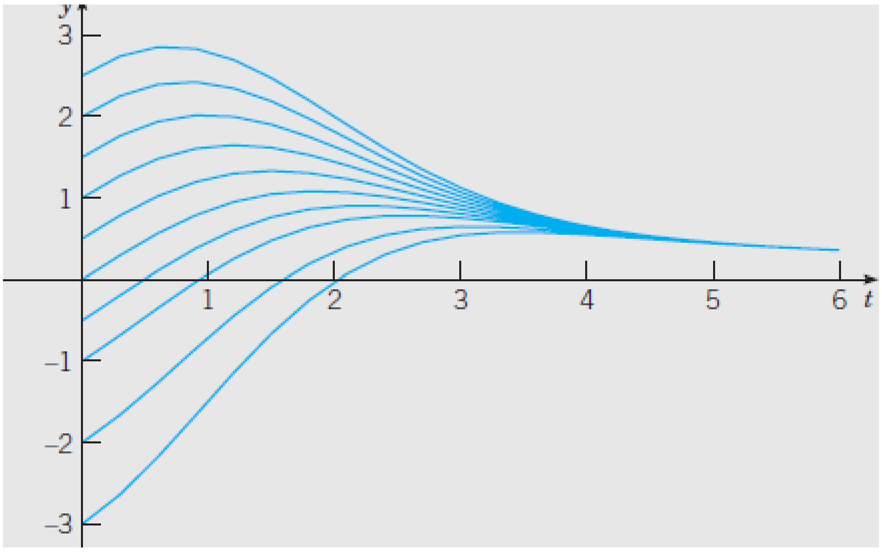


Figure 1.10: Integral curves of $2y' + ty = 2$

We have $p(t) = t/2$, $g(t) = 1$. The integrating factor is $\mu(t) = exp(t^2/4)$. Hence the solution of the given initial value problem is:

$$\begin{aligned} y(t) &= \frac{1}{\mu(t)}\left[\mu(a)y(a) + \int_a^t \mu(s)g(s)\right] \\ &= exp(-t^2/4)\left[\mu(0)y(0) + \int_0^t exp(t^2/4)\right] \\ &= exp(-t^2/4)\left[1 + \int_0^t exp(t^2/4)\right] \end{aligned}$$

1.2.13 Homogeneous Equations

Equations of the type

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \tag{1.43}$$

are called homogeneous differential equations. For example,

$$\frac{dy}{dx} = \frac{x^3 + y^3}{x^3 - y^3} = \frac{1 + (y/x)^3}{1 - (y/x)^3}$$

A homogeneous equation can be converted to a variable separable equation using a transformation of variables. Let $v = y/x$ be the new dependant variable, while x is still the independent variable. Hence

$$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting into the differential equation (1.43), we get:

$$v + x\frac{dv}{dx} = f(v) \Rightarrow x\frac{dv}{dx} = f(v) - v$$

Case 1. If $f(v) - v = 0$, then $y = cx$ is a solution, where c is the solution of the equation $f(v) - v = 0$.

Case 2. If $f(v) - v \neq 0$, then separating variables, we get:

$$\frac{dv}{f(v) - v} = \frac{dx}{x}$$

Integrating both sides gives the general solution:

$$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x} + c$$

Example 5. Solve $\frac{dy}{dx} = \frac{y-x}{y+x}$.

The given equation can be written as:

$$\frac{dy}{dx} = \frac{y - x}{y + x} = \frac{(y/x) - 1}{(y/x) + 1}$$

which is a homogeneous equation. Letting $v = y/x$,

$$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx},$$

the equation becomes

$$v + x\frac{dv}{dx} = \frac{v - 1}{v + 1},$$

$$x\frac{dv}{dx} = \frac{v - 1}{v + 1} - v = \frac{-1 - v^2}{v + 1}$$

Since $v^2 + 1 \neq 0$, separating variables gives:

$$\frac{v + 1}{v^2 + 1}dv = -\frac{1}{x}dx$$

Integrating both sides gives:

$$\int \frac{v + 1}{v^2 + 1}dv = -\int \frac{1}{x}dx + c,$$

$$\int \frac{v}{v^2+1}dv + \int \frac{1}{v^2+1}dv = -\int \frac{1}{x}dx + c,$$
$$\frac{1}{2}\ln(v^2+1) + tan^{-1}(v) = -\ln x + c$$

Replacing v by the original variables x and y results in the general solution

$$\frac{1}{2}\ln((y/x)^2+1) + tan^{-1}(y/x) = -\ln x + c$$

Therefore

$$\ln(x^2+y^2) + 2tan^{-1}(y/x) = c$$

1.2.14 Exact differential equations and integrating factors

Consider the differential equation of the form

$$\frac{dy}{dx} = -M(x,y)N(x,y),\ N(x,y) \neq 0 \text{ or } M(x,y)dx + N(x,y)dy = 0, \tag{1.44}$$

where $\frac{\partial M}{\partial x}$ and $\frac{\partial N}{\partial y}$ are continuous. Suppose the solution of (1.44) is $u(x,y) = c$, where c is a constant. Taking differential gives:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = dc = 0 \Rightarrow \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \tag{1.45}$$

Equation (1.45) should be same as (1.44) if $u(x,y) = c$ is the solution of (1.44), except for a common factor $\mu(x,y)$, that is, the coefficients of dx and dy in equations (1.44) and (1.45) are propotional

$$\frac{(\partial u/\partial x)}{M(x,y)} = \frac{(\partial u/\partial y)}{N(x,y)} = \mu(x,y) \Rightarrow \partial u/\partial x = \mu M, \partial u/\partial y = \mu N$$

Substituting into equation (1.45) gives:

$$\mu M dx + \mu N dy = 0 \tag{1.46}$$

Since the left hand side is an exact differential of some function, $u(x,y)$,

$$du(x,y) = 0 \Rightarrow u(x,y) = c$$

Hence, if one could find a function $\mu(x,y)$, called an integrating factor multiplying it to equation (1.44) yield an exact differential equation (1.45), which means that the left hand side is the exact differential of some function. The resulting differential equation can be easily solved.

Example 6. Solve $(y + 2xy^2)dx + (2x + 3x^2y)dy = 0$

Note that y is an integrating factor of the differential equation. Multiplying both sides of the equation by y, we get:

$$(y^2 + 2xy^3)dx + (2xy + 3x^2y^2)dy = 0$$

The left hand side is the exact differential of $u(x, y) = xy^2 + x^2y^3$. Hence,

$$d(xy^2 + x^2y^3) = 0 \Rightarrow xy^2 + x^2y^3 = c$$

1.2.15 Exact differential equations

If the differential equation:

$$M(x, y)dx + N(x, y)dy = 0 \tag{1.47}$$

is exact, then there is a function $u(x, y)$ such that

$$du = M(x, y)dx + N(x, y)dy \tag{1.48}$$

But, by definition of differential,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \tag{1.49}$$

Comparing equations (1.48) and (1.50) gives

$$M = \frac{\partial u}{\partial x}, N = \frac{\partial u}{\partial y} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}, \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

If $\frac{\partial M}{\partial y}$ and $\frac{\partial M}{\partial x}$ are continuous one has

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \tag{1.50}$$

Hence, a necessary condition for exactness is, from equations (1.50),

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

It can be shown that this condition is also sufficient.

Example 7. Solve $(6xy^2 + 4x^3y)dx + (6x^2y + x^4 + e^y)dy = 0)$

The differential equation is of the form:

$$M(x,y)dx + N(x,y)dy = 0$$

where $M = (6xy^2 + 4x^3y)$ and $N = (6x^2y + x^4 + e^y)$.
Test for exactness:

$$\frac{\partial M}{\partial y} = 12xy + 4x^3, \frac{\partial N}{\partial x} = 12xy + 4x^3$$

Therefore

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{ The differential equation is exact.}$$

Two methods are introduced in the following to find the general solution.

Method 1: Since the differential equation is exact, there is a function $u(x,y)$ such that

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = (6xy^2 + 4x^3y)dx + (6x^2y + x^4 + e^y)dy$$

That is,

$$\frac{\partial u}{\partial x} = 6xy^2 + 4x^3y \tag{1.51}$$

$$\frac{\partial u}{\partial y} = 6x^2y + x^4 + e^y \tag{1.52}$$

To determine $u(x,y)$, integrate equation (4.29) with respect to x

$$\begin{aligned} u(x,y) &= \int (6xy^2 + 4x^3y) + f(y) \\ &= 3x^2y^2 + x^4y + f(y) \end{aligned} \tag{1.53}$$

Differentiating equation (1.53) with respect to y and comparing with equation (1.52) yield

$$\frac{\partial u}{\partial y} = 6x^2y + x^4 + \frac{d}{dy}(f(y)) \tag{1.54}$$

$$= 6x^2y + x^4 + e^y \tag{1.55}$$

Hence

$$\frac{d}{dy}(f(y)) = e^y \Rightarrow f(y) = e^y$$

Substituting into equation (1.53) leads to

$$u(x,y) = 3x^2y^2 + x^4y + e^y$$

Hence the general solution is given by:

$$3x^2y^2 + x^4y + e^y = c$$

Method 2: The essence of Method is to determine function $u(x, y)$ by

1. integrating the coefficient with respect to x

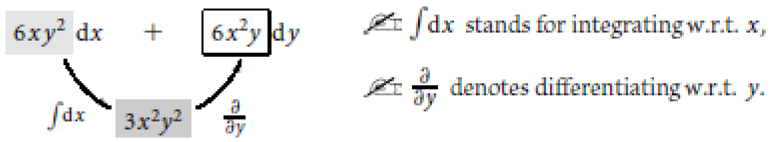
2. differentiating the result with respect to y and comparing y corresponding with the coefficient of dy. The method is illustrated step by step as follows:

   (a) Pick up a term, for example $6xy^2dx$

      i. Since the term has dx, integrate the coefficient $6xy^2$ with respect to x to yield $3x^2y^2$.

      ii. Differentiate the result with respect to y to yield the coefficient of dy term, that is, $6x^2y$

      iii. the two terms $6xy^2dx + 6x^2ydy$ are grouped together.

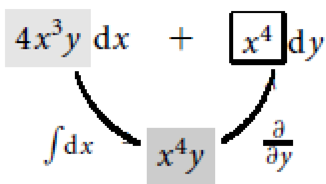


   (b) Pick up one of the remaining terms, for example $4x^3ydx$.

      i. Similarly, since the term has dx, integrate the coefficient $4x^3y$ with respect to x to yield x^4y.

      ii. Differentiate the result with respect to y yield the coefficient of dy term, that is, x^4.

      iii. The two terms $4x^3ydx + x^4dy$ are grouped together.

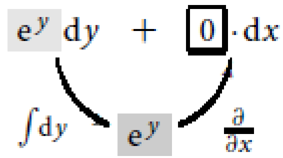


   (c) Pick up one of the remaining terms. Since there is only one term left, e^ydy is picked.

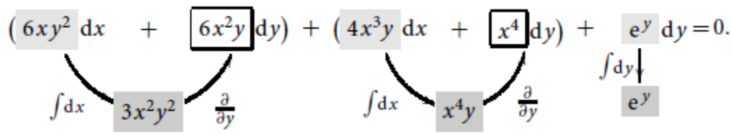
      i. Since the term has dy, integrate the coefficient e^y with respect to y to yield e^y.

ii. Differentiate the result with respect to x to yield the coefficient of dx term, that is, 0.

iii. The term $e^y dy$ is in a group by itself



(d) All terms on the left hand side of the equation have now been grouped



(e) Steps 1 to 3 can be combined to give a single expression as follows:

(f) Hence

$$d(3x^2y^2 + x^4y + e^y) = 0$$

which gives the general solution:

$$3x^2y^2 + x^4y + e^y = c$$

Example 8. Solve $\frac{dy}{dx} = \frac{y \sin x - e^x \sin 2y}{\cos x + 2e^x \cos 2y}$

The given differential equation can be written as:

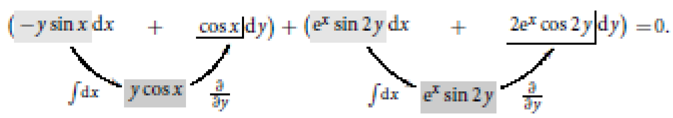
$$\underbrace{-y\sin x + e^x \sin 2y}_{M(x,y)}\, dx + \underbrace{\cos x + 2e^x \cos 2y}_{N(x,y)}\, dy$$

Test for exactness:

$$\frac{\partial M}{\partial y} = -\sin x + 2e^x \cos 2y, \frac{\partial N}{\partial x} = -\sin x + 2e^x \cos 2y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{The differential equation is exact}$$

The general solution is obtained by grouping:



Hence, by summing up the terms in the second row, we get the general solution:

$$y \cos x + e^x \sin 2y = c$$

Example 9. Solve $2x(3x + y - ye^{-x^2})dx + (x^2 + 3y^2 + e^{-x^2})dy = 0$

Note that

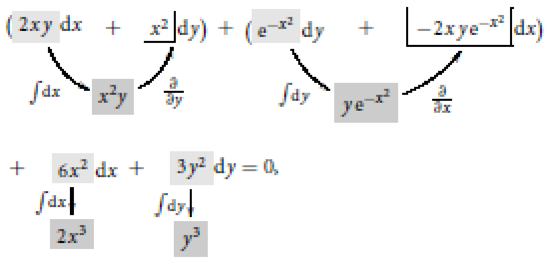
$$M = 2x(3x + y - ye^{-x^2}), N = (x^2 + 3y^2 + e^{-x^2})$$

Test for exactness:

$$\frac{\partial M}{\partial y} = 2x - 2xe^{-x^2}, \frac{\partial N}{\partial x} = 2x - 2xe^{-x^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{ The differential equation is exact}$$

The general solution is obtained by grouping:



Hence the general solution is

$$x^2y + ye^{-x^2} + 2x^3 + y^3 = c$$

Remark. When applying the method of grouping terms, whether to pick a term $f(x, y)dx$ or $g(x, y)dy$ first depends on whether it is easier to compute:

$$\int f(x, y)dx \quad \text{or} \quad \int g(x, y)dy$$

1.2.16 Integrating factors

Consider the differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

1. If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the differential equation is exact.

2. If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the differential equation can be rendered exact by multiplying by a function $\mu(x, y)$, known as an integrating factor, that is,

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0 \tag{1.56}$$

is exact. To find an integrating factor $\mu(x, y)$, apply the exactness condition on the equation (1.56)

$$\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x}$$

That is,

$$M\frac{\partial \mu}{\partial y} + \mu\frac{\partial M}{\partial y} = N\frac{\partial \mu}{\partial x} + \mu\frac{\partial N}{\partial x}$$

This implies that

$$\mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y} \tag{1.57}$$

This is a partial differential equation for the unknown function $\mu(x, y)$, which is more difficult to solve than the original ordinary differential equation. However, for some special cases, equation (1.57) can be solved for an integrating factor.

Special Cases:

If μ is a function of x only, that is, $\mu = \mu(x)$, then

$$\frac{\partial \mu}{\partial x} = \frac{d\mu}{\partial x}, \frac{\partial \mu}{\partial y} = 0$$

and equation (1.57) becomes

$$N\frac{d\mu}{\partial x} = \mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \tag{1.58}$$

This implies that

$$\frac{1}{\mu}\frac{d\mu}{\partial x} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \tag{1.59}$$

Since $\mu(x)$ is a function of x only, the left hand side is a function of x only. Hence, if an integrating factor of the form $\mu = \mu(x)$ is to exist, the right hand side must be a function of x only. Observe that equation (1.59) is variable separable, which can be solved easily by integration

$$\ln \mu = \int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx \tag{1.60}$$

This implies that

$$\mu(x) = exp\left[\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx\right] \tag{1.61}$$

Interchanging M and N, and x and y in equation (1.57), one obtains an integrating factor for another special case

$$\mu(y) = exp\left[\underbrace{\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy}_{\text{function of } y \text{ only}}\right] \tag{1.62}$$

Consider the differential equation $M(x, y)dx + N(x, y)dy = 0$

(a) If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x only,

$$\mu(x) = exp\left[\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx\right]$$

(b) If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a function of y only,

$$\mu(y) = exp\left[\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\right]$$

Example 10. Solve $3(x^2 + y^2)dx + x(x^2 + 3y^2 + 6y)dy = 0$.

Comparing with the standard form, we get:

$$M(x, y) = 3(x^2 + y^2), N(x, y) = x^3 + 3xy^2 + 6xy.$$

Test for exactness

$$\frac{\partial M}{\partial y} = 6y, \frac{\partial N}{\partial x} = 3x^2 + 3y^2 + 6y,$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{The differential equation is not exact.}$$

Since

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{3(x^2 + y^2)}[(3x^2 + 3y^2 + 6y) - 6y] = 1$$

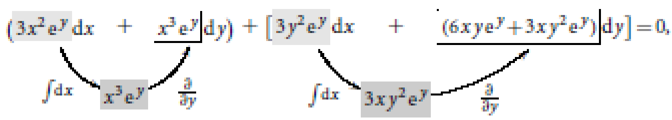
That is $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of y alone. Therefore

$$\mu(y) = exp\left[\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\right] = e^y$$

Multiplying the differential equation by the integrating factor $\mu(y) = e^y$ yields:

$$3(x^2e^y + y^2e^y)dx + x(x^2e^y + 3y^2e^y + 6ye^y)dy = 0$$

Thus the general solution is determined using the method of grouping terms:



Hence the general solution is $x^3e^y + 3xy^2e^y = c$.

Example 11. Solve $y(2x - y + 2)dx + 2(x - y)dy = 0$.

Note that

$$M = y(2x - y + 2), N = 2(x - y)$$

Test for exactness:

$$\frac{\partial M}{\partial y} = 2x - 2y + 2, \frac{\partial N}{\partial x} = 2,$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{The differential equation is not exact.}$$

Since

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = 1, \text{a function of } x \text{ alone,}$$

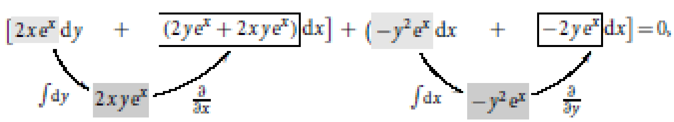
Therefore

$$\mu(x) = exp\left[\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx\right] = e^x$$

Multiplying both sides of equation by the integrating factor $\mu(x) = e^x$ yields

$$y(2xe^x - ye^x + 2e^x)dx + 2(xe^x - ye^x)dy = 0.$$

The general solution is determined by using the method of grouping terms:



Hence the general solution is:$2xye^x - y^2e^x = c$

1.2.17 Linear Equations

Consider the first order linear differential equation:

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{1.63}$$

The above equation can be written as:

$$[P(x)y - Q(x)]dx + dy = 0 \tag{1.64}$$

Note that $M(x, y) = P(x)y - Q(x), N(x, y) = 1$. Moreover, $\frac{\partial M}{\partial y} = P(x), \frac{\partial N}{\partial x} = 0$. The differential equation is not exact.

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = P(x), \text{a function of } x \text{ alone,}$$

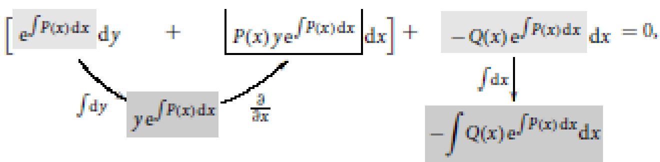
Therefore

$$\mu(x) = exp\left[\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx\right] = e^{\int P(x)dx}$$

Multiplying both sides of equation by the integrating factor $\mu(x) = e^{\int P(x)dx}$ yields

$$[P(x)y - Q(x)]e^{\int P(x)dx}dx + e^{\int P(x)dx}dy = 0 \tag{1.65}$$

The general solution can be determined using the method of grouping terms:



Hence the general solution is

$$ye^{\int P(x)dx} - \int Q(x)e^{\int P(x)dx}dx = c$$

That is,

$$y = e^{-\int P(x)dx}\left[\int Q(x)e^{\int P(x)dx}dx + c\right]$$

$$\frac{dy}{dx} + P(x)y = Q(x) \Rightarrow y = e^{-\int P(x)dx}\left[\int Q(x)e^{\int P(x)dx}dx + c\right]$$

$$\frac{dx}{dy} + P(y)x = Q(y) \Rightarrow x = e^{-\int P(y)dy}\left[\int Q(y)e^{\int P(y)dy}dy + c\right]$$

1.2.18 Existence and Uniqueness of Solutions

So far, we have discussed a number of initial value problems, each of which had a solution and apparently only one solution. This raises the question of whether this is true of all initial value problems for first order equations. In other words, does every initial value problem have exactly one solution? Further, if you are successful in finding one solution, you might be interested in knowing whether you should continue a search for other possible solutions or whether you can be sure that there are no other solutions. For linear equations, the answers to these questions are given by the following fundamental theorem.

Theorem 1. (Fundamental theorem of Existence and Uniqueness) If the functions p and g are continuous on an open interval $I = (\alpha, \beta)$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t) \tag{1.66}$$

for each t in I, and that also satisfies the initial condition

$$y(t_0) = y_0, \tag{1.67}$$

where y_0 is an arbitrary prescribed initial value.

Theorem 2. (Fundamental theorem of Existence and Uniqueness)[General case] Let the functions f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), y(t_0) = y_0.$$

Remark.

1. Let $f(t, y) = -p(t)y + g(t)$. Then $\frac{\partial f}{\partial y} = -p(t)$. So the continuity of f and $\frac{\partial f}{\partial y}$ is equivalent to the continuity of p and g. So Theorem 1is a particular case of theorem 2.

2. Here we note that the conditions stated in Theorem 2 are sufficient to guarantee the existence of a unique solution of the initial value problem 1.66 in some interval $t_0 - h < t < t_0 + h$, but they are not necessary. That is, the conclusion remains true under slightly weaker hypotheses about the function f . In fact, the existence of a solution (but not its uniqueness) can be established on the basis of the continuity of f alone.

3. An important geometrical consequence of the uniqueness parts of Theorems 1 and 2 is that the integral curves cannot intersect each other.

Example 12. Find an interval in which the initial value problem

$$ty' + 2y = 4t^2, \tag{1.68}$$

$$y(1) = 2 \tag{1.69}$$

has a unique solution.

The given equation can be rewritten as:

$$y' + (2/t)y = 4t$$

We have $p(t) = 2/t$ and $g(t) = 4t$. Note that g is continuous for all t and $p(t)$ is continuous for all $t \in (-\infty, 0) \cup (0, \infty)$. The interval $(0, \infty)$ contains the initial point; consequently, Theorem 1 guarantees that the given problem has a unique solution on the interval $(0, \infty)$. In Example 2 of Section 1.2, we found the solution of this initial value problem to be

$$y = t^2 + 1/t^2, t > 0.$$

In a similar manner we can prove that the initial value problem $ty' + 2y = 4t^2, y(-1) = 2$ has a unique solution in the interval $(-\infty, 0)$. has a unique solution.

Example 13. Prove that the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)},$$

$$y(0) = -1.$$

has a unique solution in some interval about $x = 0$.

Observe that

$$f(x, y) = \frac{3x^2 + 4x + 2}{2(y-1)}, \frac{\partial f}{\partial y} = \frac{3x^2 + 4x + 2}{2(y-1)^2}$$

Thus each of these functions is continuous everywhere except on the line $y = 1$. Consequently, a rectangle can be drawn about the initial point $(0, -1)$ in which both f and $\frac{\partial f}{\partial y}$ are continuous. Therefore Theorem 2 guarantees that the initial value problem has a unique solution in some interval about $x = 0$.

Example 14. Prove that the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)},$$

$$y(0) = 1.$$

does no have a unique solution.

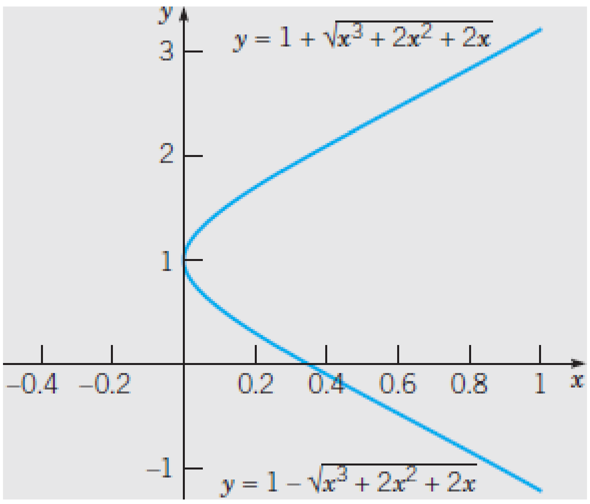
The initial point $(0, 1)$ now lies on the line $y = 1$ so no rectangle can be drawn about it within which f and $\frac{\partial f}{\partial y}$ are continuous. Consequently, Theorem 2 says nothing about possible solutions of this initial value problem. We have seen that the general solution to the differential equation $\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$, is

$$y^2 - 2y = x^3 + 2x^2 + 2x + c.$$

Further, if x = 0 and y = 1, then c=-1. Putting $c = -1$ in the above equation and solving for y, we obtain

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x}. \tag{1.70}$$

Equation (1.70) provides two functions that satisfy the given differential equation for $x > 0$ and also satisfy the initial condition $y(0) = 1$. Thus the initial value problem consisting of the given differential equation with the initial condition y(0) = 1 does not have a unique solution. The two solutions are shown in Figure



Example 15. Discuss the existence and uniqueness of solutions of the initial value problem

$$y' = y^{1/3}, y(0) = 0, t \geq 0.$$

The function $f(t, y) = y^{1/3}$ is continuous everywhere, but $\frac{\partial f}{\partial t}$ does not exist when y = 0, and hence is not continuous there. Thus Theorem 2 does not apply to this problem and no conclusion can be drawn from it. However, by the remark following Theorem 2 the continuity of f does ensure the existence of solutions, but not their uniqueness.

To understand the situation more clearly, we must actually solve the problem, which is easy to do since the differential equation is separable. Thus we have

$$y^{-1/3}dy = dt,$$

Integrating;

$$(3/2)y^{2/3} = t + c$$

That is,

$$y = [(2/3)(t + c)]^{3/2}$$

Applying the initial condition $y(0) = 0$, we get $c = 0$. Hence

$$y = \phi_1(t) = [(2/3)t]^{3/2}, \quad t \geq 0.$$

On the other hand, the function

$$y = \phi_2(t) = -[(2/3)t]^{3/2}, \quad t \geq 0$$

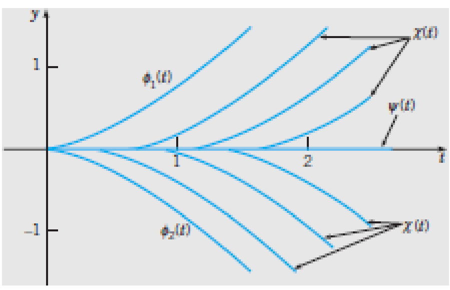
is also a solution of the initial value problem. Moreover, the function

$$y = \phi_3(t) = 0, t \geq 0$$

is yet another solution. Indeed, for an arbitrary positive t_0, the functions

$$y = \chi(t) = \begin{cases} 0 & \text{if } 0 \leq t_0, \\ \pm[(2/3)(t - t_0)]^{3/2} & \text{if } t_0 \leq t < \infty \end{cases}$$

are continuous, differentiable (in particular at $t = t_0$), and are solutions of the given initial value problem . Hence this problem has an infinite family of solutions; see Figure 2.4.1, where a few of these solutions are shown.



Example 16. Solve the initial value problem

$$y' = y^2, y(0) = 1,$$

and determine the interval in which the solution exists.

Since $f(t, y) = y^2$ and $\frac{\partial f}{\partial t} = 2y$ are continuous everywhere, Theorem 2 guarantees that this problem has a unique solution. The given equation can be written as:

$$\frac{1}{y^2} dy = dt$$

Integrating:

$$-y^{-1} = t + c$$

That is,

$$y = -\frac{1}{c + t}$$

Applying the condition $y(0) = 1$, we get $c = -1$. So $y = \frac{1}{1-t}$ is the solution of the given initial value problem. Clearly, the solution becomes unbounded as $t \to 1$; therefore, the solution exists only in the interval $-\infty < t < 1$.

Remark. $y = \frac{y_0}{1-y_0 t}$ is the solution of the initial value problem $y' = y^2, y(0) = t_0$.Observe that the solution becomes unbounded as $t \to 1/y_0$, so the interval of existence of the solution is $-\infty < t < 1/y_0$ if $y_0 > 0$, and is $1/y_0 < t < \infty$ if $y_0 < 0$.

1.2.19 Modeling with First-Order Equations

Model for the Motion of a Ball Near the Surface of the Earth

Consider the motion of an object of mass m dropped vertically at time $t = 0$ from a position as shown in figure. We assume that the force of air resistance is proportional to the velocity, v, of the object. The equation of motion of the object can be established by using Newton's Law:

$$F = ma \tag{1.71}$$

where m is the mass of the object, a is its acceleration, and F is the net force exerted on the object. Note that the acceleration, a and velocity, v are connected by the relation: $a = dv/dt$. So we can rewrite equation (1.71) in the following form:

$$F = m(dv/dt) \tag{1.72}$$

The forces that acts on the object as it falls are

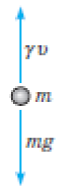


Figure 1.11: Free-body diagram of the forces on a falling object.

1. The force of gravity, mg, where g is the acceleration due to gravity.

2. The force of air resistance(drag force), $-kv$, where k is a positive constant.

Therefore the total force exerted on the object is given by $F = mg - \gamma v$. Hence equation (1.72) can be written as

$$m(dv/dt) = mg - \gamma v \tag{1.73}$$

Equation (1.73) is a mathematical model of an object falling in the atmosphere near the surface of earth. Clearly, equation (1.73) is separable.

CHAPTER 2

Second order linear differential equations

2.1 Nonhomogeneous second order linear equations

A second order ordinary differential equation has the form

$$f(y, t, y'(t), y''(t)) \tag{2.1}$$

where f is some given function. Usually, we will denote the independent variable by t and the dependant variable by y. But sometimes we will use x instead of t. Equation (2.1) is said to be linear if the function f is linear in y, y' and $y''(t)$. The general form of a second order linear differential equation is

$$a_0(t)y'' + a_1(t)y' + a_2(t)y = \varphi(t). \tag{2.2}$$

If $a_0(t) \neq 0$, then equation (2.2) can be rewritten as:

$$y'' + (a_1(t)/a_0(t))y' + (a_2(t)/a_0(t))y = \varphi(t)/a_0(t) \tag{2.3}$$

That is

$$y'' + p(t)y' + q(t)y = g(t), \tag{2.4}$$

where $p(t) = a_1(t)/a_0(t)$, $q(t) = a_2(t)/a_0(t)$ and $g(t) = \varphi(t)/a_0(t)$. If equation (2.1) is not of the form (2.2) or (2.4), then it is called nonlinear. An initial value problem consists of a differential equation such as Eq. (2.2), (2.3), or (2.4) together with a pair of initial conditions

$$y(t_0) = y_0, y'(t_0) = y'_0,$$

where y_0 and y'_0 are given numbers prescribing values for y and y' at the initial point t_0. Observe that the initial conditions for a second order equation identify not only a particular point (t_0, y_0) through which the graph of the solution must pass, but also the slope y'_0 of the graph at that point.

The most general form of a *nonhomogeneous second order linear equation* is

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = \varphi(x) \tag{2.5}$$

If D represents $\frac{d}{dx}$ and D^2 represents $\frac{d^2}{dx^2}$, then equation (2.5) may be written as:

$$\begin{aligned} & a_0(x)D^2y + a_1(x)Dy + a_2(x)y = \varphi(x) \\ \text{or} \quad & [a_0(x)D^2 + a_1(x)D + a_2(x)]y = \varphi(x) \\ \text{or} \quad & [f(D)]y = \varphi(x) \end{aligned}$$

where $f(D) = a_0(x)D^2 + a_1(x)D + a_2(x)$.
This equation is said to be in 'D-operator' form.

Definition. A second order linear differential equation is said to be *normal on an interval I* if and only if its coefficient functions $a_0(x), a_1(x), a_2(x)$ and the nonhomogeneous term $\varphi(x)$are continuous on I and the leading coefficient $a_0(x)$ is never zero for any value of x in the interval I.

Example 17. Find the interval in which the differential equation

$$(1 + x^2)y'' + xy' + y = 0$$

is normal.

Solution. Comparing the given equation with the general nonhomogeneous differential equation, we get

$$a_0(x) = (1 + x^2), \quad a_1(x) = x, \quad a_2(x) = 1, \quad \varphi(x) = 0$$

Domains of $a_0(x), a_1(x), a_2(x)$ and $\varphi(x)$ are shown in the table

Function	$a_0(x)$	$a_1(x)$	$a_2(x)$	$\varphi(x)$
Domain	$(-\infty, \infty)$	$(-\infty, \infty)$	$(-\infty, \infty)$	$(-\infty, \infty)$

From the table it follows that the domain of the differential equation is $(-\infty, \infty)$. Also note that the leading coefficient $a_0(x)$ is never zero on the interval $(-\infty, \infty)$. Hence the differential equation is normal on the interval $(-\infty, \infty)$.

Example 18. Find the interval on which the differential equation

$$\sqrt{x}y'' + 13xy' - 11y = \ln(x^2 - 100)$$

is normal.

Solution. Comparing the given equation with the general nonhomogeneous differential equation, we get

$$a_0(x) = \sqrt{x}, \quad a_1(x) = 13x, \quad a_2(x) = -11, \quad \varphi(x) = \ln(x^2 - 100)$$

Domain of definitions of the above function are shown in the following table

Function	$a_0(x)$	$a_1(x)$	$a_2(x)$	$\varphi(x)$
Domain	$[0, \infty)$	$(-\infty, \infty)$	$(-\infty, \infty)$	$(10, \infty)$

From the table we can see that the domain of the differential equation is $(10, \infty)$. Also, the leading coefficient $a_0(x) \neq 0$ for all x in the interval $(10, \infty)$. Hence the given differential equation is normal on the interval $(10, \infty)$

2.2 Homogeneous linear differential equation

The general form of a homogeneous second order linear equation is

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \tag{2.6}$$

Theorem 3 (Existence and uniqueness theorem)**.** Let the differential equation

$$a_0(x)y_2'' + a_1(x)y_2' + a_2(x)y_2 = 0$$

be normal on an interval I and let x_0 be an element in I. Then there exists one, and only one function $y(x)$ satisfying the differential equation on the interval I and the initial conditions $y(x_0) = y_0$, $y'(x_0) = y_0'$. In particular, if $y(x)$ is a solution of the differential equation which satisfies $y(x_0) = y'(x_0) = 0$, then $y \equiv 0$ for all x in the interval I.

Theorem 4 (Linearity principle)**.** If $y_1(x)$ and y_2 are any two solutions of a homogeneous second order linear differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$$

then $c_1y_1(x) + c_2y_2(x)$ is also a solution of this differential equation, where c_1 and c_2 are arbitrary constants.

Proof. Since $y_1(x)$ and $y_2(x)$ are the solutions of the differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0, \tag{2.7}$$

therefore

$$a_0(x)y_1'' + a_1(x)y_1' + a_2(x)y_1 = 0 \tag{2.8}$$

and

$$a_0(x)y_2'' + a_1(x)y_2' + a_2(x)y_2 = 0 \tag{2.9}$$

We will show that $c_1y_1(x) + c_2y_2(x)$ is a solution of equation (2.7). Inserting $y = c_1y_1(x) + c_2y_2(x)$ in the left hand side of equation (2.7), we get

$$\begin{aligned}
&a_0(x)[c_1y_1(x) + c_2y_2(x)]'' + a_2(x)[c_1y_1(x) + c_2y_2(x)]' + a_2(x)[c_1y_1(x) + c_2y_2(x)] \\
&= a_0(x)[c_1y_1''(x) + c_2y_2''(x)] + a_2(x)[c_1y_1'(x) + c_2y_2'(x)] + a_2(x)[c_1y_1(x) + c_2y_2(x)] \\
&= c_1\left[a_0(x)y_1'' + a_1(x)y_1' + a_2(x)y_1\right] + c_2[a_0(x)y_2'' + a_1(x)y_2' + a_2(x)y_2] \\
&= c_1(0) + c_2(0) = 0 \qquad \text{(by equations} \quad \text{(2.8) and(2.9))}
\end{aligned}$$

This implies that $y = c_1y_1(x)+c_2y_2(x)$ is a solution of the given differential equation.

□

2.3 Linear independence and dependence

Definition. Let f_1 and f_2 are any two functions and c_1 and c_2 are arbitrary constants. Then $c_1f_1 + c_2f_2$ is called a linear combination of f_1 and f_2.

Note The domain of $c_1f_1 + c_2f_2$ is the intersections of the domains of f_1 and f_2.

Definition. Functions f_1 and f_2 are said to be linearly dependent on an interval I if and only if there exists constants c_1 and c_2 not both zero such that for all x in I,

$$c_1f_1 + c_2f_2 = 0$$

Definition. Functions f_1 and f_2 are said to be linearly independent if and only if

$$c_1 f_1 + c_2 f_2 = 0 \quad \text{for all } x \in I \quad \Rightarrow c_1 = c_2 = 0$$

2.3.1 Test for independence

Theorem 5. Let f_1 and f_2 be non zero functions. Then f_1 and f_2 are linearly independent on an interval I if and only if f_1 and f_2 are proportional.

Proof. First, assume that f_1 and f_2 are linearly dependent. Then there exists constants c_1 and c_2 not both zero such that

$$c_1 f_1(x) + c_2 f_2(x) = 0 \quad \text{for all} \quad x \in I \tag{2.10}$$

If $c_1 = 0$, then from equation (2.10), we get $c_2 f_2(x) = 0$ for all x in the interval I. Since f_2 is a nontrivial function, therefore $c_2 = 0$. Hence $c_1 = c_2 = 0$. This shows that f_1 and f_2 are linearly independent. This is a contradiction to the assumption that f_1 and f_2 are linearly dependent. Hence $c_1 \neq 0$. In a similar manner we can show that $c_2 \neq 0$. Hence $c_1 \neq 0$ and $c_2 \neq 0$. That is, f_1 and f_2 are linearly dependent. Converse is trivial. □

Example 19. Prove that the functions $f_1(x) = e^x$ and $f_2(x) = e^{2x}$ are linearly independent for all real x.

Solution. $\frac{f_1(x)}{f_2(x)} = \frac{e^x}{e^{2x}} = e^{-x}$ is defined for all real x. That is, f_1 and f_2 are not proportional. Hence f_1 and f_2 are linearly independent .

Example 20. Prove that the functions $f_(x) = \ln(x^4)$ and $f_2(x) = \ln(x^2)$ are linearly dependent.

Solution. Note that $f_1 * x) = \ln(x^4) = 4\ln(x)$ and $f_2(x) = \ln(x^2)$. Then $\frac{f_1(x)}{f_2(x)} = \frac{4\ln(x)}{2\ln(x)} = 2$. Hence $f_1(x)$ and $f_2(x)$ are proportional. That is, $f_1(x)$ and $f_2(x)$ are linearly dependent.

Theorem 6. If the differential equation

$$a_0(x)y_2'' + a_1(x)y_2' + a_2(x)y_2 = 0$$

is normal on an interval I, then it has two linearly independent solutions $y_1(x)$ and $y_2(x)$ and any particular solution of this differential equation is a linear combination of $y_1(x)$ and $y_2(x)$.

Proof. Let x_0 be any point on the interval I. Let $y_1(x)$ and $y_2(x)$ be any two solutions of the given equation such that $y_1(x_0) = 1, y_2(x_0) = 0$, $y_1(x_0) = 0$, $y_2(x_0) = 1$. Then by existence and uniqueness theorem $y_1(x)$ and $y_2(x)$ are unique.

Claim: $y_1(x)$ and $y_2(x)$ are linearly independent:

Assume that

$$c_1y_1(x) + c_2y_2(x) = 0 \quad \text{for all } x \in I \tag{2.11}$$

But then

$$c_1y_1'(x) + c_2y_2'(x) = 0 \quad \text{for all } x \in I \tag{2.12}$$

In particular,

$$c_1y_1(x_0) + c_2y_2(x_0) = 0 \tag{2.13}$$

But then

$$c_1y_1'(x_0) + c_2y_2'(x_0) = 0 \tag{2.14}$$

Applying the initial conditions $y_1(x_0) = 1, y_2(x_0) = 0$in equation (2.11) we get

$$c_1(1) + c_2(0) = 0 \quad \Rightarrow c_1 = 0$$

Similarly applying the initial condition $y_1(x_0) = 0$, $y_2(x_0) = 1$ in equation (2.12), we get

$$c_1(0) + c_2(1) = 0 \quad \Rightarrow c_2 = 0$$

Hence $c_1 = c_2 = 0$. This implies that f_1 and f_2 are linearly independent.

Next we will show that every particular solution of the given equation is a linear combination of $y_1(x)$ and $y_2(x)$. Let $y(x)$ be any particular solution of the differential equation. Consider the linear combination

$$Y(x) = y(x) - y(x_0)y_1(x) - y'(x_0)y_2(x) \tag{2.15}$$

Then by linearity principle, $Y(x)$ is a solution of the given equation.

Differentiating equation (2.15) with respect to x, we get:

$$Y'(x) = y'(x) - y(x_0)y_1'(x) - y'(x_0)y_2'(x) \tag{2.16}$$

Putting $x = x_0$ in equations (2.15) and (2.16), we get:

$$\begin{aligned} Y(x_0) &= y(x_0) - y(x_0)y_1(x_0) - y'(x_0)y_2(x_0) \\ &= y(x_0) - y(x_0)(1) - y'(x_0)(0) \end{aligned}$$

$$= y(x_0) - y(x_0) = 0$$

and

$$\begin{aligned} Y'(x_0) &= y'(x_0) - y(x_0)y_1'(x_0) - y'(x_0)y_2'(x_0) \\ &= y'(x_0) - y(x_0)(0) - y'(x_0)(1) \\ &= y'(x_0) - y'(x_0) = 0 \end{aligned}$$

Thus we have shown that $Y(x)$ is a solution of the differential equation satisfying $Y(x_0) = Y'(x_0) = 0$. Hence by existence and uniqueness theorem, $Y(x) \equiv 0$ on I. That is

$$y(x) = y(x_0)y_1(x) + y'(x_0)y_2(x) \quad \text{for all } x \in I$$

Hence $y(x)$ is the linear combination of $y_1(x)$ and $y_2(x)$. □

Remark. (i)The above theorem guarantees the existence of two linearly independent solutions of every second order homogeneous linear differential equation. This theorem also says that there are some independent solutions of the differential equation such that every particular solution of of the equation can be expressed as a linear combination of these solutions. This theorem does not say that every particular solution can be expressed the linear combination of *any* linearly independent particular solutions.
(ii) Independent solutions of a differential equation is not unique. For example, consider the differential equation $\dfrac{d^2y}{dx^2} + \omega^2 x = 0$. Note that $y_1(x) = \sin\omega x$ and $y_2(x) = \cos\omega x$ are linearly independent solutions. Again $y_3(x) = \sin\omega x + \cos\omega x$ and $y_3(x) = \sin\omega x - \cos\omega x$ are also linearly independent solutions of this equation.

Definition. The *Wronskian* of the functions f_1 and f_2 is defined as the determinant:

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

Theorem 7. Let the differential equation

$$a_0(x)y_2'' + a_1(x)y_2' + a_2(x)y_2 = 0$$

be normal on an interval I and let $y_1(x)$ and $y_2(x)$ be two solutions of the equation. Then $W(f_1, f_2)$ is identically zero or its value is never zero on the interval.

Proof. Assume that $y_1(x)$ and $y_2(x)$ be the solutions of the differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \tag{2.17}$$

Then we have

$$a_0(x)y_1'' + a_1(x)y_1' + a_2(x)y_1 = 0 \tag{2.18}$$

and

$$a_0(x)y_2'' + a_1(x)y_2' + a_2(x)y_2 = 0 \tag{2.19}$$

Since the differential equation (2.17) is normal $a_0(x) \neq 0$ for all x in the interval I. So dividing both sides of the equations (2.18) and (2.19) by $a_0(x)$, we get:

$$y_1'' = -\frac{a_1(x)}{a_0(x)}y_1' - \frac{a_2}{a_0(x)}(x)y_1 \tag{2.20}$$

and

$$y_2'' = -\frac{a_1(x)}{a_0(x)}y_2' - \frac{a_2}{a_0(x)}(x)y_2 \tag{2.21}$$

We have

$$\begin{aligned} W(f_1, f_2) &= \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} \\ &= y_1(x)y_2'(x) - y_1'(x)y_2(x) \end{aligned} \tag{2.22}$$

Differentiating equation (2.22) with respect to x, we get,

$$\begin{aligned} W'(x) &= y_1(x)y_2''(x) - y_2'(x)y_1'(x) - y_1(x)y_2'(x) - y_2(x)y_1''(x) \\ &= y_1(x)y_2''(x) - y_2(x)y_1''(x) \end{aligned} \tag{2.23}$$

Substituting the values of $y_1''(x)$ and $y_2''(x)$ in equation (2.23), we get:

$$\begin{aligned} W'(x) &= y_1(x)\left[\frac{a_1(x)}{a_0(x)}y_1' - \frac{a_2}{a_0(x)}(x)y_1\right] - y_2(x)\left[\frac{a_1(x)}{a_0(x)}y_2' - \frac{a_2}{a_0(x)}(x)y_2\right] \\ &= -\frac{a_1(x)}{a_2(x)}W(x) \\ \textit{i.e.,}\quad & W'(x) + \frac{a_1(x)}{a_0(x)}W(x) = 0 \end{aligned} \tag{2.24}$$

This shows that $W(x)$ is a solution of a first order differential equation (2.24). If $W(x_0) = 0$ for some some $x_0 \in I$, then by existence and uniqueness theorem, $W(x) = 0$ for all x in the interval I. If $W(x_0) = k$, where k is any non zero number, then again by existence and uniqueness theorem there exists a non trivial solution $W(x)$ which satisfies $W(x_0) = k$. Hence $W(x) \neq 0$ for all x in the interval I. □

Theorem 8 (Abel's formula)**.** Let $y_1(x)$ and $y_2(x)$ be any two particular solutions of the differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$

which is normal on an interval I. Then

$$W(y_1(x), y_2(x)) = W(y_1(x_0), y_2(x_0))exp\left[-\int_{x_0}^{x} \frac{a_1(t)}{a_0(t)}dt\right]$$

where x_0 is an arbitrary point in the interval I.

Proof. From the above theorem we have

$$W'(x) + \frac{a_1(x)}{a_0(x)}W(x) = 0$$

$$i.e., \frac{W'(x)}{W(x)} + \frac{a_1(x)}{a_0(x)} = 0 \tag{2.25}$$

$$\tag{2.26}$$

Integrating both sides of the equation (2.25) between the limits x_0 to x, we get:

$$\int_{x_0}^{X} \frac{W'(x)}{W(x)}dx + \int_{x_0}^{x} \frac{a_1(t)}{a_0(t)}dt = 0$$

$$[\ln(W(x))]_{x}^{x_0} + \ln\left[exp\left(\int_{x_0}^{x} \frac{a_1(t)}{a_0(t)}\right)\right] = 0$$

$$i.e., \quad \ln(W(x)) - \ln(W(x_0)) + \ln\left[exp\left(\int_{x_0}^{x} \frac{a_1(t)}{a_0(t)}\right)\right] = 0$$

$$i.e., \quad \ln\left((W(x)\left[exp\left(\int_{x_0}^{x} \frac{a_1(t)}{a_0(t)}\right)\right]\right) = \ln(W(x_0))$$

$$i.e., \quad W(x)\left[exp\left(\int_{x_0}^{x} \frac{a_1(t)}{a_0(t)}\right)\right] = W(x_0)$$

$$i.e, \quad W(y_1(x), y_2(x)) = W(y_1(x_0), y_2(x_0))exp\left[-\int_{x_0}^{x} \frac{a_1(t)}{a_0(t)}dt\right]$$

□

Lemma 1. The system of linear equations

$$ax + by = 0$$

$$cx + dy = 0$$

have a non trivial solution (other than $x = 0, y = 0$) if and only if

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$$

Theorem 9 (Wronskian test for independence)**.** Let the differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$

be normal on an interval I and let $y_1(x)$ and $y_2(x)$ be any two particular solutions of the differential equation. Then $y_1(x)$ and $y_2(x)$ are linearly independent if and only if $W(y_1(x), y_2(x) \neq 0$ for any $x \in I$.

Proof. Assume that the differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \tag{2.27}$$

be normal in the interval I. Let $y_1(x)$ and $y_2(x)$ be any two particular solutions of the equation (2.27). Firstly, assume that $y_1(x)$ and $y_2(x)$ are linearly independent.
Claim $W(y_1(x), y_2(x)) \neq 0$ for any $x \in I$.
Suppose that $W(y_1(x_0), y_2(x_0)) = 0$ for some $x_0 \in I$. Then the system equations

$$\begin{aligned} y_1(x_0)c_1 + y_2(x_0)c_2 &= 0 \\ y_1'(x_0)c_1 + y_2'(x_0)c_2 &= 0 \end{aligned}$$

have a nontrivial solution k_1, k_2(by the above result). But then

$$\left.\begin{aligned} y_1(x_0)k_1 + y_2(x_0)k_2 &= 0 \\ y_1'(x_0)k_1 + y_2'(x_0)k_2 &= 0 \end{aligned}\right\} \tag{2.28}$$

But by the linearity property, the equation

$$y(x) = k_1y_1(x) + k_2y_2(x) \tag{2.29}$$

is also a solution of the differential equation (2.27) Differentiating equation (2.29) with respect to x, we get Putting $x = x_0$ in equations (2.29) and (2.30), we get:

$$y'(x) = k_1y_1'(x) + k_2y_2'(x) \tag{2.30}$$

$$\begin{aligned} y(x_0) &= k_1y_1(x_0) + k_2y_2(x_0) = 0 \\ y'(x_0) &= k_1y_1'(x_0) + k_2y_2'(x_0) = 0 (\text{by the above result}) \end{aligned}$$

Thus $y(x)$ is a solution of the differential equation (2.27) satisfying $y(x_0) = 0$ and $y'(x_0) = 0$. Hence by existence and uniqueness theorem, $y(x) = k_1y_1(x) + k_2y_2(x) \equiv$

0 on I. This shows that $y_1(x)$ and $y_2(x)$ are linearly dependent. This is a contradiction to the assumption that $y_1(x)$ and $y_2(x)$ are linearly independent.
Conversely assume that $W(y_1(x), y_2(x)) \neq 0$ for any value of x in I.
Claim $y_1(x)$ and $y_2(x)$ are linearly independent. Assume that

$$c_1y_1(x) + c_2y_2(x) = 0 \quad \text{for all} x \in I$$

This implies that

$$c_1y_1'(x) + c_2y_2'(x) = 0 \quad \text{for all} x \in I$$

In particular,

$$\left.\begin{array}{r} c_1y_1(x_0) + c_2y_2(x_0) = 0 \\ c_1y_1'(x_0) + c_2y_2'(x_0) = 0 \end{array}\right\} \tag{2.31}$$

Since $W(y_1(x_0), y_2(x_0)) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0$, the system of equations (2.31) have only trivial solution $c_1 = 0, c_2 = 0$. This shows that $y_1(x)$ and $y_2(x)$ are linearly independent. ☐

Theorem 10. If the differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$

is normal on an interval I and if $y_1(x)$ and $y_2(x)$ are *any* two linearly independent solutions then every solution of the differential equation over I is a linear combination of $y_1(x)$ and $y_2(x)$.

Proof. Assume that $y_1(x)$ and $y_2(x)$ be any two linearly independent solutions of the differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \tag{2.32}$$

Let $y(x)$ be any particular solution of equation (2.32). Since $y_1(x)$ and $y_2(x)$ are linearly independent, the value of $W(y_1(x), y_2(x))$ is never zero on the interval I. Let x_0 be a fixed point in I. Then by linearity property, the linear combination

$$Y(x) = y(x) - \frac{\begin{vmatrix} y(x_0) & y_2(x_0) \\ y'(x_0) & y_2'(x_0) \end{vmatrix}}{\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}} y_1(x) + \frac{\begin{vmatrix} y(x_0) & y_1(x_0) \\ y'(x_0) & y_1'(x_0) \end{vmatrix}}{\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}} y_2(x) \tag{2.33}$$

is also a solution of the equation (2.32). Equation (2.33) can also be written as:

$$Y(x) = \frac{\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}}{\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}} y(x) - - \frac{\begin{vmatrix} y(x_0) & y_2(x_0) \\ y'(x_0) & y_2'(x_0) \end{vmatrix}}{\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}} y_1(x) + \frac{\begin{vmatrix} y(x_0) & y_1(x_0) \\ y'(x_0) & y_1'(x_0) \end{vmatrix}}{\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}} y_2(x)$$

$$= \frac{W(y_1(x_0), y_2(x_0)}{W(y_1(x_0), y_2(x_0)} y(x) - \frac{W(y(x_0), y_2(x_0)}{W(y_1(x_0), y_2(x_0)} y_1(x) + \frac{W(y(x_0), y_(x_0)}{W(y_1(x_0), y_2(x_0)} y_2(x)$$

$$= \frac{1}{W(y_1(x_0), y_2(x_0)} \begin{vmatrix} y(x) & y_1(x) & y_2(x) \\ y(x_0) & y_1(x_0) & y_2(x_0) \\ y'(x_0) & y_1'(x_0) & y_2'(x_0) \end{vmatrix} \tag{2.34}$$

Therefore

$$Y'(x) = \quad = \frac{1}{W(y_1(x_0), y_2(x_0)} \begin{vmatrix} y'(x) & y_1'(x) & y_2'(x) \\ y(x_0) & y_1(x_0) & y_2(x_0) \\ y'(x_0) & y_1'(x_0) & y_2'(x_0) \end{vmatrix} \tag{2.35}$$

Putting $x = x_0$ in equations (2.34) and (2.35), we get $Y(x_0) = 0$ and $Y'(x_0) = 0$. Hence by existence and uniqueness theorem

$$Y(x) \equiv 0 \quad \text{for all} x \in I$$

$$i.e., y(x) = \frac{W(y(x_0), y_2(x_0))}{W(y_1(x_0), y_2(x_0))} y_1(x) + \frac{W(y_1(x_0), y_2(x_0))}{W(y_1(x_0), y_2(x_0))} y_2(x)$$

That is, $y(x)$ is a linear combination of $y_1(x)$ and $y_2(x)$. □

Definition. Let a second order homogeneous linear differential be normal on an interval I. Then two of its solutions are called fundamental solution (basis for all solutions) if and only if they are linearly independent on the interval.

Definition. Let $y_1(x)$ and $y_2(x)$ be a fundamental solutions of a second order linear differential equation which is normal on an interval I. Then the solution $y = c_1 y_1(x) + c_2 y_2(x)$ is called a complete solution (general solution) of the differential equation.

2.4 Solutions of Nonhomogeneous equations

Theorem 11. Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$

which is normal in an interval I and let Y be any specific solution of the nonhomogeneous linear equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = \varphi(x).$$

Then $y(x) = c_1y_1(x) + c_2y_2(x) + Y(x)$ is a complete solution of the homogeneous equation.

Proof. Let $y(x)$ be any arbitrary solution and let $Y(x)$ be any specific solution of the differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = \varphi(x) \tag{2.36}$$

Then we have

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = \varphi(x) \tag{2.37}$$

and

$$a_0(x)Y'' + a_1(x)Y' + a_2(x)Y = \varphi(x) \tag{2.38}$$

Subtracting equation (2.37) from (2.38), we get:

$$a_0(x)\left[y''(x) - Y''(x)\right] + a_1(x)\left[y'(x) - Y'(x)\right] + a_2(x)\left[y(x) - Y(x)\right] = 0$$
$$a_0(x)\left[y(x) - Y(x)\right]'' + a_1(x)\left[y(x) - Y(x)\right]' + a_2(x)\left[y(x) - Y(x)\right] = 0$$

This equation shows that $y(x) - Y(x)$ is a solution of the equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = \varphi(x)$$

But every complete solution of equation (2.36) is of the form $y = c_1y_1(x) + c_2y_2(x)$, therefore

$$y(x) - Y(x) = c_1y_1(x) + c_2y_2(x)$$

for some constants c_1 and c_2.

$$i.e., \quad y(x) = c_1y_1(x) + c_2y_2(x) + Y(x)$$

is a complete solution of the differential equation (2.36) □

Remark. (i) The solution $Y(x)$ is called the particular integral of the nonhomogeneous differential equation.
(ii) The complete solution $c_1y_1(x) + c_2y_2(x)$ is called the complementary function of the homogeneous differential equation.

Rules for solving a second order nonhomogeneous linear equation

(i) Find two independent solutions $y_1(x)$ and $y_2(x)$ of the homogeneous differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$

(ii) Find the complementary function $c_1y_1(x) + c_2y_2(x)$

(iii) Find the particular solution of the nonhomogeneous linear differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = \varphi(x)$$

(iv) The complete solution of the nonhomogeneous differential equation is given by

$$y(x) = c_1y_1(x) + c_2y_2(x) + Y$$

2.5 Linear equations with constant coefficients

The general form a homogeneous linear differential equation with constant coefficients is of the form:

$$ay'' + by' + cy = 0, \tag{2.39}$$

where a, b, and c are given constants. Some examples are

$$y'' + 7y' + 6y = 0$$
$$y'' - 6y' + 3y = 0$$
$$y'' - 8y' + 15y = 0.$$

Since the coefficients are constants, they are, trivially, continuous functions on the entire real line. Consequently, we can take the entire real line as the interval of interest, and be confident that any solutions derived will be valid on all of $(-\infty, \infty)$
.

2.5.1 Exponential Solutions with First-Order Equations

Let us look for clues on how to solve our second-order equations by first looking at solving a first-order, homogeneous linear differential equation with constant coefficients, say,

$$2y' + 6y = 0. \tag{2.40}$$

Since we are considering 'linear' equations, let's solve it using the method developed for first order linear equations: First divide through by the first coefficient, 2 , to get

$$y' + 3y = 0. \tag{2.41}$$

The integrating factor is then

$$\mu = e^{\int 3} dx = e^{3x}$$

Multiplying through and proceeding as usual with first-order linear equations:

$$e^{3x}[y' + 3y] = 0e^{3x}$$

$$i.e., \qquad \underbrace{e^{3x}[y' + 3y]}_{d/dx(ye^{3x})} = 0$$

$$i.e., \qquad \frac{d}{dx}(ye^{3x}) = 0$$

$$i.e., \qquad (ye^{3x}) = c$$

$$i.e., \qquad y = ce^{-3x}$$

So a general solution to

$$2y' + 6y = 0.$$

is

$$y = ce^{-3x}.$$

Clearly, there is nothing special about the numbers used here. Replacing 2 and 6 with constants a and b in the above would just as easily have given us the fact that a general solution to

$$ay' + by = 0$$

is

$$y = ce^{rx}$$

where $r = -b/a.$. Thus we see that all solutions to first-order homogeneous linear equations with constant coefficients are given by constant multiples of exponential functions.

2.5.2 Exponential Solutions with Second-Order Equations

Consider the second order equation:

$$ay'' + by' + cy = 0 \tag{2.42}$$

where a, b and c are constants. From our experience with the first-order case, it seems reasonable to expect at least some of the solutions to be exponentials. So let us find all such solutions by setting

$$y = e^{rx}$$

where r is a constant to be determined, plugging this formula into our differential equation, and seeing if a constant r can be determined. For example,

$$y'' - 5y' + 6y = 0$$

Letting $y = e^{rx}$ yields

$$D^2(e^{rx}) - 5D(e^{rx}) + e^{rx} = 0$$

$$i.e., \qquad D^2(e^{rx}) - 5D(e^{rx}) + e^{rx} = 0$$

$$i.e., \qquad r^2(e^{rx}) - 5r(e^{rx}) + e^{rx} = 0$$

$$i.e., \qquad e^{rx}(r^2 - 5r + 1) = 0$$

Since e^{rx} can never be zero, we can divide it out, leaving the algebraic equation

$$r^2 - 5r + 6 = 0.$$

Before solving this for r , let us pause and consider the more general case. More generally, letting $y = e^{rx}$ in

$$ay'' + by' + cy = 0$$

yields

$$aD^2(e^{rx}) + bD(e^{rx}) + c(e^{rx}) = 0$$

$$i.e., \qquad ar^2(e^{rx}) + br(e^{rx}) + c(e^{rx}) = 0$$

$$i.e., \qquad e^{rx}(ar^2 + br + c) = 0$$

Since e^{rx} can never be zero, we can divide it out, leaving us with the algebraic equation

$$ar^2 + br + c = 0 \tag{2.43}$$

Equation (2.43) is called called the characteristic equation for differential equation (2.42). Note the similarity between the original differential equation and its characteristic equation. The characteristic equation is nothing more that the algebraic equation obtained by replacing the various derivatives of y with corresponding powers of r :

$$ay'' + by' + cy = 0(\text{original differential equation})$$

$$ar^2 + br + c = 0(\text{characteristic equation})$$

The nice thing is that the characteristic equation is easily solved for r by either factoring the polynomial or using the quadratic formula. These values for r must then be the values of r for which $y = e^{rx}$ are (particular) solutions to our original differential equation. In our example, letting $y = e^{rx}$ in

$$y'' - 5y' + 6y = 0$$

lead to the characteristic equation

$$r^2 - 5r + 6 = 0,$$

which factors to

$$(r - 2)(r - 3) = 0.$$

Hence,

$$r - 2 = 0 \text{ or } r - 3 = 0.$$

So the possible values of r are

$$r = 2 \quad \text{and} \quad r = 3,$$

which, in turn, means

$$y_1 = e^{2x} \quad \text{and} \quad y_2 = e^{3x}$$

are solutions to our original differential equation. Clearly, neither of these functions is a constant multiple of the other.

2.5.3 The Basic Approach, Summarized

To solve a second-order homogeneous linear differential equation

$$ay'' + by' + cy = 0$$

in which a , b and c are constants, start with the assumption that $y(x) = e^{rx}$. where r is a constant to be determined. Plugging this formula for y into the differential equation yields, after a little computation and simplification, the differential equation's characteristic equation for r ,

$$ar^2 + br + c = 0.$$

Alternatively, the characteristic equation can simply be constructed by replacing the derivatives of y in the original differential equation with the corresponding powers of r . Observe that the solution to the polynomial equation

$$ar^2 + br + c = 0$$

can always be obtained via the quadratic formula

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Notice how the nature of the value r depends strongly on the value under the square root, $b^2 - 4ac$. There are three possibilities:

1. If $b^2 - 4ac > 0$, then $\sqrt{b^2 - 4ac}$ is some positive value, and we have two distinct real values for r ,

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

2. If $b^2 - 4ac = 0$, then

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{0}}{2a} = -b/2a,$$

and we only have one real root for our characteristic equation, namely,

$$r = -\frac{b}{2a}.$$

3. If $b^2 - 4ac < 0$, then the quantity under the square root is negative, and, thus, this square root gives rise to an imaginary number.

Whatever the case, if we find r_0 to be a root of the characteristic polynomial, then, by the very steps leading to the characteristic equation, it follows that

$$y_0(x) = e^{r_0 x}$$

is a solution to our original differential equation.

2.5.4 Case 1: Two Distinct Real Roots

Suppose the characteristic equation for

$$ay'' + by' + cy = 0$$

has two distinct (i.e., different) real solutions r_1 and r_2 . Then we have that both $y_1 = e^{r_1 x}$ and $y2 = e^{r_2 x}$ are solutions to the differential equation. Since we are assuming r_1 and r_2 are not the same, it should be clear that neither y_1 nor y_2 is a constant multiple of the other. Hence

$$\{e^{r_1 x}, e^{r_2 x}\}$$

is a linearly independent set of solutions to our second-order, homogeneous linear differential equation (Fundemental solution). The theorem on solutions to second-order, homogenous linear differential equations, tells us that

$$y(x) = c_1 e^{r_1 x} + c2 e^{r_2 x}$$

is a general solution to our differential equation.

Lemma 2. Let a , b and c be constants with $a \neq 0$. If the characteristic equation for

$$ay'' + by' + cy = 0$$

has two distinct real solutions r_1 and r_2 , then

$$y_1(x) = e^{r_1 x} \text{ and } y_2(x) = e^{r_2 x}.$$

are two solutions to this differential equation. Moreover, $\{e^{r_1 x}, e^{r_2 x}\}$ is a fundamental set for the differential equation, and

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

is a general solution.

2.5.5 Case 2: Only One Root Using Reduction of Order

If the characteristic polynomial only has one root r , then

$$y_1(x) = e^{rx}$$

is one solution to our differential equation. This, alone, is not enough for a general solution, but we can use this one solution with the reduction of order method to get the full general solution. Let us do one example this way. Consider the differential equation

$$y'' - 6y' + 9y = 0.$$

The characteristic equation is

$$r^2 - 6r + 9 = 0,$$

which factors nicely to

$$(r - 3)^2 = 0,$$

giving us $r = 3$ as the only root. Consequently, we have

$$y_1(x) = e^{3x}$$

as one solution to our differential equation. To find the general solution, we start the reduction of order method as usual by letting

$$y(x) = y_1(x)u(x) = e^{3x}u(x).$$

The derivatives are then computed,

$$y'(x) = [e^{3x}u]' = 3e^{3x}u + e^{3x}u'$$

and

$$\begin{aligned} y'' &= [3e^{3x}u + e^{3x}u]' \\ &= 9e^{3x}u + 3e^{3x}u' + 3e^{3x}u' + e^{3x}u'' \\ &= 9e^{3x}u + 6e^{3x}u' + e^{3x}u'' \end{aligned}$$

and plugged into the differential equation,

$$0 = y'' - 6y' + 9y$$

$$=[9e^{3x}u + 6e^{3x}u' + e^{3x}u''] - 6[3e^{3x}u + e^{3x}u''] + 9[e^{3x}u]$$
$$=e^{3x}[9u + 6u' + u'' - 18u - 6u' + 9u] = u''$$

Thus we have

$$u'' = 0$$

Integrating:

$$u' = A$$

Again integrating:

$$u = Ax + B$$

Thus

$$y = e^{3x}u = e^{3x}(Ax + B) = Axe^{3x} + Be^{3x}$$

is the general solution. Let us consider the most general case where the characteristic equation

$$ar^2 + br + c = 0$$

has only one root. As noted when we discussed the possible of solutions to the characteristic polynomial (see page 341), this means

$$r = -\frac{b}{2a}$$

Let us go through the reduction of order method, keeping this fact in mind. Start with the one known solution

$$y_1(x) = e^{rx} \quad \text{where} r = -b/2a$$

Set

$$y(x) = y_1(x)u(x) = e^{rx}u(x),$$

compute the derivatives,

$$y'(x) = [e^{rx}u]' = re^{rx}u + e^{rx}u'$$

$$y'' = [re^{rx}u + e^{rx}u]'$$
$$= r^2e^{rx}u + re^{rx}u' + re^{rx}u' + e^{rx}u''$$

$$= r^2e^{rx}u + r^2e^{rx}u' + e^{rx}u''$$

and plug these into the differential equation,

$$\begin{aligned} 0 &= ay'' + by' + cy \\ &= a[r^2e^{rx}u + 2re^{rx}u' + e^{rx}u''] + b[re^{rx}u + e^{rx}u'] + c[e^{rx}u] \\ &= e^{rx}[ar^2u + 2aru' + au'' + bru + bu' + cu] \end{aligned}$$

Dividing out the exponential and grouping together the coefficients for u, u' and u'' , we get

$$0 = au'' + [2ar + b]u' + [ar^2 + br + c]u$$

Since r satisfies the characteristic equation,

$$ar^2 + br + c = 0,$$

the "u term" drops out, as it should. Moreover, because $r = -b/2a$,

$$2a^r + b = 2a[-b/2a] + b = -b + b = 0$$

and the "u term" also drops out, just as in the example. Dividing out the a (which, remember, is a nonzero constant), the differential equation for u simplifies to

$$u'' = 0$$

Integrating twice yields

$$u(x) = Ax + B,$$

and, thus,

$$y(x) = y_1(x)u(x) = e^{rx}[Ax + B] = Axe^{rx} + Be^{rx}.$$

Lemma 3. Let a , b and c be constants with $a \neq 0$. If the characteristic equation for

$$ay'' + by' + cy = 0$$

has only one solution r , then

$$y_1(x) = e^{rx} \quad \text{and} \quad y_2(x) = xe^{rx}.$$

are two solutions to this differential equation. Moreover, $\{e^{rx}, xe^{rx}\}$ is a fundamental set for the differential equation, and

$$y(x) = c_1e^{rx} + c_2xe^{rx}$$

is a general solution.

2.5.6 Case 3: Complex Roots

If $b^2 - 4ac$ is negative, then the characteristic equation (**??**) has complex roots

$$\lambda_1 = \frac{-b + i\sqrt{4ac - b^2}}{2a} = \alpha + i\beta \quad \text{and} \quad \lambda_2 = \frac{-b - i\sqrt{4ac - b^2}}{2a} = \alpha - i\beta$$

where $\alpha = (-b/2a)$ and $\beta = (\sqrt{4ac - b^2}/2a)$. Then $y_1(x) = e^{\alpha + i\beta x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$ and $y_2(x) = e^{\alpha + i\beta x} = e^{\alpha x}(\cos\beta x - i\sin\beta x)$ are solutions of the differential equation. Then $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are independent solutions of the equation (2.42) by the following Lemma.

Lemma 4. Let $y(x) = u(x) + iv(x)$ be a complex valued solution of the differential equation

$$ay'' + by' + cy = 0$$

where a, b and c are real numbers. Then $y_1(x) = u(x)$ and $y_2(x) = v(x)$ are two real valued solutions of the equation. In other words, both real and imaginary parts of a complex valued valued solution are solutions of the equation.

Proof. Since $u + iv$ is a solution of the differential equation

$$ay'' + by' + cy = 0$$

therefore

$$\begin{aligned} & a[u(x) + iv(x)]'' + b[u(x) + iv(x)]' + c[u(x) + iv(x)] = 0 \\ \textit{i.e.,} \quad & a[u''(x) + iv''(x)] + b[u'(x) + iv'(x)] + c[u(x) + iv(x)] = 0 \\ \textit{i.e.,} \quad & [au''(x) + bu'(x) + cu(x)] + i[av''(x) + bv'(x) + cv(x)] = 0 \end{aligned}$$

Equating real imaginary parts on both sides, we get:

$$au''(x) + bu'(x) + cu(x) = 0 \quad \text{and} \quad av''(x) + bv'(x) + cv(x) = 0$$

Thus $u(x)$ and $v(x)$ are solutions of $ay'' + by' + cy = 0$ □

Theorem 12. Let a , b and c be real-valued constants with $a \neq= 0$. Then the characteristic polynomial for

$$ay'' + by' + cy = 0$$

will have either one or two has two distinct real roots or will have two complex roots that are complex conjugates of each other. Moreover:

1. If there are two distinct real roots r_1 and r_2 , then

$$\{e^{r_1 x}, e^{r_2 x}\}$$

is a fundamental set of solutions to the differential equation, and

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

is a general solution.

2. If there is only one real root r , then

$$\{e^{rx}, xe^{rx}\}$$

is a fundamental set of solutions to the differential equation, and

$$y(x) = c_1 e^{rx} + c_2 x e^{rx}$$

is a general solution.

3. If there is there is a conjugate pair of roots $r = \alpha \pm i\beta$, then both

$$\{e^{(\alpha+i\beta)x}, e^{(\alpha-i\beta)x}\} \quad \text{and} \quad \{e^{\alpha x}\cos\beta x, e^{\alpha x}\sin\beta x\}$$

are fundamental sets of solutions to the differential equation, and either

$$y(x) = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$$

or

$$y(x) = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x$$

can be used as a general solution.

2.6 Method of Undetermined Coefficients

2.7 Basic Ideas

In this chapter, we will discuss a method for finding particular solutions to nonhomogeneous differential equations.

Example 21. Consider

$$y'' - 2y' - 3y = 36e^{5x}$$

Since all derivatives of e^{5x} equal some constant multiple of e^{5x} , it should be clear that, if we let

$$y(x) = \text{some multiple of} e^{5x},$$

then

$$y'' - 2y' - 3y = \text{some other multiple of } e^{5x}.$$

So let us let A be some constant "to be determined", and try

$$y_p(x) = Ae^{5x}$$

as a particular solution to our differential equation: particular solution to our differential equation:

$$y_p'' - 2yp' + 3y_p = 36e^{5x}$$

$$i.e., \qquad [Ae^{5x}]'' - 2[Ae^{5x}]' - 3[Ae^{5x}] == 36e^{5x}$$

This implies

$$12Ae5x = 36e5x$$

That is $A = 3$. So our "guess", $y_p(x) = Ae^{5x}$, satisfies the differential equation only if $A = 3$. Thus,

$$y_p(x) = 3e^{5x}$$

is a particular solution to our nonhomogeneous differential equation.

Example 22. Find the general solution of the differential equation

$$y'' - 2y' - 3y = 36e^{5x}.$$

Solution. From the last example, we know

$$y_p(x) = 3e^{5x}$$

is a particular solution to the differential equation. The corresponding homogeneous equation is

$$y'' - 2y' - 3y = 0.$$

Its characteristic equation is

$$r^2 - 2r - 3 = 0,$$

which factors as

$$(r + 1)(r - 3) = 0.$$

So $r = -1$ and $r = 3$ are the possible values of r , and

$$y_h(x) = c_1 e^{-x} + c_2 e^{3x}$$

is the general solution to the corresponding homogeneous differential equation. So

$$y(x) = y_p(x) + y_h(x) = 3e^{5x} + c_1 e^{-x} + c_2 e^{3x}.$$

is a general solution to our nonhomogeneous differential equation.

Example 23. Consider the initial-value problem

$$y'' - 2y' - 3y = 36e^{5x} \quad \text{with} \quad y(0) = 9 \quad \text{and} \quad y'(0) = 25.$$

Solution. From above, we know the general solution to the differential equation is

$$y(x) = 3e^{5x} + c_1 e^{-x} + c_2 e^{3x}.$$

Its derivative is

$$y'(x) = 15e^{5x} - c_1 e^{x} + 3c_2 e^{3x}$$

This, with our initial conditions, gives us

$$c_1 + c_2 = 6$$

$$-c_1 + 3c_2 = 10$$

Solving this system , we get:

$$c_1 = 2 \quad \text{and} \quad c_2 = 4$$

So the solution to the given differential equation that also satisfies the given initial conditions is

$$y(x) = 3e^{5x} + c_1 e^{-x} + c_2 e^{3x} = 3e^{5x} + 2e^{-x} + 4e^{3x}$$

In all of the following, we are interested in finding a particular solution $y_p(x)$ to

$$ay'' + by' + cy = g \tag{2.44}$$

where a, b and c are constants and g is the indicated type of function.

Exponentials If, for some constants C and α ,

$$g(x) = Ce^{\alpha x}$$

then a good first guess for a particular solution to differential equation (2.44) is

$$y_p(x) = Ae^{\alpha x}$$

where A is a constant to be determined.

Sines and Cosines Consider

$$y'' - 2y' - 3y = 65\cos(2x). \tag{2.45}$$

A naive first guess for a particular solution might be

$$y_p(x) = A\cos(2x),$$

where A is some constant to be determined. Unfortunately, here is what we get when plug this guess into the differential equation:

$$y_p'' - 2y_p' - 3y_p = 65\cos(2x)$$

$$i.e., \quad [A\cos(2x)]'' - 2[A\cos(2x)]' - 3[A\cos(2x)] = 65\cos(2x)$$

$$A[-7\cos(2x) + 4\sin(2x)] = 65\cos(2x).$$

But there is no constant A satisfying this last equation for all values of x . So our naive first guess will not work.

Since our naive first guess resulted in an equation involving both sines and cosines, let us add a sine term to the guess and see if we can get all the resulting sines and cosines in the resulting equation to balance. That is, assume

$$y_p(x) = A\cos(2x) + B\sin(2x)$$

where A and B are constants to be determined. Plugging this into the differential equation:

$$y_p'' - 2y_p' - 3y_p = 65\cos(2x)$$

$$i.e., \quad [A\cos(2x)+B\sin(2x)]''-2[A\cos(2x)+B\sin(2x)]'-3[A\cos(2x)+B\sin(2x)] = 65\cos(2x)$$

$$i.e., \quad (-7A-4B)\cos(2x) + (4A-7B)\sin(2x) = 65\cos(2x)$$

For the cosine terms on the two sides of the last equation to balance, we need

$$-7A - 4B = 65,$$

and for the sine terms to balance, we need

$$4A - 7B = 0.$$

Thus, $A = -7$ and $B = -4$, and a particular solution to the differential equation is given by

$$y_p(x) = A\cos(2x) + B\sin(2x) = -7\cos(2x) - 4\sin(2x)$$

This example illustrates that, typically, if g(x) is a sine or cosine function (or a linear combination of a sine and cosine function with the same frequency) then a linear combination of both the sine and cosine can be used for $y_p(x)$. Thus, we have the following rule: If, for some constants A, B and α,

$$g(x) = Acos(\alpha x) + Bsin(\alpha x)$$

then a good first guess for a particular solution to differential equation (2.44) is

$$y_p(x) = Acos(\alpha x) + Bsin(\alpha x)$$

where A and B are constants to be determined.

Polynomials Let us find a particular solution to

$$y'' - 2y' - 3y = 9x^2 + 1.$$

Now consider, if y is any polynomial of degree N , then y, y' and y'' are also polynomials of degree N or less. So the expression "$y'' - 2y' - 3y$" would then be a polynomial of degree N . Since we want this to match the right side of the above differential equation, which is a polynomial of degree 2 , it seems reasonable to try a polynomial of degree N with $N = 2$. So we "guess"

$$y_p(x) = Ax^2 + Bx + C.$$

In this case

$$y_p'(x) = 2Ax + B, y_p''(x) = 2A$$

Plugging these into the differential equation $y_p'' - 2y_p' - 3y_p = 9x^2 + 1$, we get

$$-3Ax^2 + [-4A - 3B]x + [2A - 2B - 3C] = 9x^2 + 1$$

For the last equation to hold, the corresponding coefficients to the polynomials on the two sides must equal, giving us the following system:

$$x^2 \text{ terms} : -3A = 9$$
$$x \text{ terms} : -4A - 3B = 0.$$
$$\text{constant terms} : 2A - 2B - 3C = 1$$

So,$A = -3, B = 4$ And the particular solution is

$$y_p(x) = Ax^2 + Bx + C = -3x^2 + 4x - 5$$

Generalizing from this example, we can see that the rule for the first guess for $y_p(x)$ when g is a polynomial is:
If

$$g(x) = \text{a polynomial of degree K},$$

then a good first guess for a particular solution to differential equation (2.44) is a K^{th}-degree polynomial

$$y_p(x) = A_0x^K + A_1x^{K-1} + \ldots + A_{K-1}x + A_K$$

where the A_ks are constants to be determined.

Products of Exponentials, Polynomials, and Sines and Cosines If g is a product of the simple functions discussed above, then the guess for y_p must take into account everything discussed above. That leads to the following rule:

If, for some pair of polynomials $P(x)$ and $Q(x)$, and some pair of constants α and β,

$$g(x) = P(x)e^{\alpha x}\cos(\beta x) + Q(x)e^{\alpha x}\sin(\beta x)$$

then a good first guess for a particular solution to differential equation eqrefgu is

$$\begin{aligned} y_p(x) = {} & [A_0x^K + A_1x^{K-1} + \cdots + A_{K-1}x + A_K]e^{\alpha x}\cos(\beta x) \\ & + [B_0x^K + B_1x^{K-1} + \cdots + B_{K-1}x + B_K]e^{\alpha x}\sin(\beta x) \end{aligned}$$

where the A_ks and B_ks are constants to be determined and K is the highest power of x appearing in polynomial $P(x)$ or $Q(x)$.

Example 24. Find a particular solution to

$$y'' - 2y' - 3y = 65x\cos(2x),$$

Solution. we should start by assuming it is of the form

$$y_p(x) = [A_0x + A_1]cos(2x) + [B_0x + B_1]\sin(2x).$$

Putting the value of y_p in the given equation and simplifying we get:

$$\begin{aligned} & [-2A_0 - 7A_1 + 4B_0 - 4B_1]\cos(2x) + [-7A_0 - 4B_0]x\cos(2x) \\ & + [-4A_0 + 4A_1 - 2B_0 - 7B_1]\sin(2x) + [4A_0 - 7B_0]x\sin(2x) = 65x\cos(2x) \end{aligned}$$

Comparing the terms on either side of the last equation, we get the following system:

$$\begin{aligned} \cos(2x)\text{terms} : & -2A_0 - 7A_1 + 4B_0 - 4B_1 = 0 \\ x\cos(2x)\text{terms} : & -7A_0 - 4B_0 = 65 \\ \sin(2x)\text{terms} : & -4A_0 + 4A_1 - 2B_0 - 7B_1 = 0 \\ x\sin(2x)\text{terms} : & 4A_0 - 7B_0 = 0 \end{aligned}$$

Solving this system yields

$$A_0 = -7, A_1 = -158/65, B_0 = -4, B_1 = 244/65.$$

So a particular solution to the differential equation is given by

$$yp(x) = [-7x - 158/65]\cos(2x) + [-4x + 244/65]\sin(2x)$$

2.7.1 When the First Guess Fails

Consider

$$y'' - 2y' - 3y = 28e^{3x}.$$

Our first guess is

$$y_p(x) = Ae^{3x}.$$

Plugging it into the differential equation:$y_p'' - 2y_p' - 3y_p = 28e^{3x}$, we get

$$0 = 28e^{3x}$$

No value for A can make this equation true! So our first guess fails.
Why did it fail? Because the guess, Ae^{3x} was already a solution to the corresponding homogeneous equation

$$y'' - 2y' - 3y = 0.$$

If the first guess for $y_p(x)$ contains a term that is also a solution to the corresponding homogeneous differential equation, then consider

$$x \times \text{“the first guess''}$$

as a “second guess”. If this (after multiplying through by the x) does not contain a term satisfying the corresponding homogeneous differential equation, then set

$$y_p(x) = \text{“second guess”} = x \times \text{“the first guess”}.$$

If, however, the second guess also contains a term satisfying the corresponding homogeneous differential equation, then set

$$y_p(x) = \text{“the third guess”}$$

where

$$\text{“third guess''} = x \times \text{“the second guess”} = x^2 \times \text{“the first guess”}.$$

I should emphasize that the second guess is used only if the first fails (i.e., has a term that satisfies the homogeneous equation). If the first guess works, then the second (and third) guesses will not work. Likewise, if the second guess works, the third guess is not only unnecessary, it will not work. If, however the first and second guesses fail, you can be sure that the third guess.

2.8 Method of variation of parameters

The method of *variation of parameters* is a powerful method used to find a particular integral of a linear differential equation once its complementary function is known. Consider the general linear second order linear differential equation

$$y'' + ay' + by = f(x) \tag{2.46}$$

Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the homogeneous differential equation:

$$y'' + ay' + by = 0 \tag{2.47}$$

Then the complementary function is:

$$y = c_1 y_1(x) + c_2 y_2(x) \tag{2.48}$$

The idea underlying the method of variation of parameters is to replace the constants c_1 and c_2 by the unknown functions $u_1(x)$ and $u_2(x)$, and then find a particular integral of the form:

$$y = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{2.49}$$

Two equations are needed in order to determine $u_1(x)$ and $u_2(x)$, and the first of these is obtained as follows:
Differentiating equation (2.49), we get:

$$y'(x) = u_1(x)y_1'(x) + u_2(x)y_2'(x) + u_1'(x)y_1(x) + u_2'(x)y_2(x) \tag{2.50}$$

We have to find $u_1(x)$ and $u_2(x)$ such that the last two terms in the above equation vanish. That is

$$y'(x) = u_1(x)y_1'(x) + u_2(x)y_2'(x) \tag{2.51}$$

subject to the condition

$$u_1'(x)y_1(x) + u_2'(x)y_2(x) = 0 \tag{2.52}$$

Equation (2.51) is the *first* condition to be imposed on $u_1(x)$ and $u_2(x)$, and a second condition is obtained as follows:
Differentiating equation (2.51) gives:

$$y''(x) = u_1(x)y_1''(x) + u_2(x)y_2''(x) + u_1'(x)y_1'(x) + u_2'(x)y_2'(x), \tag{2.53}$$

Substituting (2.49), (2.51), and (2.53) into (2.47),followed by grouping gives:

$$u_1[y_1'' + ay_1' + by_1] + u_2[y_2'' + ay_2' + by_2] + \tag{2.54}$$

$$u_1'y_1' + u_2'y_2' = f(x) \tag{2.55}$$

Since $y_1(x)$ and $y_2(x)$ are solutions of the equation (2.47), the first two terms in the above equation vanish identically. Hence equation (2.55) reduces to the following form:

$$u_1'y_1' + u_2'y_2' = f(x) \tag{2.56}$$

So we get a *second* condition on $u_1(x)$ and $u_2(x)$. The functions $u_1(x)$ and $u_2(x)$ can now be found by solving equations (2.52) and (2.56).
Multiplying equation (2.52) by y_2', equation (2.56) by y_2, and subtracting gives:

$$\begin{aligned}
[y_1y_2' - y_1'y_2]u_1'(x) &= -f(x)y_2 \\
i.e., \quad W(y_1, y_2)u_1'(x) &= -f(x)y_2 \\
i.e., \quad u_1'(x) &= -\frac{f(x)y_2}{W(y_1, y_2)} \\
\therefore \quad u_1(x) &= -\int \frac{f(x)y_2}{W(y_1, y_2)}\, dx
\end{aligned} \tag{2.57}$$

Again multiplying equation (2.52) by y_1', equation (2.56) by y_1, and subtracting gives:

$$\begin{aligned}
[y_1y_2' - y_1'y_2]u_2'(x) &= f(x)y_1 \\
i.e., \quad W(y_1, y_2)u_2'(x) &= -f(x)y_1 \\
i.e., \quad u_2'(x) &= \frac{f(x)y_1}{W(y_1, y_2)} \\
\therefore \quad u_2(x) &= \int \frac{f(x)y_1}{W(y_1, y_2)}\, dx
\end{aligned} \tag{2.58}$$

Finally, substituting (2.57) and (2.58) into (2.49), we get:

$$y(x) = -y_1(x)\int \frac{f(x)y_2}{W(x)}\, dx + y_2(x)\int \frac{f(x)y_1}{W(x)}\, dx$$

2.8.1 Rule for the method of variation of parameters

1. Write the differential equation in the standard form

$$y'' + ay' + by = f(x)$$

2. Find two linearly independent solutions $y_1(x)$ and $y_2(x)$

3. Substitute y_1 and y_2 into

$$y_p(x) = -y_1(x)\int \frac{f(x)y_2}{W(x)}\,dx + y_2(x)\int \frac{f(x)y_1}{W(x)}\,dx$$

4. The general solution is

$$y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x)$$

Example 25. Find the general solution of the second order differential equation

$$y'' + 2y' + y = xe^{-x}$$

by the method of variation of parameters.

Solution. The characteristic equation is

$$\lambda^2 + 2\lambda + 1 = 0$$

The characteristic equation has repeated root $\lambda = -1$. Thus, the complementary function is

$$y_c(x) = c_1e^{-x} + c_2xe^{-x}$$

Two linearly independent solutions are thus

$$y_1(x) = e^{-x} \qquad \text{and} \qquad y_2(x) = xe^{-x}$$

The nonhomogeneous term is $f(x) = xe^{-x}$. The Wronskian

$$\begin{aligned} W(x) &= \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \\ &= \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & -xe^{-x} + e^{-x} \end{vmatrix} \\ &= e^{-x}(e^{-x} - xe^{-x}) + e^{-x}xe^{-x} = e^{-2x} \end{aligned}$$

Thus the particular integral y_p is given by

$$y_p(x) = -y_1(x)\int \frac{f(x)y_2}{W(x)}\,dx + y_2(x)\int \frac{f(x)y_1}{W(x)}\,dx$$

$$= -e^{-x}\int x^2dx + xe^{-x}\int xdx = \frac{1}{6}x^3e^{-x}.$$

Thus, the general solution is

$$y_c(x) = c_1e^{-x} + c_2xe^{-x} + \frac{1}{6}x^3e^{-x}$$

Example 26. Find the general solution of the differential equation

$$y'' + y = \csc x$$

Solution. The characteristic equation is

$$\lambda^2 + 1 = 0$$

The roots of the characteristic equation are $\lambda_1 = i$ and $\lambda_2 = -i$. Thus, the complementary function is

$$y_c(x) = c_1 \cos x + c_2 \sin x$$

Two linearly independent solutions are

$$y_1(x) = \cos x \qquad \text{and} \qquad y_2(x) = \sin x$$

The Wronskian $W(x) = y_1y_2' - y_1'y_2 = (\cos x)(\cos x) - (-\sin x)(\sin x) = 1$, and $f(x) = 1/\sin x$. Therefore

$$\begin{aligned} y_p(x) &= -y_1(x)\int \frac{f(x)y_2}{W(x)}\,dx + y_2(x)\int \frac{f(x)y_1}{W(x)}\,dx \\ &= . - \cos x\int dx + \sin x\int \cot x dx \\ &= -x\cos x + \sin x \ln|\sin x| \end{aligned}$$

Hence the general solution is

$$y_c(x) = c_1 \cos x + c_2 \sin x + -x\cos x + \sin x \ln|\sin x|$$

2.9 Mechanical and Electrical Vibrations

Imagine a horizontal spring with one end attached to an immobile wall and the other end attached to some object of interest which can slide along the floor, as in figure 2.1. For brevity, this entire assemblage of spring, object, wall, etc. will be called a mass/spring system. Let us assume that:

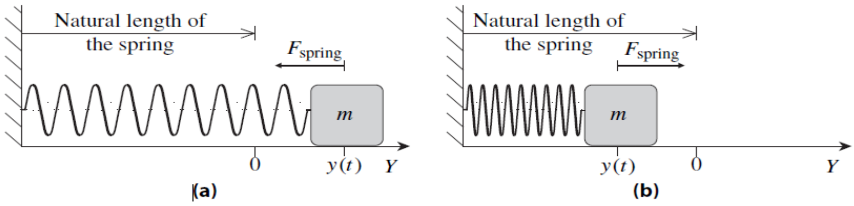


Figure 2.1: The mass/spring system with the direction of the spring force F_{spring} on the mass (a) when the spring is extended ($y(t) > 0$), and (b) when the spring is compressed ($y(t) < 0$).

1. The object can only move back and forth in the one horizontal direction.
2. Newtonian physics apply.
3. The total force acting on the object is the sum of:
   (a) The force from the spring responding to the spring being compressed and stretched.
   (b) The forces resisting motion because of air resistance and friction between the box and the floor.
   (c) Any other forces acting on the object.
4. The spring is an "ideal spring" with no mass. It has some natural length at which it is neither compressed nor stretched, and it can be both stretched and compressed. (So the coils are not so tightly wound that they are pressed against each other,making compression impossible.)

Our goal is to describe how the position of the object varies with time, and to see how this objects motion depends on the different parameters of our mass/spring system (the object's mass, the strength of the spring, the slipperiness of the floor, etc.). To set up the general formulas and equations, we'll first make the following traditional symbolic assignments:

m = the mass (in kilograms) of the object,

t = the time (in seconds) since the mass/spring system was set into motion, and

$y =$ the position (in meters) of the object when the spring is at its natural length.

This means our Yaxis is horizontal (nontraditional, maybe, but convenient for this application), and positioned so that $y = 0$ is the "equilibrium position" of the object. Let us also direct the Yaxis so that the spring is stretched when $y > 0$, and compressed when $y < 0$ (again, see figure 2.1).

Modeling the Forces

The motion of the object is governed by Newtons law $F = ma$ with F being the force acting on the box and

$$a = a(t) = \text{acceleration of the box at time} t = \frac{d^2y}{dt^2}$$

By our assumptions,

$$F = F_{\text{resist}} + F_{\text{spring}} + F_{\text{other}}$$

where

$F_{\text{resist}} =$ force due to the air resistance and friction,

$F_{\text{spring}} =$ force from the spring due to it being compressed or stretched, and

$F_{\text{other}} =$ any other forces acting on the object.

Thus

$$F_{\text{resist}} + F_{\text{spring}} + F_{\text{other}} = F = ma = m\frac{d^2y}{dt^2}$$

The above equation can be rewritten as:

$$m\frac{d^2y}{dt^2} - F_{\text{resist}} - F_{\text{spring}} = F_{\text{other}} \tag{2.59}$$

Observe that

$$F_{\text{resist}} = -\gamma \times \text{velocity of the box} = -\gamma\frac{dy}{dt}$$

where γ is some nonnegative constant. Because of the role it will play in determining how much the resistive forces "dampens" the motion, we call the damping constant. It will be large if the air resistance is substantial (possibly because the mass/spring system is submerged in water instead of air) or if the object does not slide easily on the floor. It will be small if there is little air resistance and the floor is very slippery. And it will be zero if there is no air resistance and no friction with the floor (a very idealized situation).

Now consider what we know about the spring force, F_{spring} . At any given time t , this force depends only on how much the spring is stretched or compressed at that time, and that, in turn, is completely described by $y(t)$. Hence, we can describe the spring force as a function of y , $F_{\text{spring}} = F_{\text{spring}}(y)$. Moreover:

1. If $y = 0$, then the spring is at its natural length, neither stretched nor compressed, and exerts no force on the box. So $F_{\text{spring}} = 0$.

2. If $y > 0$, then the spring is stretched and exerts a force on the box pulling it backwards. So $F_{\text{spring}}(y)$ ¡ 0 whenever $y > 0$.

3. Conversely, if $y < 0$, then the spring is compressed and exerts a force on the box pushing it forwards. So $F_{\text{spring}}(y) > 0$ whenever $y < 0$.

Knowing nothing more about the spring force, we might as well model it using the simplest mathematical formula satisfying the above:

$$F_{\text{spring}}(y) = -\kappa y \tag{2.60}$$

where κ is some positive constant.

Formula (2.60) is the famous Hooke's law for springs. Experiment has shown it to be a good model for the spring force, provided the spring is not stretched or compressed too much. The constant κ in this formula is called the spring constant. It describes the "stiffness" of the spring (i.e., how strongly it resists being stretched), and can be determined by compressing or stretching the spring by some amount y_0 , and then measuring the corresponding force F_0 at the end of the spring. Hookes law then says that

$$\kappa = -\frac{F_0}{y_0}$$

And because κ is a positive constant, we can simplify things a little bit more to

$$\kappa = \frac{|F_0|}{|y_0|}$$

2.10 The Mass/Spring Equation and its Solutions

Replacing $F_{\text{resist}} = \gamma \frac{dy}{dt}$ and $F_{\text{spring}}(y) = -\kappa y$ in equation (2.59), we get:

$$m\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \kappa y = F_{\text{other}} \tag{2.61}$$

This is the differential equation for $y(t)$, the position y of the object in the system at time t.

For the rest of this section, let us assume the object is moving "freely" under the influence of no forces except those from friction and from the spring's compression and expansion. Thus, for the rest of this section, we will restrict our interest to the above differential equation with $F_{\text{other}} = 0$,

$$m\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \kappa y = 0 \tag{2.62}$$

This is a second-order, homogeneous, linear differential equation with constant coefficients; so we can solve it by the methods discussed in the previous sections.

Keep in mind that the mass, m, and the spring constant, κ , are positive constants for a real spring. On the other hand, the damping constant,γ, can be positive or zero. This is significant. Because $\gamma = 0$ when there is no resistive force to dampen the motion, we say the mass/spring system is undamped when $\gamma = 0$. We will see that the motion of the mass in this case is relatively simple.

If, however, there is a nonzero resistive force to dampen the motion, then $\gamma > 0$. Accordingly, in this case, we say mass/spring system is damped. We will see that there are three subcases to consider, according to whether $\gamma^2 - 4\kappa m$ is negative, zero or positive. Lets now carefully examine, case by case, the solutions that can arise.

Undamped Systems

If $\gamma = 0$, differential equation (2.63) reduces to

$$m\frac{d^2y}{dt^2} + \kappa y = 0 \tag{2.63}$$

The corresponding characteristic equation,

$$mr^2 + \kappa = 0,$$

has roots

$$r_{1,2} = \pm\frac{\sqrt{-\kappa m}}{m} = \pm\omega_0$$

where $\omega_0 = \sqrt{\kappa/m}$. We know the general solution to our differential equation is given by

$$y(t) = c_1 cos(\omega_0 t) + c_2 sin(\omega_0 t) \tag{2.64}$$

where c_1 and c_2 are arbitrary constants. However, for graphing purposes (and a few other purposes) it is convenient to write our general solution in yet another form.

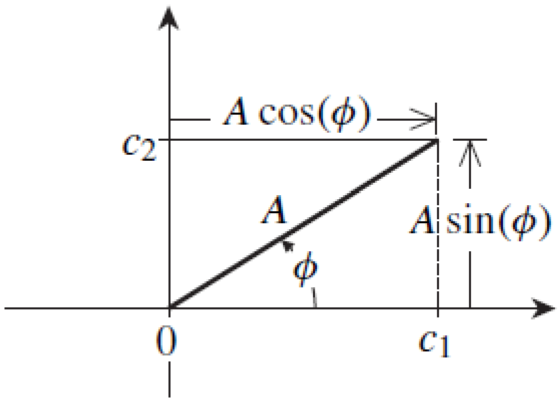


Figure 2.2: Expressing c_1 and c_2 as $A\cos(\phi)$ and $A\sin(\phi)$

To derive this form, plot (c_1, c_2) as a point on a Cartesian coordinate system, and let A and ϕ be the corresponding polar coordinates of this point (see figure 2.2). That is, let

$$A = \sqrt{c_1^2 + c_2^2}$$

and let ϕ be the angle in the range $[0, 2\pi)$ with

$$c_1 = A\cos(\phi) \quad \text{and} \quad c_2 = A\sin(\phi)$$

Using this and the well-known trigonometric identity

$$cos(\theta \pm \phi) = cos(\theta)cos(\phi) \mp sin(\theta)sin(\phi)$$

we get

$$\begin{aligned} c_1 cos(\omega_0 t) + c_2 sin(\omega_0 t) &= A\cos(\phi)\cos(\omega_0 t)] + A\sin(\phi)\sin(\omega_0 t) \\ &= A[\cos(\phi)\cos(\omega_0 t)] + \sin(\phi)\sin(\omega_0 t)] \\ &= A\cos(\omega_0 t - \phi.) \end{aligned}$$

Thus, our general solution is given by either

$$y(t) = c_1 cos(\omega_0 t) + c_2 sin(\omega_0 t) \tag{2.65}$$

or, equivalently,

$$y(t) == A\cos(\omega_0 t - \phi.) \tag{2.66}$$

where

$$\omega_0 = \sqrt{\frac{\kappa}{m}}$$

and other constants are related by

$$A = \sqrt{c_1^2 + c_2^2}, \cos(\phi) = c_1/A, \sin(\phi) = c_2/A$$

Damped Systems

If $\gamma > 0$, then all coefficients in our differential equation

$$m\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \kappa y = 0 \tag{2.67}$$

are positive. The corresponding characteristic equation is

$$mr^2 + r + \kappa = 0$$

and its solutions are given by:

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\kappa m}}{2m} \tag{2.68}$$

As we saw in the last section, the nature of the differential equations solution, $y = y(t)$, depends on whether $\gamma^2 - 4\kappa m$ is positive, negative or zero. And this, in turn, depends on the positive constants γ, κ and mass m as follows:

$$\gamma < 2\sqrt{\kappa m} \Leftrightarrow \gamma^2 - 4\kappa m < 0$$
$$\gamma = 2\sqrt{\kappa m} \Leftrightarrow \gamma^2 - 4\kappa m = 0$$
$$\gamma > 2\sqrt{\kappa m} \Leftrightarrow \gamma^2 - 4\kappa m > 0$$

We say that a mass/spring system is, respectively, underdamped , critically damped or overdamped if and only if

$$0 < \gamma < 2\sqrt{\kappa m}, \gamma = 2\sqrt{\kappa m}, \quad \text{and} \gamma > 2\sqrt{\kappa m}$$

Since we've already considered the case where $\gamma = 0$, the first damped cases considered will be the underdamped mass/spring systems (where $0 < \gamma < 2\sqrt{\kappa m}$).

Underdamped Systems ($0 < \gamma < 2\sqrt{\kappa m}$)

In this case,

$$\sqrt{\gamma^2 - 4\kappa m} = \sqrt{-|\gamma^2 - 4\kappa m|} = i\sqrt{|\gamma^2 - 4\kappa m|} = i\sqrt{|4\kappa m - \gamma^2|}$$

and formula (2.68) for the $r_{1,2}$ can be written as:

$$r_{1,2} = -\alpha \pm i\omega$$

where $\alpha = \gamma/2m$ and $\omega = \frac{\sqrt{4\kappa m-\gamma^2}}{2m}$.

Note that α and ω are positive real values. Hence the general solution to our differential equation is

$$y(t) = c_1 e^{-\alpha t}\cos(\omega t) + c_2 e^{-\alpha t}\sin(\omega t).$$

Factoring out the exponential and applying the same analysis to the linear combination of sines and cosines as was done for the undamped case, we get that the position y of the box at time t is given by any of the following:

$$y(t) = e^{-\alpha t}[c_1 cos(\omega t) + c_2 sin(\omega t)],$$
$$y(t) = Ae^{-\alpha t}\cos(\omega t - \phi)$$

These two formulas are equivalent, and the arbitrary constants are related, as before, by

$$A = \sqrt{(c_1)^2 + (c_2)^2}, \cos(\phi) = c_1/A, \quad \text{and} \quad \sin(\phi) = c_2/A$$

Critically damped Systems ($\gamma = 2p\kappa m$)

In this case,

$$\sqrt{\gamma^2 - 4\kappa m} = 0$$

and

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\kappa m}}{2m} = \frac{-\gamma \pm \sqrt{0}}{2m} = -\sqrt{\kappa/m}$$

So the corresponding general solution to our differential equation is

$$y(t) = c_1 e^{-\alpha t} + c_2 t e^{-\alpha t}$$

where $\alpha = \sqrt{\kappa/m}$. Factoring out the exponential yields

$$y(t) = e^{-\alpha t}[c_1 + c_2 t]$$

Overdamped Systems ($2\sqrt{\kappa m} < \gamma$)

In this case, it is first worth observing that

$$\gamma > \sqrt{\gamma^2 - 4\kappa m} > 0$$

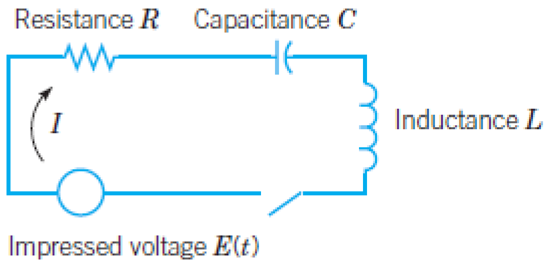


Figure 2.3: A simple electric circuit.

Consequently, the formula for $r_{1,2}$ is given by:

$$r_{1,2} = \frac{-\gamma + \sqrt{\gamma^2 - 4\kappa m}}{2m}$$

Taking $\alpha = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\kappa m}}{2m}$ and $\beta = \frac{-\gamma - \sqrt{\gamma^2 - 4\kappa m}}{2m}$. Then α and β are positive values. Hence, the corresponding general solution of the differential equation is

$$y(t) = c_1 e^{-\alpha t} + c_2 e^{-\beta t}$$

Electric Circuits

A second example of the occurrence of second order linear differential equations with constant coefficients is their use as a model of the flow of electric current in the simple series circuit shown in Figure2.3. The current I, measured in amperes (A), is a function of time t. The resistance R in ohms $(|Omega)$, the capacitance C in farads (F), and the inductance L in henrys (H) are all positive and are assumed to be known constants. The impressed voltage E in volts (V) is a given function of time. Another physical quantity that enters the discussion is the total charge Q in coulombs (C) on the capacitor at time t. The relation between charge Q and current I is

$$I = dQ/dt. \tag{2.69}$$

The flow of current in the circuit is governed by Kirchhoffs second law:*In a closed circuit the impressed voltage is equal to the sum of the voltage drops in the rest of the circuit.*

According to the elementary laws of electricity, we know that The voltage drop across the resistor is IR.

The voltage drop across the capacitor is Q/C.

The voltage drop across the inductor is LdI/dt.
Hence, by Kirchhoffs law,

$$LdI/dt + RI + (1/C)Q = E(t). \tag{2.70}$$

The units have been chosen so that 1 volt = 1 ohm 1 ampere = 1 coulomb/1 farad= 1 henry 1 ampere/1 second. Substituting for I from equation (2.69), we obtain the differential equation

$$LQ'' + RQ' + (1/C)Q = E(t)$$

for the charge Q. The initial conditions are

$$Q(t_0) = Q_0, Q'(t_0) = I(t_0) = I_0.$$

CHAPTER 3

Laplace Transforms

3.1 Introduction

We have already come across instances where a mathematical transformation has been used to simplify the solution of a problem. For example, the logarithm is used to simplify multiplication and division problems. To multiply or divide two numbers, we transform them into their logarithms, add or subtract these and then perform the inverse transformation(that is antilogarithm) to obtain the product or quotient of original numbers. The purpose of using a transformation is to create a new domain in which it is easier to handle the problem being investigated. Once results have been obtained in the new domain, they can be inverse-transformed to give the desired results in the original domain.

The Laplace transform is an example of a class called integral transforms and it changes a real variable function $f(t)$ into a function $F(s)$ of a variable s through

$$F(s) = \int_0^\infty e^{-st} f(t)dt$$

where s is a complex variable.

3.2 Definitions and basic theory

Definition. The improper integral $\int_a^\infty f(t)\ dt$ is defined as:

$$\int_a^\infty f(t)\ dt = \lim_{x\to\infty} \int_a^x f(t)\ dt$$

Similarly

$$\int_{-\infty}^a f(t)\ dt = \lim_{x\to -\infty} \int_x^a f(t)\ dt$$

Definition. Let $f\colon \mathbb{R} \to \mathbb{R}$ and let $a \in \mathbb{R}$. If $\int_{-\infty}^a f(t)dt$ and $\int_a^\infty f(t)dt$ exit then the improper integral $\int_{-\infty}^\infty f(t)dt$ is defined as:

$$\int_{-\infty}^\infty f(t)dt = \int_{-\infty}^a f(t)dt + \int_a^\infty f(t)dt$$

Note:

$$\int_{-\infty}^\infty f(t)dt \neq \lim_{x\to\infty} \int_{-x}^x f(t)dt$$

Definition. An improper integral of the form

$$\boldsymbol{I\{f(t)\} = \int_{-\infty}^{\infty} K(s,t)f(t)\ dt} \tag{3.1}$$

is called *integral transform* of $f(t)$ if it is convergent. The function $K(s,t)$ is called the *kernel* of the transform and s is a complex number, called the *parameter* of the transform.

Remarks. If we define

$$K(s,t) = \begin{cases} e^{-st}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

then (3.1) becomes

$$I\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

This transform is called the *Laplace transform* of $f(t)$.

When

$$K(s,t) = \begin{cases} \sqrt{\frac{2}{\pi}}\sin st, & t \geq 0 \\ 0, & t < 0 \end{cases},$$

equation (3.1) becomes

$$\boldsymbol{I\{f(t)\} = \mathscr{F}_s\{f(t)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin st\ dt}$$

This transform is called the *Fourier sine transform.*

Similarly, when

$$K(s,t) = \begin{cases} \sqrt{\frac{2}{\pi}} \cos st, & t \geq 0 \\ 0, & t < 0 \end{cases},$$

we get the *Fourier cosine transform*:

$$\boldsymbol{I\{f(t)\} = \mathscr{F}_c\{f(t)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos st \; dt}$$

Definition. If a function $f(t)$ is defined for all t in the interval $[0, \infty)$, then the *Laplace Transform* of $f(t)$ is defined as

$$\boldsymbol{\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty f(t) e^{-st} dt}$$

3.2.1 Common notations used for the Laplace transform

The following are various commonly used notations for the Laplace transform of $f(t)$.

(i) $\mathscr{L}\{f(t)\}$ or $L\{f(t)\}$

(ii) $\mathscr{L}(f(t))$ or Lf

(iii) $\bar{f}(s)$ or $\tilde{f}(s)$ or $\tilde{f}$

Also, the letter p is sometimes used instead of s as the parameter. In this book the original function is denoted by $f(t)$ and its Laplace transform is denoted by $\mathscr{L}\{f(t)\}$

Notes:

(b) The symbol $\mathscr{L}$ denote the Laplace transform operator, when it operates on a function $f(t)$, it transforms into a function $F(s)$ of the variable s. We say the operator transform the function $f(t)$ in the t domain(usually called time domain) into the function $F(s)$ in the s domain(usually called frequency domain). This relationship is shown in figure 3.1

(c) Laplace transforms does not exist for all functions. For example , consider the function $f(x) = \dfrac{1}{t}$. Then

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st}(1/t) ds$$

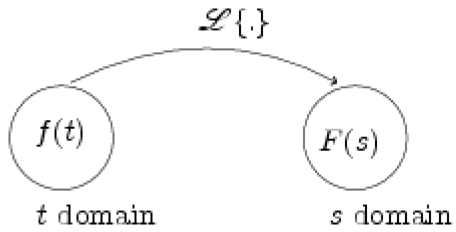


Figure 3.1: The Laplace transform operator

This integral does not exists. Hence the Laplace transform of $f(t) = \dfrac{1}{t}$ does not exists.

(d) The ordered pair $(f(t), F(s))$ is called a Laplace transform pair.

3.3 Existence of Laplace transform

Throughout this chapter we assume that s is a real positive variable.

Definition. A function $f(t)$ is said to be **exponential** order as $t \to \infty$ if there exits a constant α such that $\lim_{t\to\infty} e^{-\alpha t} f(t)$ is finite. That is, there exits a real number α and positive constants M and T such that

$$|f(t)| < Me^{\alpha t} \qquad \text{for all} \quad t > T$$

Example 27. Prove that the function $f(t) = t^n$ is of exponential order as $t \to \infty$, n being a positive integer.

Solution.

$$\begin{aligned}
\lim_{t\to\infty} e^{-\alpha t} f(t) &= \lim_{t\to\infty} e^{-\alpha t} t^n \\
&= \lim_{t\to\infty} \frac{t^n}{e^{\alpha t}} \qquad \left(\frac{\infty}{\infty} \quad \text{form}\right) \\
&= \lim_{t\to\infty} \frac{n!}{\alpha^n e^{\alpha t}} \quad \text{(by L Hospital rule)} \\
&= 0
\end{aligned}$$

Hence t^n is of exponential order as $t \to \infty$.

Example 28. Prove that the function $f(t) = \dfrac{1}{t}$ is of exponential order as $t \to \infty$.

Solution.

$$\begin{aligned}\lim_{t\to\infty} e^{-\alpha t} f(t) &= \lim_{t\to\infty} e^{-\alpha t}\left(\frac{1}{t}\right)\\ &= \lim_{t\to\infty} \frac{1}{e^{\alpha t} t}\\ &= 0\end{aligned}$$

Hence $f(t) = \dfrac{1}{t}$ is of exponential order as $t \to \infty$.

Example 29. Prove that the function $f(t) = e^{t^2}$ is not of exponential order as $t \to \infty$.

Solution.

$$\begin{aligned}\lim_{t\to\infty} e^{-\alpha t} f(t) &= \lim_{t\to\infty} e^{-\alpha t} e^{t^2}\\ &= \lim_{t\to\infty} e^{(t^2-\alpha t)} = \lim_{t\to\infty} e^{(t-\alpha/2)^2} e^{\alpha^2/4}\\ &= \infty\end{aligned}$$

Hence $f(t) = e^{t^2}$ is not of exponential order as $t \to \infty$.

Example 30. Prove that $f(t) = e^{3t}$ is of exponential order, with $\alpha \geq 3$.

Solution.

$$\begin{aligned}\lim_{t\to\infty} e^{-\alpha t} f(t) &= \lim_{t\to\infty} e^{-\alpha t} e^{3t}\\ &= \lim_{t\to\infty} e^{(3t-\alpha t)} = \lim_{t\to\infty} e^{t(3-\alpha)}\\ &= 0 \quad (\because \alpha \geq 3)\end{aligned}$$

Hence $f(t) = e^{3t}$ is of exponential order when $\alpha \geq 3$.

Definition. A function $f(t)$ is said to be piecewise continuous on an interval $[a, b]$ if it has only a finite number of discontinuities with in $[a, b]$ and elsewhere the function is continuous and bounded. For example the following function is piecewise continuous in the interval $[0, 3]$.

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 2, & 1 \leq t < 2 \\ 3, & 2 \leq t < 3 \end{cases}$$

The graph of $f(t)$ is shown in figure 3.2

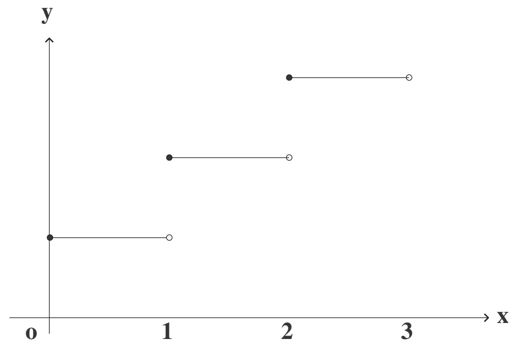


Figure 3.2: A piecewise continuous function

Lemma 5. Let $f(t)$ be piecewise continuous. Then, the improper integral $\int_0^\infty f(t)dt$ exits if $\int_0^\infty |f(t)|dt$ exists.

Theorem 13. If $f(t)$ is piecewise continuous and of exponential order, then its Laplace transform exists for all s sufficiently large. That is, if $f(t)$ is piecewise continuous, and $|f(t)| \leq Me^{-\alpha t}$, then $F(s)$ exists for $s > \alpha$.

Proof. Since $f(t)$ is piecewise continuous, the integral $\int_0^a f(t)dt$ exits for all a. Now

$$\begin{aligned}
\int_0^a |e^{-st} f(t)|dt &\leq M \int_0^a e^{-st} e^{\alpha t} dt \\
&= \int_0^a e^{(\alpha - s)t} dt = M \left[\frac{e^{(\alpha - s)t}}{\alpha - s} \right]_0^a \\
&= \frac{M}{\alpha - s} \left[e^{(\alpha - s)a} - 1 \right] \\
\therefore \quad \lim_{a \to \infty} \int_0^a |e^{-st} f(t)|dt &\leq \lim_{a \to \infty} \frac{M}{\alpha - s} \left[e^{(\alpha - s)a} - 1 \right] = \frac{M}{s - \alpha} \\
\textit{i.e.,} \quad \int_0^\infty |e^{-st} f(t)|dt &\leq \frac{M}{s - \alpha}
\end{aligned}$$

Therefore

$$\int_0^\infty |e^{-st} f(t)|dt$$

exists. Hence by lemma $\int_0^\infty e^{-st} f(t)dt$ exists. □

Remark The above conditions are sufficient conditions for the existence of Laplace transform, but not necessary conditions. That is, Laplace transforms can be found for functions that does not satisfies the conditions of the above theorem.

Theorem 14. If $f(t)$ is piecewise continuous and of exponential order, then

$$\lim_{s\to\infty} \mathscr{L}\{f(t)\} = 0.$$

Proof. We have

$$\int_0^\infty e^{-st} f(t)dt \leq \int_0^\infty |e^{-st} f(t)|dt \leq \frac{M}{s-\alpha}$$

Taking $\lim_{s\to\infty}$, we get:

$$\lim_{s\to\infty} \int_0^\infty e^{-st} f(t)dt \leq \lim_{s\to\infty} \int_0^\infty |e^{-st} f(t)|dt \leq \lim_{s\to\infty} \frac{M}{s-\alpha} = 0.$$

Hence $\lim_{s\to\infty} \int_0^\infty e^{-st} f(t)\ dt = 0$. That is, $\lim_{s\to\infty} \mathscr{L}\{f(t)\} = 0$. □

Example 31. Prove that $F(s) = (s^2-1)/(s^2-1)$ is not a Laplace transform of an ordinary function.

Solution.

$$\begin{aligned}\lim_{s\to\infty} F(s) &= \lim_{s\to\infty} \left(\frac{s^2-1}{s^2+1}\right) \\ &= \lim_{s\to\infty} \left(\frac{1-1/s^2}{1+1/s^2}\right) = 1\end{aligned}$$

Since $\lim_{s\to\infty} F(s) \neq 0$, therefore $F(s)$ is not the Laplace transform of any ordinary function.

3.3.1 Properties of the Laplace transform

In this section we consider some properties of Laplace transform that will enable us to find further transform pairs $\{f(t), F(s)\}$ without having to compute them directly using the definition.

Theorem 15 (Linearity Property)**.** If the Laplace transforms of $f(t)$ and $g(t)$ exist, then for all values of the constants c_1 and c_2,

$$\boldsymbol{\mathscr{L}\{c_1 f(t) + c_2 g(t)\} = c_1 \mathscr{L}\{f(t)\} + c_2 \mathscr{L}\{g(t)\}}$$

Proof. By definition,

$$\mathscr{L}\{c_1 f(t) + c_2 g(t)\} = \int_0^\infty \{c_1 f(t) + c_2 g(t)\} e^{-st}\ dt$$

$$
\begin{aligned}
&= \int_0^\infty c_1 f(t)e^{-st}\ dt + c_2 g(t)e^{-st}\ dt \\
&= c_1 \int_0^\infty f(t)e^{-st}\ dt + c_2 \int_0^\infty g(t)e^{-st}\ dt \\
&= c_1 \mathscr{L}\{f(t)\} + c_2 \mathscr{L}\{g(t)\}
\end{aligned}
$$

□

This property may be extended to a linear combination of any finite number of functions.

3.4 The unit step function

Definition. The *Unit Step function* (*or Heaviside's unit function*) is defined by

$$
u(t-a) = \begin{cases} 1, & t > a \\ 0 & t < a \end{cases}
$$

The graph of the unit step function is shown in figure 3.3

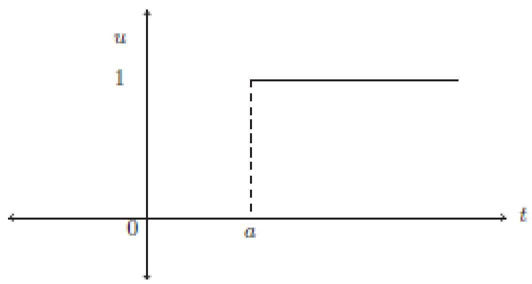


Figure 3.3: The unit step function

Theorem 16. The Laplace transform of the unit step function $u(t-a)$ is $\dfrac{e^{-as}}{s}$

Proof. By definition,

$$
\begin{aligned}
\mathscr{L}\{u(t-a)\} &= \int_0^\infty e^{-st} u(t-a)\ dt \\
&= \int_0^a e^{-st} u((t-a)\ dt + \int_a^\infty e^{-st} u(t-a)\ dt \\
&= \int_0^a e^{-st}\ (0)\ \ dt + \int_a^\infty e^{-st}\ (1)\ \ dt \\
&= \int_a^\infty e^{-st}\ dt
\end{aligned}
$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^{t\to\infty} = \frac{e^{-as}}{s}$$

□

Remarks :

(i) The unit step function is discontinuous at $t = a$ and yet has a continuous Laplace transform, namely $\dfrac{e^{-as}}{s}$

(ii) If a function $f(t)$ is defined on the interval $(-\infty, \infty)$, then

$$f(t)u(t) = \begin{cases} 0, & t < 0 \\ f(t) & t > 0 \end{cases}$$

3.5 The unit impulse function

Definition. The unit impulse function is defined as

$$p(t) = u(t-a) - u(t-b), \qquad \text{with} \quad b > a \geq 0$$

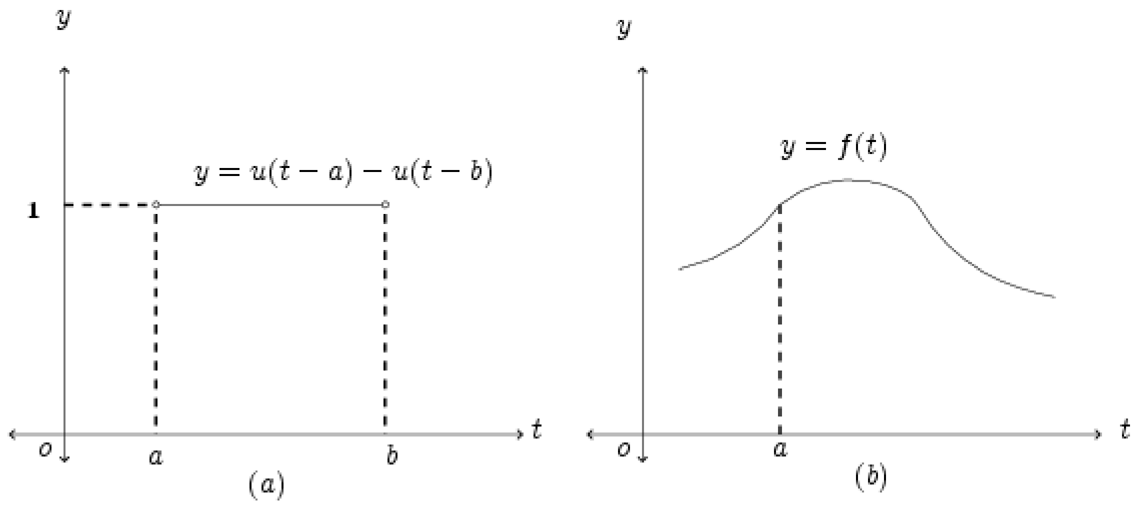


Figure 3.4: (a) The unit impulse function $p(t) = u(t-a) - u(t-b)$. (b) The function $y = f(t)$.

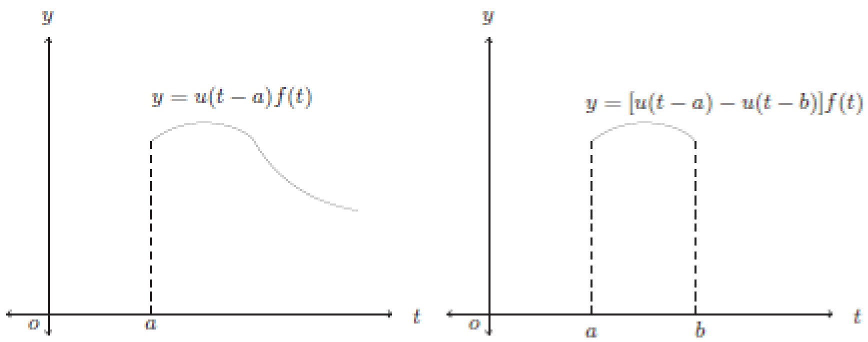


Figure 3.5: The effect on $f(t)$ of multiplication by $u(t-a)$ and $u(t-a)-u(t-b)$.

Theorem 17. The laplace transform of the unit impulse function $p(t)$ is $\dfrac{e^{-as}-e^{-bs}}{s}$.

Proof.

$$\begin{aligned}
\mathscr{L}L\{p(t)\} &= \int_0^\infty e^{-st}p(t) \quad dt \\
&= \int_0^a e^{-st}p(t)dt + \int_a^b e^{-st}p(t)dt + \int_b^\infty e^{-st}p(t)dt \\
&= \int_0^a e^{-st}(0)dt + \int_a^b e^{-st}(1)dt + \int_b^\infty e^{-st}(0)dt \\
&= \int_a^b e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_a^b \\
&= \frac{e^{-as}-e^{-bs}}{s}
\end{aligned}$$

□

3.6 Laplace transforms of the elementary functions

Example 32. What is $\mathscr{L}\{1\}$?

Solution. By definition,

$$\begin{aligned}
\mathscr{L}\{1\} &= \int_0^\infty e^{-st}\ 1\ dt \\
&= -\left[\frac{e^{-st}}{s}\right]_0^{t\to\infty} = \frac{1}{s}
\end{aligned}$$

Example 33. What is $\mathscr{L}\{t^n\}$?

Solution. By definition,

$$\begin{aligned}
\mathscr{L}\{t^n\} &= \int_0^\infty e^{-st}\ t^n\ dt \\
&= \int_0^\infty e^{-st}\ t^{(n+1)-1}\ dt \\
&= \frac{\Gamma(n+1)}{s^{n+1}} \qquad \left(\because \int_0^\infty e^{-kx} x^{n-1} = \frac{\Gamma n}{k^n}\right)
\end{aligned}$$

Remark. If n is a positive integer, $\Gamma(n+1) = n!$. Therefore

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Example 34. What is $\mathscr{L}\{e^{at}\}$?

Solution.

$$\begin{aligned}
\mathscr{L}\{e^{at}\} &= \int_0^\infty e^{-st} e^{at} dt \\
&= \int_0^\infty e^{-(s-a)t} dt = \left[-\frac{e^{-(s-a)t}}{s-a}\right]_0^{t\to\infty} \\
&= \frac{1}{s-a}, \quad \text{provided } (s-a) > 0
\end{aligned}$$

Remark. $\mathscr{L}\{e^{-at}\} = \dfrac{1}{s+a}$provided $(s+a) > 0$

Example 35. What are $\mathscr{L}\{\cos at\}$ and $\mathscr{L}\{\sin at\}$?

Solution. We have $\mathscr{L}\{e^{at}\} = \dfrac{1}{s-a}$. Therefore

$$\begin{aligned}
\mathscr{L}\{e^{iat}\} &= \frac{1}{s-ia} = \frac{s+ia}{(s-ia)(s+ia)} \\
&= \frac{s+ia}{s^2+a^2} \\
&= \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}
\end{aligned}$$

$$\begin{aligned}
\therefore \quad \mathscr{L}\{\cos at + i\sin at\} &= \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2} \quad (\because e^{iat} = \cos at + i\sin at) \\
\textit{i.e.,} \quad \mathscr{L}\{\cos at\} + i\mathscr{L}\{\sin at\} &= \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}
\end{aligned}$$

Equating real and imaginary parts, we get

$$\mathscr{L}\{\cos at\} = \frac{s}{s^2+a^2} \quad \text{and} \quad \mathscr{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

Example 36. What is $\mathscr{L}\{\cosh at\}$?

Solution.

$$\mathscr{L}\{\cosh at\} = \mathscr{L}\left\{\frac{e^{at}+e^{-at}}{2}\right\} = \frac{1}{2}\left[\mathscr{L}\{e^{at}\}+\mathscr{L}\{e^{-at}\}\right]$$
$$= \frac{1}{2}\left[\frac{1}{s-a}+\frac{1}{s+a}\right] = \frac{s}{s^2-a^2}$$

Example 37. What is $\mathscr{L}\{\sinh at\}$?

Solution.

$$\mathscr{L}\{\sinh at\} = \mathscr{L}\left\{\frac{e^{at}-e^{-at}}{2}\right\} = \frac{1}{2}\left[\mathscr{L}\{e^{at}\}-\mathscr{L}\{e^{-at}\}\right]$$
$$= \frac{1}{2}\left[\frac{1}{s-a}-\frac{1}{s+a}\right] = \frac{a}{s^2-a^2}$$

3.7 Shifting theorems

Theorem 18 (First Shifting Theorem)**.** If $\mathscr{L}\{f(t)\} = F(s)$, then

$$\boldsymbol{\mathscr{L}\{e^{at}f(t)\} = F(s-a)}$$

Proof. By the definition of Laplace transform, we have

$$\mathscr{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)\,dt$$
$$= \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a)$$

□

Note The graphs of $\mathscr{L}\{\sin t\}$ and $\mathscr{L}\{e^{2t}\sin t\}$ are shown in figure 3.6

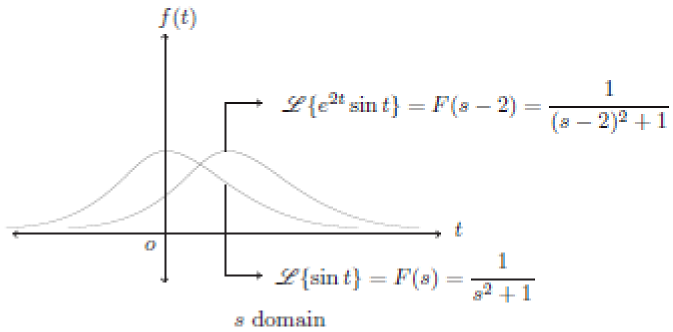


Figure 3.6: Graphs of $F(s)$ and $F(s-2)$

Remark. If $\mathscr{L}\{f(t)\} = F(s)$, then $\boldsymbol{\mathscr{L}\{e^{-at}f(t)\} = F(s+a)}$

Theorem 19 (Second Shifting Theorem)**.** If $\mathscr{L}\{f(t)\} = F(s)$, then

$$\boldsymbol{\mathscr{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)}$$

Proof. By definition,

$$\begin{aligned}
\mathscr{L}\{f(t-a)u(t-a)\} &= \int_0^\infty e^{-st} f(t-a)u(t-a)\ dt \\
&= \int_0^a e^{-st} f(t-a)u(t-a)\ dt + \int_a^\infty e^{-st} f(t-a)u(t-a)\ dt \\
&= \int_0^a e^{-st} f(t-a)\ (0)\ dt + \int_a^\infty e^{-st} f(t-a)(1)\ dt \\
&= 0 + \int_a^\infty e^{-st} f(t-a)\ dt \\
&= \int_a^\infty e^{-st} f(t-a)\ dt \qquad (3.2)
\end{aligned}$$

Letting $x = t - a$ in equation (3.2). Then $dt = dx$. Also when $t = a, x = t - a = a - a = 0$ and when $t = \infty, x = t - a = \infty - a = \infty$. Therefore equation (3.2) becomes:

$$\begin{aligned}
\mathscr{L}\{f(t-a)u(t-a)\} &= \int_a^\infty e^{-st} f(t-a)\ dt \\
&= \int_0^\infty e^{-s(x+a)} f(x)\ dx \\
&= e^{-sa} \int_0^\infty e^{-sx} f(x) dx = e^{-as}F(s).
\end{aligned}$$

□

Remark. The second shifting theorem can also be stated as:

$$\int_a^\infty e^{-st} f(t-a) dt = e^{-as}F(s)$$

Theorem 20 (Scaling theorem)**.** If $\mathscr{L}\{f(t)\} = F(s)$, then

$$\boldsymbol{\mathscr{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)}, \quad a > 0$$

Proof. By definition,

$$\begin{aligned}
\mathscr{L}\{f(at)\} &= \int_0^\infty e^{-st} f(at)\ dt \\
&= \int_0^\infty e^{-s(x/a)} f(x) \frac{dx}{a} \quad \text{(setting } at = x)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{a}\int_0^\infty e^{-(s/a)x} f(x)\ dx \\
&= \frac{1}{a}\int_0^\infty e^{-kx} f(x)\ dx \quad (k = s/a) \\
&= \frac{1}{a}\int_0^\infty e^{-kt} f(t)\ dt \quad \left(\because \int_a^b f(x)dx = \int_a^b f(t)dt\right) \\
&= \frac{1}{a}F(k) = \frac{1}{a}F\left(\frac{s}{a}\right) \quad (\because k = s/a)
\end{aligned}
$$

□

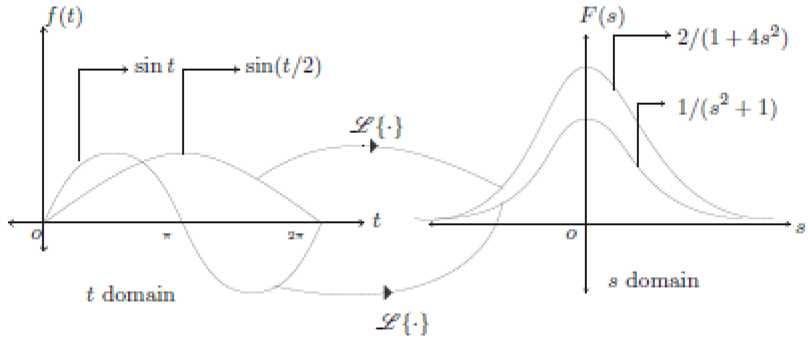


Figure 3.7: Laplace transforms of $\sin t$ and $\sin(t/2)$

3.7.1 Laplace transforms of the form $e^{at}f(t)$

Example 38. What is $\mathscr{L}\{e^{at}t^n\}$?

Solution. We have $\mathscr{L}\{t^n\} = \dfrac{\Gamma(n+1)}{s^{n+1}}$

$$\therefore \quad \mathscr{L}\{e^{at}t^n\} = \frac{\Gamma(n+1)}{(s-a)^{n+1}} \quad \text{(by first shifting theorem)}$$

Remark. $\mathscr{L}\{e^{-at}t^n\} = \dfrac{\Gamma(n+1)}{(s+a)^{n+1}}$

Example 39. What is $\mathscr{L}\{e^{at}\sin bt\}$?

Solution. We have $\mathscr{L}\{\sin bt\} = \dfrac{b}{s^2-b^2}$. Therefore

$$\mathscr{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2-b^2}\text{(by first shifting theorem)}$$

Remark. $\mathscr{L}\{e^{-at}\sin bt\} = \dfrac{b}{(s+a)^2-b^2}$

Example 40. What is $\mathscr{L}\{e^{at}\cos bt\}$?

Solution. We have $\mathscr{L}\{\cos bt\} = \dfrac{s}{s^2+b^2}$.

$$\therefore \quad \mathscr{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2} \quad \text{(by first shifting theorem)}$$

Remark. $\mathscr{L}\{e^{-at}\cos bt\} = \dfrac{s+a}{(s+a)^2+b^2}$

Example 41. What is $\mathscr{L}\{e^{at}\cosh bt\}$?

Solution. We have

$$\mathscr{L}\{\cosh bt\} = \frac{s}{s^2-b^2}$$
$$\therefore \quad \mathscr{L}\{e^{at}\cosh at\} = \frac{s-a}{(s-a)^2-b^2}$$

Remark. $\mathscr{L}\{e^{-at}\cosh bt\} = \dfrac{s+a}{(s+a)^2-b^2}$

Example 42. What is $\mathscr{L}\{e^{at}\sinh bt\}$?

Solution. We have

$$\mathscr{L}\{\sinh bt\} = \frac{b}{s^2-b^2}$$
$$\therefore \quad \mathscr{L}\{e^{at}\sinh at\} = \frac{b}{(s-a)^2-b^2}$$

Remark. $\mathscr{L}\{e^{-at}\sinh bt\} = \dfrac{b}{(s+a)^2-b^2}$

$f(t)$	$\mathscr{L}\{f(t)\}$
$e^{at}t^n$	$\Gamma(n+1)/s^{n+1}$
$e^{at}\sin bt$	$b/[(s-a)^2+b^2]$
$e^{at}\cos bt$	$(s-a)/[(s-a)^2-b^2]$
$e^{at} cosh bt$	$(s-a)/[(s-a)^2-b^2]$
$\mathscr{L}\{e^{-at}\sinh bt\}$	$b/[(s+a)^2-b^2]$

Table 2 Laplace transform pairs

3.8 Laplace transforms of the derivatives

Theorem 21. Suppose that $f(t)$ and $f'(t)$ have Laplace transforms. Then

$$\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0)$$

Proof. By definition,

$$\begin{aligned} \mathscr{L}\{f'(t)\} &= \int_0^\infty f'(t)e^{-st}dt \\ &= \left[e^{-st}f(t)\right]_0^\infty - \int_0^\infty e^{-st}(-s)f(t)\ dt \\ &= -f(0) + s\mathscr{L}\{f(t)\} \quad \text{(assuming that } \lim_{t\to\infty} e^{-st}f(t) = 0) \\ \therefore \quad \mathscr{L}\{f'(t)\} &= s\mathscr{L}\{f(t)\} - f(0) \end{aligned}$$

□

Theorem 22. Suppose Laplace transforms of $f(t), f'(t), f''(t)$ exist. Then

$$\mathscr{L}\{f''(t)\} = s^2\mathscr{L}\{f(t)\} - sf(0) - f'(0)$$

Proof. By definition,

$$\begin{aligned} \mathscr{L}\{f''(t)\} = s\mathscr{L}\{[f'(t)]'\} &= s\mathscr{L}\{f'(t)\} - f'(0) \\ &= s\left[s\mathscr{L}\{f(t) - f(0)\}\right] - f'(0) \\ &= s^2\mathscr{L}\{f(t)\} - sf(0) - f'(0) \end{aligned}$$

□

Theorem 23. Suppose Laplace transforms of $f(t)$ and $f^k(t)(n = 1, 2, \cdots, n)$ exist. Then

$$\mathscr{L}\{f^{(n)}\} = s^n\mathscr{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

where $f^(k)(t)$ denotes the n th derivative of the function $f(t)$.

3.9 Laplace transform of the integral

Theorem 24 (s- Divided transform)**.**

$$\mathscr{L}\left\{\int_0^t f(u)du\right\} = \frac{\mathscr{L}\{f(t)\}}{s}$$

Proof. Let

$$g(t) = \int_0^t f(u)du \tag{3.3}$$

From equation (3.3) it follows that

$$\begin{aligned}
g(0) &= 0 \\
g'(t) &= \frac{d}{dt}\left(\int_0^t f(u)du\right) = f(t) \\
\therefore \quad \mathscr{L}\{g'(t)\} &= \mathscr{L}\{f(t)\} \\
i.e., \quad s\mathscr{L}\{g(t)\} - g(0) &= \mathscr{L}\{f(t)\} \\
i.e., \quad s\mathscr{L}\{g(t)\} &= \mathscr{L}\{f(t)\} \quad (\because g(0) = 0) \\
\therefore \quad \mathscr{L}\{g(t)\} &= \frac{\mathscr{L}\{f(t)\}}{s} \\
\therefore \quad \mathscr{L}\left[\int_0^t f(u)du\right] &= \mathscr{L}\{g(t)\} = \frac{\mathscr{L}\{f(t)\}}{s}
\end{aligned}$$

□

3.10 Multiplication by t^n

Theorem 25.

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\left[\mathscr{L}\{f(t)\}\right], \quad \text{for} \quad n = 1, 2, 3, \cdots$$

Proof. The proof is by induction on n. First we will prove that the result is true for $n = 1$. We have

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)dt \\
\therefore \quad \frac{d}{ds}\mathscr{L}\{f(t)\} &= \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt \\
&= \int_0^\infty \frac{\partial}{\partial s}\{e^{-st} f(t)\}dt \\
&= -\int_0^\infty t e^{-st} f(t)dt \\
\therefore \quad (-1)\frac{d}{ds}\mathscr{L}\{f(t)\} &= \int_0^\infty e^{-st}\{tf(t)\}dt = \mathscr{L}\{tf(t)\}
\end{aligned}$$

Hence

$$\mathscr{L}\{tf(t)\} = (-1)\frac{d}{ds}\left[\mathscr{L}\{f(t)\}\right]$$

This proves that the theorem is true for $n = 1$.

Next assume that the theorem is true for $n = r$. Then

$$\mathscr{L}\{t^r f(t)\} = (-1)^r \frac{d^r}{ds^r} \mathscr{L}\{f(t)\}$$

Therefore

$$\begin{aligned}(-1)^r \frac{d^r}{ds^r} \left[\mathscr{L}\{f(t)\}\right] &= \mathscr{L}\{t^r f(t)\} \\ &= \int_0^\infty e^{-st} t^r f(t) dt\end{aligned}$$

Differentiating both sides with respect to t, we get:

$$\begin{aligned}(-1)^r \frac{d^{r+1}}{ds^{r+1}} \mathscr{L}\{f(t)\} &= \frac{d}{ds} \int_0^\infty e^{-st} t^r f(t) dt \\ &= \int_0^\infty \frac{\partial}{\partial s} \left[e^{-st} t^r f(t)\right] \\ &= \int_0^\infty -t e^{-st} t^r f(t) dt \\ i.e., \quad (-1)^{r+1} \frac{d^{r+1}}{ds^{r+1}} \left[\mathscr{L}\{f(t)\}\right] &= \int_0^\infty e^{-st} t^{r+1} f(t) dt \\ &= \mathscr{L}\{t^{r+1} f(t)\}\end{aligned}$$

This shows that the result is true for $n = r + 1$. Hence by mathematical induction the result is true for all positive integral values of n. □

3.11 Division by t

Theorem 26 (Transform Integration)**.** If $\mathscr{L}\{f(t)\} = F(s)$, then

$$\mathscr{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s) ds$$

Proof. By definition,

$$F(s) = \int_0^\infty e^{-st} f(t) dt \tag{3.4}$$

Integrating (3.4) with respect to s from s to ∞, we get:

$$\begin{aligned}\int_0^\infty F(s) ds &= \int_s^\infty \left[\int_0^\infty e^{-st} f(t) dt\right] ds \\ &= \int_0^\infty \int_s^\infty e^{-st} f(t) dt \quad \text{(changing order of integration)}\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt \\
&= \int_0^\infty \left[\frac{f(t)}{t}\right] e^{-st} dt \\
&= \mathscr{L}\left[\frac{f(t)}{t}\right]
\end{aligned}
$$

□

3.12 The laplace transforms of periodic functions

Theorem 27. If a function $f(t)$ is periodic with period k on $[0, \infty)$, then

$$\mathscr{L}\{f(t)\} = \frac{\int_0^k f(t)e^{-st}dt}{1-e^{-ks}}$$

Proof. Since $f(t)$ is periodic with period k,

$$f(t+nk) = f(t), \quad n = 0, 1, 2, \cdots \tag{3.5}$$

By definition,

$$
\begin{aligned}
\mathscr{L}\{f(t)\} &= \int_0^\infty f(t)dt \\
&= \int_0^k f(t)e^{-st}dt + \int_k^{2k} f(t)e^{-st}dt + \int_{2k}^{3k} f(t)e^{-st}dt + \cdots \\
&= \sum_{n=0}^\infty \int_{nk}^{(n+1)k} f(t)e^{-st}dt \\
&= \sum_{n=0}^\infty \int_0^k f(x+nk)e^{-s(x+nk)}dt \quad (\text{Putting} \quad t = x+nk) \\
&= \sum_{n=0}^\infty \int_0^k f(x)e^{-sx}e^{-nsk}dx \quad (\text{by equation}(3.5)) \\
&= \sum_{n=0}^\infty e^{-nsk} \int_0^k f(x)e^{-sx}dx \\
&= (1+e^{-sk}+e^{-2sk}+\cdots)\int_0^k f(x)e^{-sx}dx \\
&= \frac{\int_0^k f(x)e^{-sx}dx}{1-e^{-ks}} \left(\because 1+x+x^2+x^3+\cdots = \frac{1}{1-x} \text{ if } |x|<1\right) \\
&= \frac{\int_0^k f(t)e^{-st}dt}{1-e^{-ks}}
\end{aligned}
$$

□

Theorem 28.

$$\mathscr{L}\{\mathscr{L}\{f(t)\}\} = \int_0^\infty \frac{f(t)dt}{t+u}$$

Proof.

$$\begin{aligned}
\mathscr{L}\{\mathscr{L}\{f(t)\}\} &= \mathscr{L}\left[\int_0^\infty e^{-st} f(t)dt\right] \\
&= \int_0^\infty e^{-us}\left[\int_0^\infty e^{-st} f(t)dt\right] ds \\
&= \int_0^\infty \int_0^\infty f(t)e^{-s(t+u)} dsdt \\
&= \int_0^\infty f(t)\left[\frac{e^{-s(t+u)}}{-(t+u)}\right]_{s=u}^\infty \\
&= \int_0^\infty \frac{f(t)dt}{t+u}
\end{aligned}$$

□

3.13 Limit theorems

Theorem 29 (The initial value theorem)**.** Let $\mathscr{L}L\{f(t)\} = F(s)$ be the Laplace transform of an n times differentiable function $f(t)$. Then

$$f^(r)(0) = \lim_{s\to\infty}\{s^{r+1}F(s) - s^r f(0) - s^{r-1}f'(0) - \cdots - sf^{(r-1)}(0)\}$$

$$r = 0, 1, 2\cdots, n$$

In particular

$$\begin{aligned}
f(0) &= \lim_{s\to\infty}\{sF(s)\} \\
f'(0) &= \lim_{s\to\infty}\{s^2F(s) - sf(0)\} \\
f''(0) &= \lim_{s\to\infty}\{s^3F(s) - s^2f(0) - sf'(0)\}.
\end{aligned}$$

Proof. We have

$$\mathscr{L}L\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0) \quad (3.6)$$

Replacing n by $r+1$ and rewriting , we get:

$$f^{(r)}(0) = s^{r+1}F(s) - s^r f(0) - \cdots - sf^{(r-1)}(0) - \mathscr{L}L\{f^{(r+1)}(t)\}$$

$$\therefore \quad \lim_{s\to\infty} f^{(r)}(0) = \lim_{s\to\infty}\left[s^{r+1}F(s) - s^r f(0) - \cdots - sf^{(r-1)}(0) - \mathscr{L}L\{f^{(r+1)}(t)\}\right]$$

$$\begin{aligned} i.e., \quad f^{(r)}(0) &= \lim_{s\to\infty}\left[s^{r+1}F(s) - s^r f(0) - \cdots - sf^{(r-1)}(0)\right] \\ &\quad - \lim_{s\to\infty} \mathscr{L}L\{f^{(r+1)}(t)\} \end{aligned} \tag{3.7}$$

Assume that $f^{(r+1)}$ satisfies the sufficiency conditions for the existence of a Laplace transform. Then

$$\lim_{s\to\infty} \mathscr{L}L\{f^{(r+1)}(t)\} = 0$$

Hence equation (3.7) becomes

$$f^{(r)}(0) = \lim_{s\to\infty}\left[s^{r+1}F(s) - s^r f(0) - \cdots - sf^{(r-1)}(0)\right]$$

□

Theorem 30. Prove the final value theorem

$$\lim_{t\to\infty} f^r(t) = \lim_{s\to 0}\left[s^{r+1}F(s) - s^r f(0) - s^{(r-1)}f'(0)\cdots - sf^{(r-1)}(0) - f^r(0)\right]$$

Proof. We have

$$\mathscr{L}L\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - \cdots - -f^{(n-1)}(0) \tag{3.8}$$

Replacing n by $r+1$, we get:

$$\begin{aligned} \mathscr{L}L\{f^{(r+1)}(t)\} &= s^{(r+1)}F(s) - s^r f(0) - \cdots - f^{(r)}(0) \\ i.e., \quad \int_0^\infty e^{-st} f^{(r+1)}(t)dt &= s^{(r+1)}F(s) - s^r f(0) - \cdots - f^{(r)}(0) \end{aligned}$$

Taking $s \to 0$ on both sides, we get:

$$\begin{aligned} \lim_{s\to 0}\int_0^\infty e^{-st} f^{(r+1)}(t)dt &= \lim_{s\to\infty} s^{(r+1)}F(s) - s^r f(0) - \cdots - f^{(r)}(0) \\ i.e., \quad \int_0^\infty f^{(r+1)}(t)dt &= \lim_{s\to 0} s^{(r+1)}F(s) - s^r f(0) - \cdots - f^{(r)}(0) \\ i.e., \quad \lim_{t\to\infty} f^{(r)}(t) - f^r(0) &= \lim_{s\to 0} s^{(r+1)}F(s) - s^r f(0) - \cdots - f^{(r)}(0) \\ i.e., \quad \lim_{t\to\infty} f^{(r)}(t) &= \lim_{s\to 0} s^{(r+1)}F(s) - s^r f(0) - \cdots - sf^{(r-1)}(0) \end{aligned}$$

□

3.14 The delta function

Definition. The delta function is defined as

$$\delta(t-a) = \lim_{h\to 0} \frac{1}{h}\left[u(t-a) - (u-a-h)\right]$$

Delta function can be considered as the limit of a rectangular "pulse" of height $1/h$ and width h in the limit as $h \to 0$. Thus the area of the graph representing the pulse remains 1 as $h \to 0$. The graphical representation of $\delta(t-a)$ is shown in figure 3.8.

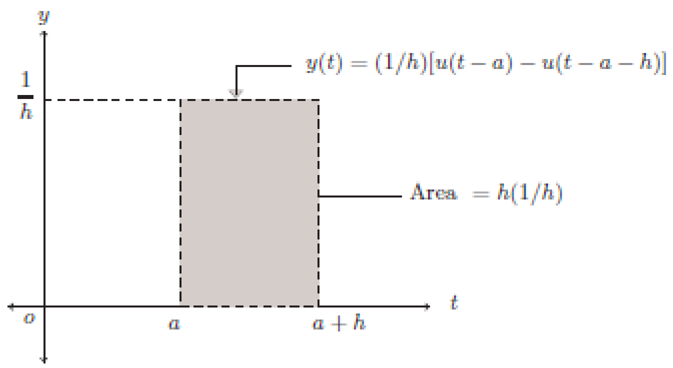


Figure 3.8: $\delta(t-a) = \lim_{h\to 0} y(t)$

Theorem 31 (Filtering property of the delta function)**.** Let $f(t)$ be defined and integrable over all intervals contained within $0 \le t < \infty$, and let it be continuous in a neighbourhood of a. Then for $a \ge 0$

$$\int_0^\infty f(t)\delta(t-a)dt = f(a)$$

Proof. From the definition of the delta function,

$$\int_0^\infty f(t)\delta(t-a)dt = \lim_{h\to 0}\int_a^{a+h} \frac{f(t)}{h}dt,$$

so applying the mean value theorem for integrals we have

$$\int_0^\infty f(t)\delta(t-a)dt = \lim_{h\to 0}\left[h\left(\frac{1}{h}\right)f(\xi)\right],$$

where $a < \xi < a+h$. In the limit as $h \to 0$ the variable $\xi \to a$, showing that

$$\int_0^\infty f(t)\delta(t-a)dt = f(a),$$

and the theorem is proved. □

Theorem 32. The Laplace transform of the delta function is e^{-as}

$$\begin{aligned}\mathscr{L}L\{\delta(t-a)\} &= \int_0^\infty e^{-st}\delta(t-a)dt \\ &= e^{-as}(\text{ by filtering property})\end{aligned}$$

As a special case we have $\mathscr{L}\{\delta(t)\} = 1$.

3.15 Worked problems

3.15.1 Worked problems on standard Laplace transforms

Example 43. Find the Laplace transform of the function $t^3 - 4t + 5 + 3\sin t$

Solution.

$$\begin{aligned}\mathscr{L}\{t^3 - 4t + 5 + 3\sin t\} &= \mathscr{L}\{t^3\} - 4\mathscr{L}\{t\} + \mathscr{L}\{1\} + 3\mathscr{L}\{\sin t\} \\ &= \frac{3!}{s^4} - \frac{1}{s^2} + 5\frac{1}{s} + \frac{1}{s^2+4} \\ &= \frac{5s^5 + 2s^4 + 20s^3 - 10s^2 + 24}{s^4(s^2+4)}\end{aligned}$$

Example 44. Find the Laplace transform of the function $e^{2t} + 4t^3 - 2\sin t + 3\cos 3t$

Solution.

$$\begin{aligned}\mathscr{L}\{e^{2t} + 4t^3 - 2\sin t + 3\cos 3t\} &= \mathscr{L}\{e^{2t}\} + 4\mathscr{L}\{t^3\} - 2\mathscr{L}\{\sin t\} + 3\mathscr{L}\{\cos 3t\} \\ &= \frac{1}{s-2} + 4\frac{3!}{s^4} - 2\frac{3}{s^2+3^2} + 3\frac{s}{s^2+3^2} \\ &= \frac{1}{s-2} + \frac{24}{s^4} + \frac{3(s-2)}{s^2+9}\end{aligned}$$

Example 45. Find the Laplace transform of the function $\sin^2 3t$.

Solution. We have

$$\sin^2 3t = \frac{1-\cos 6t}{2}$$

Therefore

$$\begin{aligned}\mathscr{L}\{\sin^2 3t\} &= \frac{1}{2}\mathscr{L}\{1\} - \frac{1}{2}\mathscr{L}\{\cos 6t\} \\ &= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s}{s^2+36} \\ &= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+36}\right] = \frac{18}{s(s^2+36)}\end{aligned}$$

Example 46. Find the Laplace transform of the function $\cos(at+b)$.

Solution.

$$\begin{aligned}
\cos(at+b) &= \cos at \cos b + \sin at \sin b \\
\therefore \quad \mathscr{L}\{\cos(ax+b)\} &= \cos b \mathscr{L}\{\cos at\} + \sin b \mathscr{L}\{\sin at\} \\
&= \cos b \left(\frac{s}{s^2+a^2}\right) - \sin b \left(\frac{a}{s^2+a^2}\right) \\
&= \frac{s\cos b - a \sin b}{s^2+a^2}
\end{aligned}$$

Example 47. Find the Laplace transform of the function $\cos^3 2t$.

Solution. We have

$$\begin{aligned}
\cos 3x &= 4\cos^3 x - 3\cos x \\
\therefore \quad \cos^3 x &= \frac{1}{4}[\cos 3x - 3\cos x]
\end{aligned}$$

Putting $x = 2t$ in the above equation, we get

$$\begin{aligned}
\cos^3 6t &= \frac{1}{4}[\cos 6t - 3\cos 2t] \\
\therefore \quad \mathscr{L}\{\cos^3 6t\} &= \frac{1}{4}[\mathscr{L}\{\cos 6t\} - 3\mathscr{L}\{\cos 2t\}] \\
&= \frac{1}{4}\left[\frac{s}{s^2+36} - \frac{3s}{s^2+4}\right] \\
&= \frac{s}{4}\left[\frac{s^2+4+3s^2+108}{(s^2+36)(s^2+4)}\right] = \frac{s(s^2+28)}{(s^2+36)(s^2+4)}
\end{aligned}$$

Example 48. Find the Laplace transform of the function $\sin^6 t$.

Solution. If n is even we have

$$\cos^n t = \frac{1}{2^{n-1}}\left[\binom{n}{0}\cos nt + \binom{n}{1}\cos(n-2)t + \binom{n}{2}\cos(n-4)t + \cdots + \frac{1}{2}\binom{n}{\frac{n}{2}}\right]$$

Putting $n = 6$ in the above result, we get:

$$\begin{aligned}
\cos^6 t &= \frac{1}{2^5}\left[\binom{6}{0}\cos 6t + \binom{6}{1}\cos 4t + \binom{6}{2}\cos 2t + \binom{6}{3}\right] \\
&= \frac{1}{32}[\cos 6t + 6\cos 4t + 15\cos 2t + 10] \\
\therefore \quad \mathscr{L}\{\cos^6 t\} &= \frac{1}{32}[\mathscr{L}\{\cos 6t\} + 6\mathscr{L}\{\cos 4t\} + 15\mathscr{L}\{\cos 2t\} + 10\mathscr{L}\{1\}] \\
&= \frac{1}{32}\left[\frac{s}{s^2+36} + \frac{6s}{s^2+16} + \frac{15s}{s^2+4} + \frac{10}{s}\right]
\end{aligned}$$

Example 49. Find the Laplace transform of the function $\cos^7 t$.

Solution. If n is odd, we have

$$\cos^n t = \frac{1}{2^{n-1}}\left[\binom{n}{0}\cos nt + \binom{n}{1}\cos(n-2)t + \binom{n}{2}\cos(n-4)t + \cdots + \frac{1}{2}\binom{n}{\frac{n-1}{2}}\cos t\right]$$

Putting $n = 7$ in the above result, we get:

$$\begin{aligned}
\cos^7 t &= \frac{1}{2^6}\left[\binom{7}{0}\cos 7t + \binom{7}{1}\cos 5t + \binom{7}{2}\cos 3t + \binom{7}{3}\cos 2t + \binom{7}{4}\cos t\right] \\
&= \frac{1}{2^6}\left[\cos 7t + 7\cos 5t + 21\cos 3t + 35\cos t\right] \\
\therefore \quad \mathscr{L}\{\cos^7 t\} &= \frac{1}{2^6}\left[\mathscr{L}\{\cos 7t\} + 7\mathscr{L}\{\cos 5t\} + 21\mathscr{L}\{\cos 2t\} + 35\mathscr{L}\{\cos t\}\right] \\
&= \frac{1}{2^6}\left[\frac{s}{s^2+49} + \frac{7s}{s^2+25} + \frac{21s}{s^2+4} + \frac{35}{s^2+1}\right]
\end{aligned}$$

Example 50. Find the Laplace transform of the function

$$f(t) = |t-1| + |t+1|, \ t \geq 0$$

Solution.

$$\begin{aligned}
f(t) &= |t-1| + |t+1| \\
&= \begin{cases} 2, & 0 \leq t \leq 1 \\ 2t, & t \geq 1 \end{cases} \\
\therefore \quad \mathscr{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)\ dt \\
&= \int_0^1 e^{-st}(2)\ dt + \int_1^\infty e^{-st}(2t)\ dt \\
&= 2\left[\frac{e^{-st}}{-s}\right]_0^1 + 2\left[t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{(-s)^2}\right)\right]_1^\infty \\
&= -\frac{2}{s}\left[e^{-s} - 1\right] + 2\left[\frac{e^{-s}}{s^2}\right] = \frac{2}{s}\left[1 + \frac{e^{-s}}{s}\right]
\end{aligned}$$

Example 51. Find the Laplace transform of the function

$$f(t) = |t-1| + |t-2|$$

Solution.

$$f(t) = |t-1| + |t-2|$$

$$= \begin{cases} -2t+3, & 0 \le t \le 1 \\ 1, & 1 \le t \le 2 \\ 2t-3, & t \le 2 \end{cases}$$

Therefore

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t) dt \\
&= \int_0^1 e^{-st}(-2t+3)\ dt + \int_1^2 e^{-st}(1)\ dt + \int_2^\infty e^{-st}(2t-3)\ dt \\
&= \left[(-2t+3)\left(\frac{e^{-st}}{-s}\right) - (-2)\left(\frac{e^{-st}}{(-s)^2}\right)\right]_0^1 + \left[\frac{e^{-st}}{-s}\right]_1^2 + \\
&\quad \left[(2t-3)\left(\frac{e^{-st}}{-s}\right) - (2)\left(\frac{e^{-st}}{(-s)^2}\right)\right]_2^\infty \\
&= \left[\frac{e^{-s}}{-s} + 2\frac{e^{-s}}{s^2} + \frac{1}{s} - \frac{2}{s^2}\right] + \left[-\frac{e^{-2s}}{s} + \frac{e^{-s}}{s}\right] - \\
&\quad \left[-\frac{e^{-2s}}{-s} - 2\frac{e^{-2s}}{s^2}\right] \\
&= \frac{2e^{-s}}{s^2} + \frac{1}{s} + \frac{2e^{-s}}{s^2}
\end{aligned}$$

Example 52. Find the Laplace transform of the function

$$f(t) = \begin{cases} e^t & 0 < t < 1 \\ 0 & t > 1 \end{cases}$$

Solution.

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)\ dt \\
&= \int_0^1 e^{-st} e^t\ dt + \int_1^\infty e^{-st}(0)\ dt \\
&= \int_0^1 e^{-(s-1)t}\ dt = \left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^1 \\
&= -\frac{e^{-(s-1)}}{(s-1)} + \frac{1}{s-1} = \frac{1}{(s-1)}\left[1 - e^{(1-s)}\right]
\end{aligned}$$

Example 53. Find the laplace transform of the function $\sin at \cos bt$.

Solution.

$$\sin at \cos at = \frac{1}{2}\left[\sin(a+b)t + \sin(a-b)t\right]$$

$$
\begin{aligned}
\therefore \quad \mathscr{L}\{\sin at \cos at\} &= \frac{1}{2}\left[\mathscr{L}\{\sin(a+b)t\} + \mathscr{L}\{\sin(a-b)t\}\right] \\
&= \frac{1}{2}\left[\frac{a+b}{s^2+(a+b)^2} + \frac{a-b}{s^2+(a-b)^2}\right] \\
&= \frac{1}{2}\left[\frac{(a+b)[s^2+(a-b)^2] + (a-b)[s^2+(a+b)^2]}{[s^2+(a+b)^2][s^2+(a-b)^2]}\right] \\
&= \frac{1}{2}\left[\frac{(a+b+a-b)s^2 + (a^2-b^2)(a-b) + (a^2-b^2)(a+b)}{[s^2+(a+b)^2][s^2+(a-b)^2]}\right] \\
&= \frac{1}{2}\left[\frac{2as^2 + (a^2-b^2)(a-b+a+b)}{[s^2+(a+b)^2][s^2+(a-b)^2]}\right] \\
&= \frac{1}{2}\left[\frac{2as^2 + 2a(a^2-b^2)}{[s^2+(a+b)^2][s^2+(a-b)^2]}\right] \\
&= \left[\frac{as^2 + a(a^2-b^2)}{[s^2+(a+b)^2][s^2+(a-b)^2]}\right]
\end{aligned}
$$

3.15.2 Problems involving first shift theorem

Example 54. Find the Laplace transform of $e^{-t}(3\sinh 2t - 5\cosh 2t)$

Solution.

$$
\begin{aligned}
\mathscr{L}\{3\sinh 2t - 5\cosh 2t\} &= 3\mathscr{L}\{\sinh 2t\} - 5\mathscr{L}\{\cosh 2t\} \\
&= 3\left[\frac{2}{s^2-4}\right] - 5\left[\frac{s}{s^2-4}\right] \\
&= \frac{6-5s}{s^2-4} \\
\therefore \quad \mathscr{L}\{e^{-t}(3\sinh 2t - 5\cosh 2t)\} &= \frac{6-5(s+1)}{(s+1)^2-4} \quad (\because \mathscr{L}\{e^{-at}f(t)\} = f(s+a)) \\
&= \frac{1-5s}{s^2+2s-3}
\end{aligned}
$$

Example 55. Find the Laplace transform of $e^{-3t}t^3$.

Solution. We have $\mathscr{L}\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}$. Therefore by first shift theorem

$$\mathscr{L}\{e^{-3t}t^3\} = \frac{6}{(s+3)^4}$$

Example 56. Find the Laplace transform of $e^{-t}\sin^2 t$.

Solution. We have

$$
\begin{aligned}
\sin^2 t &= \frac{1-\cos 2t}{2} \\
\therefore \quad \mathscr{L}\{\sin^2 t\} &= \frac{1}{2}\left[\mathscr{L}\{1\} - \mathscr{L}\{\cos 2t\}\right]
\end{aligned}
$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

Therefore by first shift theorem

$$\begin{aligned}\mathscr{L}\{e^{-t}\sin^2 t\} &= \frac{1}{2}\left[\frac{1}{s+1} - \frac{s+1}{(s+1)^2+4}\right] \\ &= \frac{2}{(s+1)(s^2+2s+5)}\end{aligned}$$

Example 57. Find the Laplace transform of $e^{-2t}\sin 4t$.

Solution. We have $\mathscr{L}\{\sin 4t\} = \dfrac{4}{s^2+16}$. Therefore by first shift theorem

$$\mathscr{L}\{e^{-2t}\sin 4t\} = \frac{4}{(s+2)^2+16} = \frac{4}{s^2+4s+20}$$

Example 58. Find the laplace transform of $\cosh at \sin at$.

Solution.

$$\begin{aligned}\cosh at \sin at &= \left[\frac{e^{at}-e^{-at}}{2}\right]\sin at \\ &= \frac{1}{2}\left[e^{at}\sin at + e^{-at}\sin at\right] \\ \therefore \quad \mathscr{L}\{\cosh at \sin at\} &= \frac{1}{2}\left[\mathscr{L}\{e^{at}\sin at\} + \mathscr{L}\{e^{-at}\sin at\}\right] \\ &= \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2} + \frac{a}{(s+a)^2+a^2}\right] \\ &= \frac{1}{2}\left[\frac{a}{s^2-2as+2a^2} + \frac{a}{s^2+2as+2a^2}\right] \\ &= \frac{a}{2}\left[\frac{1}{[(s^2+2a^2)-2as]} + \frac{1}{[(s^2+2a^2)+2as]}\right] \\ &= \frac{a}{2}\left[\frac{s^2+2a^2+2as+s^2+2a^2-2as}{[(s^2+2a^2)-2as][(s^2+2a^2)-2as]}\right] \\ &= \frac{a}{2}\left[\frac{2(s^2+2a^2)}{[(s^2+2a^2)^2-(2as)^2]}\right] \\ &= \frac{a(s^2+2a^2)}{s^4+4a^4}\end{aligned}$$

Alitter

$$\begin{aligned}\mathscr{L}\{\cosh at\}&\frac{s}{s^2-a^2} \\ \therefore \quad \mathscr{L}\{e^{iat}\cosh at\} &= \frac{(s-ia)}{(s-ia)^2-a^2}\end{aligned}$$

$$
\begin{aligned}
&= \frac{s-ia}{[(s^2-2a^2)^2-i2as]}\\
&= \frac{(s-ia)[(s^2-2a^2)^2+i2as]}{[(s^2-2a^2)^2-i2as]}\\
&= \frac{(s^3-2a^2s+2as)+i(as^2+2a^3)}{(s^2-2a^2)^2+4a^2s^2}\\
i.e., \mathscr{L}\{\cosh at\cos at + i\cosh at\sin at\} &= \frac{(s^3-2a^2s+2as)}{s^4+4a^4} + i\frac{(as^2+2a^3)}{s^4+4a^4}
\end{aligned}
$$

Equating real and imaginary parts, we get:

$$(i)\ \mathscr{L}\{\cosh at\cos at\} = \frac{(s^3-2a^2s+2as)}{(s^2-2a^2)^2+4a^2s^2}$$

$$(ii)\ \mathscr{L}\{\cosh at\sin at\} = \frac{(as^2+2a^3)}{(s^2-2a^2)^2+4a^2s^2}$$

Example 59. Prove the following results

(i) $\mathscr{L}\{\sinh at\cos at\} = \dfrac{a(s^2-2a^2)}{s^4+4a^4}$

(ii) $\mathscr{L}\{\sinh at\sin at\} = \dfrac{2a^2s}{s^4+4a^4}$

Solution. We have

$$
\begin{aligned}
\mathscr{L}\{\sinh at\} &= \frac{a}{s^2-a^2}\\
\therefore\quad \mathscr{L}\{e^{iat}\sinh at\} &= \frac{a}{(s-ia)^2-a^2}\\
&= \frac{a}{(s^2-2a^2)-i2sa}\\
&= \frac{a(s^2-2a^2+i2as)}{(s^2-2a^2)^2+4s^2a^2}
\end{aligned}
$$

$$
\begin{aligned}
i.e.,\quad \mathscr{L}\{(\cos at + i\sin at)\sinh at\} &= \frac{a(s^2-2a^2+i2as)}{(s^2-2a^2)^2+4s^2a^2}\\
i.e.,\quad \mathscr{L}\{\cos at\sinh at\} + i\mathscr{L}\{\sin at\sinh at\} &= \frac{a(s^2-2a^2+i2as)}{(s^2-2a^2)^2+4s^2a^2}\\
&= \frac{a(s^2-2a^2)}{s^4+4a^4} + i\frac{2a^2s}{s^4+4a^4}
\end{aligned}
$$

Equating real and imaginary parts, we get:

$$
\begin{aligned}
\text{(i)}\mathscr{L}\{\sinh at\cos at\} &= \frac{a(s^2-2a^2)}{s^4+4a^4}\\
\text{(ii)}\mathscr{L}\{\sinh at\sin at\} &= \frac{2a^2s}{s^4+4a^4}
\end{aligned}
$$

Example 60. Find the laplace transform of $\sinh 3t \cos^2 t$.

Solution. We have

$$\begin{aligned}
\cos^2 t &= \frac{1+\cos 2t}{2} \\
\therefore \quad \mathscr{L}\{\cos^2 t\} &= \frac{1}{2}\left[\mathscr{L}\{1\} + \mathscr{L}\{\cos 2t\}\right] \\
&= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]
\end{aligned}$$

Let $f(t) = \sinh 3t \cos^2 t$. Therefore

$$\begin{aligned}
f(t) &= \left[\frac{e^{3t} - e^{-3t}}{2}\right]\cos^2 t \\
&= \frac{1}{2}\left[e^{3t}\cos^2 t - e^{-3t}\cos^2 t\right] \\
\therefore \quad \mathscr{L}\{f(t)\} &= \frac{1}{2}\left[\mathscr{L}\{e^{3t}\cos^2 t\} - \mathscr{L}\{e^{-3t}\cos^2 t\}\right] \\
&= \frac{1}{4}\left[\frac{1}{s-3} + \frac{s-3}{(s-3)^2+4} - \frac{1}{s+3} - \frac{s+3}{(s+3)^2+4}\right] \\
&= \frac{1}{4}\left[\frac{1}{s-3} - \frac{1}{s+3} + \frac{s-3}{(s-3)^2+4} - \frac{s+3}{(s+3)^2+4}\right] \\
&= \frac{1}{4}\left[\frac{s+3-s+3}{(s-3)(s+3)} + \frac{s-3}{(s-3)^2+4} - \frac{s+3}{(s+3)^2+4}\right] \\
&= \frac{1}{4}\left[\frac{6}{s^2-9} + \frac{s-3}{[(s^2+13)-6s]} - \frac{s+3}{[(s^2+13)+6s]}\right] \\
&= \frac{1}{4}\left[\frac{6}{s^2-9} + \frac{(s-3)[(s^2+13)+6s] - (s+3)[(s^2+13)-6s]}{[(s^2+13)-6s][(s^2+13)+6s]}\right] \\
&= \frac{1}{4}\left[\frac{6}{s^2-9} + \frac{[(s-3)-(s+3)]s^2 + 6[s-3+s+3] + 13[s-3-s-3]}{(s^2+13)^2-36s^2}\right] \\
&= \frac{1}{4}\left[\frac{6}{s^2-9} + \frac{-6s^2+6(2s)+13(-6)}{(s^2+13)^2-36s^2}\right] \\
&= \frac{3}{2}\left[\frac{1}{s^2-9} - \frac{s^2+(2s)-13)}{(s^2+13)^2-36s^2}\right]
\end{aligned}$$

Example 61. Prove that $\mathscr{L}\{(1+te^{-t})^3\} = \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^2} + \frac{6}{(s+3)^4}$

Solution.

$$\begin{aligned}
\mathscr{L}\{(1+te^{-t})^3\} &= \mathscr{L}\{1 + 3t^2e^{-t} + 3t2e^{-2t} + t^3e^{-3t}\} \\
&= \mathscr{L}\{1\} + 3\mathscr{L}\{te^{-t}\} + 3\mathscr{L}\{t^2e^{-2t}\} + \mathscr{L}\{t^3e^{-3t}\}
\end{aligned}$$

Applying the results $\mathscr{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$ and $\mathscr{L}\{e^{at}f(t)\} = F(s-a)$ in the above equation, we get:

$$\begin{aligned}\mathscr{L}\{(1+te^{-t})^3\} &= \frac{1}{s} + \frac{3\cdot 1!}{(s+1)^2} + \frac{3\cdot 2!}{(s+2)^3} + \frac{3!}{(s+3)^4} \\ &= \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^2} + \frac{6}{(s+3)^4}\end{aligned}$$

Example 62. Find the Laplace transform of the function $\mathscr{L}\{3e^{(-1/2)t}\sin^2 t\}$.

Solution. We have $\sin^2 t = \dfrac{1-\cos 2t}{2}$. Hence

$$\begin{aligned}\mathscr{L}\{3e^{(-1/2)t}\sin^2 t\} &= \mathscr{L}\left\{3e^{(-1/2)t}\left(\frac{1-\cos 2t}{2}\right)\right\} \\ &= \frac{3}{2}\mathscr{L}\{e^{(-1/2)t}\} - \frac{3}{2}\mathscr{L}\{e^{(-1/2)t}\cos 2t\} \\ &= \frac{3}{2}\left(\frac{1}{s+1/2}\right) - \frac{3}{2}\left(\frac{s+1/2}{(s+1/2)^2+2^2}\right) \\ &= \frac{3}{2s+1} - \frac{3(s+1/2)}{2(s^2+s+1/4+4)} \\ &= \frac{3}{2s+1} - \frac{6s+3}{4s^2+4s+17} \\ &= \frac{3(4s^2+4s+17)-(6s+3)(2s+1)}{(2s+1)(4s^2+4s+17)} \\ &= \frac{48}{(2s+1)(4s^2+4s+17)}\end{aligned}$$

3.15.3 Problems involving graphing functions

Example 63. Sketch $f(t) = u(t-2)(1+t)$

Solution. The graph of the given function is given below:

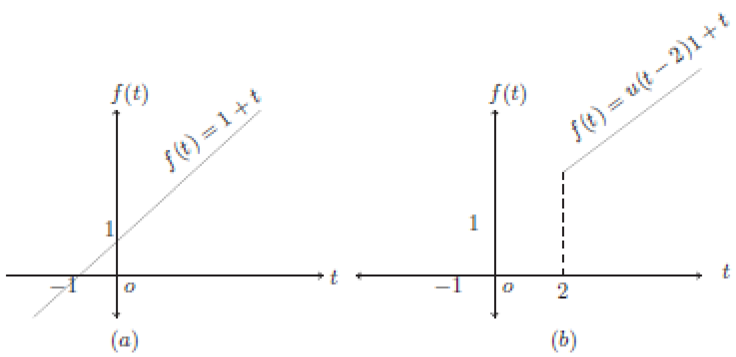


Figure 3.9: (a) The graph of $f(t) = (1+t)$ (b) The graph of $f(t) = u(t-2)(1+t)$

3.15.4 Problems involving the Laplace transform of periodic functions

Example 64. Find the Laplace transform of the square wave shown in figure 3.10

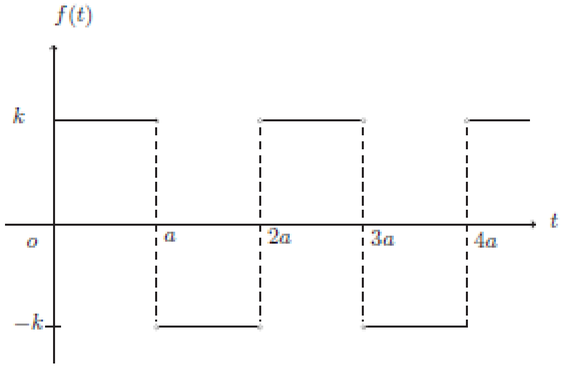


Figure 3.10: A square wave with period $2a$

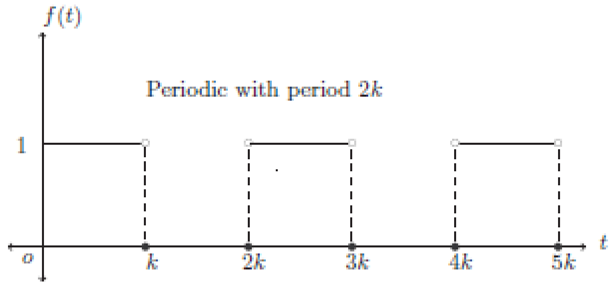
Solution.

$$\begin{aligned}\int_0^\infty e^{-st}f(t)dt &= \int_0^\infty e^{-st}f(t)dt + \int_0^\infty e^{-st}f(t)dt \\ &= \int_0^\infty e^{-st}kdt + \int_0^\infty e^{-st}(-k)dt \\ &= \frac{k}{s}(1-e^{-as}) + \frac{k}{s}(e^{-2as}-e^{-as}) \\ &= \frac{k}{s}(1+e^{-2as}-2e^{-as}) = \frac{k}{s}(1-e^{-as})^2.\end{aligned}$$

Then

$$\begin{aligned}\mathscr{L}L\{f(t)\} &= \frac{\int_0^{2a} e^{-st}f(t)dt}{1-e^{-2as}} \\ &= \frac{k(1+e^{-as})^2}{s(1-e^{-2as})} = \frac{k(1+e^{-as})^2}{s(1-e^{-as})(1+e^{-as})} \\ &= \frac{k(1-e^{-as})}{s(1+e^{-as})} = \frac{k(e^{as/2}-e^{-as/2})}{s(e^{as/2}+e^{-as/2})} \\ &= \frac{k\sinh(as/2)}{s\cosh(as/2)} = \frac{k}{s}\tanh(as/2)\end{aligned}$$

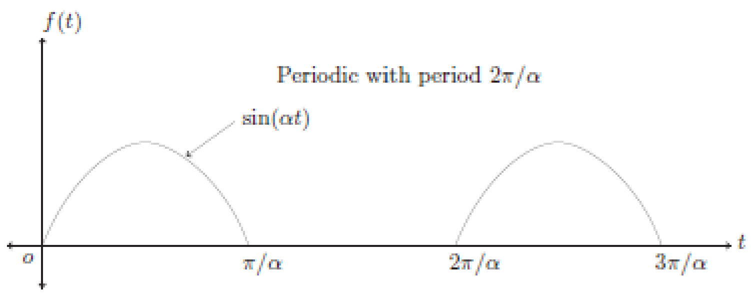
Example 65. Find the Laplace transform the function whose graph is given below:



Solution.

$$\int_0^{2k} e^{-st} f(t)\ dt = \int_0^{k} e^{-st} f(t)\ dt + \int_k^{2k} e^{-st} f(t)\ dt$$
$$= \int_0^{k} e^{-st}(1)\ dt + \int_k^{2k} e^{-st}(0)\ dt$$
$$= \int_0^{k} e^{-st}\ dt$$
$$= \left[\frac{e^{-st}}{-s}\right]_0^k = \frac{1-e^{-ks}}{s}$$
$$\therefore \quad \mathscr{L}\{f(t)\} = \frac{1}{1-e^{-2ks}} \int_0^{2k} e^{-st} f(t)\ dt$$
$$= \frac{1}{1-e^{-2ks}} \frac{1-e^{-ks}}{s}$$
$$= \frac{1-e^{-ks}}{s(1-e^{-ks})(1+e^{-ks})}$$
$$= \frac{1}{s(1+e^{-ks})}$$

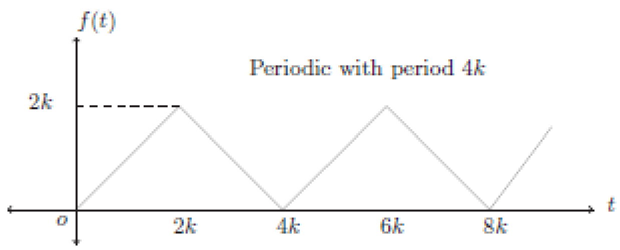
Example 66. Find the Laplace transform the function whose graph is given below:



Solution.

$$\begin{aligned}
\int_0^{2\pi/\alpha} e^{-st} f(t)\ dt &= \int_0^{\pi/\alpha} e^{-st} f(t)\ dt + \int_{\pi/\alpha}^{2\pi/\alpha} e^{-st} f(t)\ dt \\
&= \int_0^{\pi/\alpha} e^{-st} \sin(\alpha t)\ dt + \int_{\pi/\alpha}^{2\pi/\alpha} e^{-st}(0)\ dt \\
&= \int_0^{\pi/\alpha} e^{-st} \sin(\alpha t)\ dt \\
&= \left[\frac{e^{-st}}{\alpha^2+s^2}\{(-s)\sin(\alpha t) - \alpha\cos(\alpha t)\}\right]_0^{\pi/\alpha} \\
&= \frac{1}{\alpha^2+s^2}\left[e^{-\pi s/\alpha}\alpha + \alpha\right] \\
&= \frac{\alpha}{\alpha^2+s^2}\left[1 + e^{-\pi s/\alpha}\right] \\
\therefore \quad \mathscr{L}\{f(t)\} &= \frac{1}{1-e^{-2\pi/\alpha}} \int_0^{2\pi/\alpha} e^{-st} f(t)\ dt \\
&= \frac{1}{\alpha^2+s^2}\frac{\alpha(1+e^{-\pi s/\alpha})}{1-e^{-2\pi s/\alpha}} \\
&= \frac{\alpha}{(\alpha^2+s^2)(1-e^{-\pi s/\alpha})}
\end{aligned}$$

Example 67. Find the Laplace transform the function whose graph is given below:



Solution. The equation of the line in the interval $[0, 2k]$ is given by:

$$\frac{t-0}{2k-0} = \frac{y-0}{2k-0} \quad \text{(by two point form of a line)}$$

$$i.e., \quad y = t$$

Similarly equation of the line in the interval $[2k, 4k]$ is given by:

$$\frac{t-4k}{2k-4k} = \frac{y-0}{2k-0} i.e., \quad y = 4k - t$$

Hence equation of the curve in the interval $[0, 4k]$ is:

$$f(t) = \begin{cases} t, & 0 \le t \le 2k \\ 4k - t, & 2k \le t \le 4k \end{cases}$$

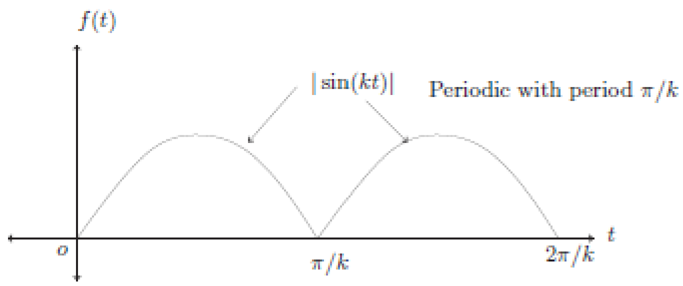
Therefore

$$\begin{aligned}
\int_0^{4k} e^{-st} f(t)\ dt &= \int_0^{2k} e^{-st} f(t)\ dt + \int_{2k}^{4k} e^{-st} f(t)\ dt \\
&= \int_0^{2k} e^{-st} t\ dt + \int_{2k}^{4k} e^{-st}(4k - t)\ dt \\
&= \left[t\frac{e^{-st}}{(-s)} - (1)\frac{e^{-st}}{(-s)^2} \right]_0^{2k} + \left[(4k - t)\frac{e^{-st}}{(-s)} - (-1)\frac{e^{-st}}{(-s)^2} \right]_{2k}^{4k} \\
&\qquad \text{(by Kroneckers formula)} \\
&= \frac{1}{s^2}\left[1 - 2e^{-2ks} + e^{-4ks} \right] = \frac{1}{s^2}(1 - e^{-2ks})^2
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \frac{1}{1 - e^{-4ks}} \int_0^{4k} e^{-st} f(t)\ dt \\
&= \frac{1}{1 - e^{-4ks}} \frac{(1 - e^{-2ks})^2}{s^2} = \frac{1 - e^{-2ks}}{s^2(1 + e^{-2ks})} \\
&= \frac{1}{s^2} \frac{e^{ks} - e^{-ks}}{e^{ks} + e^{-ks}} = \frac{\tanh ks}{s^2}
\end{aligned}$$

Example 68. Find the Laplace transform the function whose graph is given below:

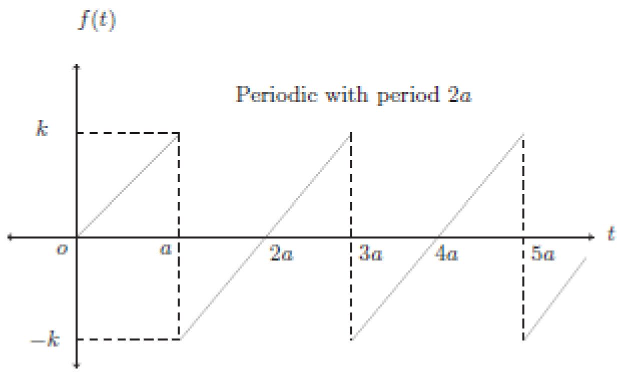


Solution.

$$\int_0^{\pi/k} e^{-st} f(t)\ dt = \int_0^{\pi/k} |\sin(kt)|\ dt$$

$$= \int_0^{\pi/k} e^{-st}\sin(kt)\ dt\,(\because\ |\sin(kt)\text{positive in the interval}\quad [0,\pi/k])$$
$$= \left[\frac{e^{-st}}{s^2+k^2}\{(-s)\sin(kt) - k\cos(kt)\}\right]_0^{\pi/k}$$
$$= \frac{1}{s^2+k^2}\left[e^{-\pi s/k}(k)+k\right] = \frac{(1+e^{e^{-\pi s/k}})}{s^2+k^2}$$
$$\therefore \mathscr{L}\{f(t)\} = \frac{1}{1-e^{-\pi s/k}}\int_0^{\pi/k} e^{-st}f(t)dt$$
$$= \frac{1}{1-e^{-\pi s/k}}\left(\frac{k(1+e^{-\pi s/k})}{s^2+k^2}\right) = \left(\frac{1+e^{-\pi s/k}}{1-e^{-\pi s/k}}\right)\frac{k}{s^2+k^2}$$
$$= \left(\frac{e^{\pi s/2k}+e^{-\pi s/2k}}{e^{\pi s/2k}-e^{-\pi s/2k}}\right)\frac{k}{s^2+k^2} = \frac{k\coth(s\pi/2k)}{s^2+k^2}$$

Example 69. Find the Laplace transform the function whose graph is given below:



Solution. Equation of the line in the interval $[0,a]$ is $y(t) = \dfrac{k}{a}t$. Similarly equation of the line in the interval $[a,2a]$ is $y(t) = \dfrac{k}{a}(t-2a)$.

$$\int_0^{2a} e^{-st}f(t)\ dt = \int_0^a e^{-st}f(t)\ dt + \int_a^{2a} e^{-st}f(t)\ dt$$
$$= \int_0^a e^{-st}(k/a)t\ dt + \int_a^{2a} e^{-st}(k/a(t-2a))\ dt$$
$$= \frac{k}{a}\left[\int_0^a e^{-st}t\ dt + \int_a^{2a} e^{-st}(t-2a)\right]$$
$$= \frac{k}{a}\left[\left\{t\frac{e^{-st}}{(-s)} - (1)\frac{e^{-st}}{(-s)^2}\right\}_0^a + \left\{(t-2a)\frac{e^{-st}}{(-s)} - (1)\frac{e^{-st}}{(-s)^2}\right\}_a^{2a}\right]$$
$$= \frac{k}{a}\left[-\frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} - \frac{e^{-2as}}{s^2} - \frac{ae^{-as}}{s} + \frac{e^{-as}}{s^2}\right]$$

$$+\frac{k}{a}\left[-\frac{2ae^{-as}}{s}-\frac{e^{-2as}}{s^2}=\frac{1}{s^2}\right]=\frac{k}{as^2}\left[1-e^{-2as}-2ae^{-as}\right]$$

$$\therefore \quad \mathscr{L}\{f(t)\}=\frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st}f(t)\ dt$$

$$=\frac{k}{as^2(1-e^{-2as})}[1-e^{-2as}-2ae^{-as}]$$

Example 70. Prove that $\mathscr{L}\{\sin t\} = 1/(s^2+1)$ and use scaling theorem to show that $\mathscr{L}\{\sin at\} = a/(s^2+a^2)$.

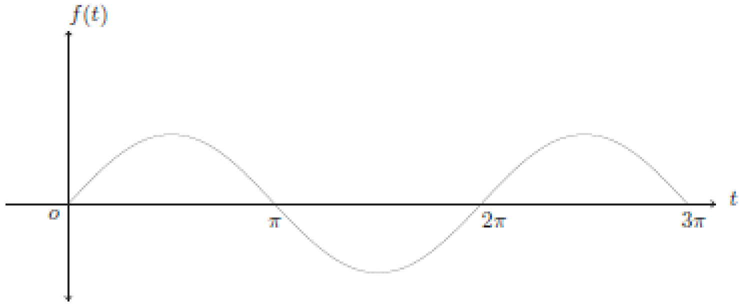


Figure 3.11: Sine wave with period 2π

Solution. The function $f(t) = \sin t$ is periodic with period 2π. Therefore

$$\mathscr{L}L\{\sin t\}=\frac{1}{(1-e^{-2\pi s})}\int_0^{2\pi} e^{-st}\sin t dt$$

$$=\frac{1}{(1-e^{-2\pi s})}\left(\frac{1}{s^2+1}-\frac{e^{-2\pi s}}{s^2+1}\right)=\frac{1}{s^2+1}$$

Therefore

$$\mathscr{L}L\{\sin at\}=\frac{1}{a}\frac{1}{[(s/a)^2+1]} \quad (\because \mathscr{L}L\{f(at\}=\frac{1}{a}F(s/a))$$

$$=\frac{a}{s^2+a^2}$$

Example 71. Find the Laplace transform of $\frac{\sin at}{t}$. Does the transform of $\frac{\cos at}{t}$ exists?

Solution. We have

$$\mathscr{L}\left(\frac{f(t)}{t}\right)=\int_s^{\infty}\mathscr{L}\{f(t)\}ds$$

$$\therefore \quad \mathscr{L}\left(\frac{\sin at)}{t}\right)=\int_s^{\infty}\mathscr{L}\{\sin at\}ds$$

$$\begin{aligned}
&= \int_s^\infty \frac{a}{s^2+a^2} ds = (\arctan(x/a))_s^\infty \\
&= \arctan(\infty) - \arctan(s/a) \\
&= \frac{\pi}{2} - \arctan sa = arccot(s/a)
\end{aligned}$$

Now

$$\begin{aligned}
\therefore \quad \mathscr{L}\left(\frac{\cos at)}{t}\right) &= \int_s^\infty \mathscr{L}\{\cos at\} ds \\
&= \int_s^\infty \frac{s}{s^2+a^2} ds = \frac{1}{2}\int_0^\infty \frac{2s}{s^2+a^2} \\
&= \frac{1}{2}\left[\ln(s^2+a^2)\right]_s^\infty \quad \left(\because \int \frac{f'(x)}{f(x)} dx = \ln f(x)\right) \\
&= \frac{1}{2}\left[\lim_{s\to\infty} \ln(s^2+a^2) - \ln(s^2+a^2)\right]
\end{aligned}$$

Since $\lim_{s\to\infty} \ln(x^2+a^2)$ is infinite , $\mathscr{L}\left(\frac{\cos at}{t}\right)$ does not exists.

Example 72. Prove that $\int_0^\infty \frac{e^{-t}\sin t}{t}\ dt = \frac{\pi}{4}$

Solution. We have

$$\begin{aligned}
\mathscr{L}\left(\frac{\sin at}{t}\right) &= cot^{-1}(s/a) \\
\therefore \quad \int_0^\infty e^{-st}\frac{\sin at}{t}\ dt &= cot^{-1}(s/a)
\end{aligned}$$

Putting $a = 1$ and $s = 1$ in the above equation, we get

$$\int_0^\infty \frac{e^{-t}\sin t}{t}\ dt = \frac{\pi}{4}$$

Example 73. Using Laplace transformation prove that $\int_0^\infty \left(\frac{\sin t}{t}\right)\ dt = \frac{\pi}{2}$.

Solution. We have

$$\begin{aligned}
\mathscr{L}\left(\frac{\sin at}{t}\right) &= cot^{-1}(s/a) \\
\therefore \quad \int_0^\infty e^{-st}\frac{\sin at}{t}\ dt &= cot^{-1}(s/a)
\end{aligned}$$

Putting $a = 1$ and $s = 0$ in the above equation, we get

$$\int_0^\infty \frac{\sin t}{t}\ dt = cot^{-1}(0) = \frac{\pi}{2}$$

Example 74. What is $\mathscr{L}\{t^2 \cos at\}$?

Solution. We have

$$\begin{aligned}
\mathscr{L}\{\cos at\} &= \frac{s}{s^2+a^2} \\
\therefore \quad \mathscr{L}\{t^2 \cos at\} &= (-1)^2 \frac{d}{ds}\left[\frac{s}{s^2+a^2}\right] \quad (\because \mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\left[\frac{s}{s^2=a^2}\right] \\
&= \frac{d}{ds}\left[\frac{(s^2+a^2)-2s^2}{(s^2+a^2)^2}\right] = \frac{d}{ds}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right] \\
&= \frac{(s^2+a^2)^2(-2s)-(a^2-s^2)\ 2\ (s^2+a^2)(2s)}{(s^2+a^2)^4} \\
&= \frac{2s(s^2-3a^2)}{(s^2+a^2)^2}
\end{aligned}$$

Example 75. Find the Laplace transform of $\sin\sqrt{t}$. Deduce the value of $\mathscr{L}\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right)$.

Solution. We know that

$$\begin{aligned}
\sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \\
\therefore \quad \sin\sqrt{t} &= \sqrt{t} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \cdots \\
\therefore \quad \mathscr{L}\{\sin\sqrt{t}\} &= \mathscr{L}\left\{t^{1/2}\right\} - \mathscr{L}\left\{\frac{t^{3/2}}{3!}\right\} + \mathscr{L}\left\{\frac{t^{5/2}}{5!}\right\} - \mathscr{L}\left\{\frac{t^{7/2}}{7!}\right\} + \cdots \\
&= \mathscr{L}\left\{t^{1/2}\right\} - \frac{1}{3!}\mathscr{L}\left\{t^{3/2}\right\} + \frac{1}{5!}\mathscr{L}\left\{t^{5/2}\right\} - \frac{1}{7!}\mathscr{L}\left\{t^{7/2}\right\} + \cdots \\
&= \frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{s^{5/2}3!} + \frac{\Gamma(7/2)}{s^{7/2}5!} - \frac{\Gamma(9/2)}{s^{9/2}7!} + \cdots \quad (\because \mathscr{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} \\
&= \frac{1/2\Gamma(1/2)}{s^{3/2}} - \frac{(3/2)(1/2)\Gamma(1/2)}{s^{5/2}3!} + \frac{(5/2)(3/2)(1/2)\Gamma(1/2)}{s^{7/2}5!} \\
&\quad - \frac{(7/2)(5/2)(3/2)(1/2)\Gamma(1/2)}{s^{9/2}7!} \quad (\because \Gamma(n) = (n-1)\Gamma(n-1)) \\
&= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{(2^2 s)} + \frac{1}{2!(2^2 s)^2} - \frac{1}{3!(2^2 s)^3} + \cdots\right] \quad (\Gamma(1/2) = \sqrt{\pi}) \\
&= \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/2^2 s} = \frac{1}{2s}\left(\frac{\pi}{s}\right)^{1/2} e^{-1/4s}
\end{aligned}$$

Next we will find the Laplace transform of $\mathscr{L}\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right)$. Let $f(t) = \sin\sqrt{t}$. Then $\mathscr{L}\{f(t)\} = \mathscr{L}\{\sin\sqrt{t}\} = \frac{1}{2s}\left(\frac{\pi}{s}\right)^{1/2} = F(s)$. Therefore

$$f(0) = \sin(0) = 0$$

$$F'(t) = \frac{d}{dt}\left[\sin\sqrt{t}\right] = \frac{1}{2}\frac{\cos\sqrt{t}}{\sqrt{t}}$$

$$\therefore \quad \mathscr{L}\{F'(t)\} = s\mathscr{L}\{f(t)\} - f(0) = sF(s) - 0 = sF(s)$$

$$= \frac{1}{2}\left(\frac{\pi}{s}\right)^{1/2} e^{-1/4s}$$

$$\therefore \quad \mathscr{L}\left[\frac{1}{2}\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \frac{1}{2}\left(\frac{\pi}{s}\right)^{1/2} e^{-1/4s}$$

$$i.e., \quad \frac{1}{2}\mathscr{L}\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \frac{1}{2}\left(\frac{\pi}{s}\right)^{1/2} e^{-1/4s}$$

$$i.e., \quad \mathscr{L}\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \left(\frac{\pi}{s}\right)^{1/2} e^{-1/4s}$$

Example 76. Find $\mathscr{L}\left\{\int_0^t \frac{\sin x}{x} dx\right\}$

Solution. We have

$$\mathscr{L}\left\{\int_0^t F(x)\ dx\right\} = \frac{\mathscr{L}\{f(t)\}}{s} \tag{3.9}$$

Also we have

$$\begin{aligned}\mathscr{L}\left\{\frac{\sin t}{t}\right\} &= \int_s^\infty \mathscr{L}\{\sin t\} ds \\ &= \int_s^\infty \frac{1}{1+s^2} ds = \left[tan^{-1}(s)\right]_s^\infty \\ &= \frac{\pi}{2} - tan^{-1}(s) = cot^{-1}(s) \end{aligned} \tag{3.10}$$

Letting $F(x) = \dfrac{\sin x}{x}$. Then from (3.9), we get:

$$\mathscr{L}\left(\int_0^t \frac{\sin x}{x}\right) = \frac{cot^{-1}(s)}{s}$$

Example 77. Show that $\int_0^\infty \frac{f(t)}{t} dt = \int_0^\infty f(x)dx$, assume that the integral converge and $\mathscr{L}\{f(t)\} = F(s)$ and hence prove that

$$\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$$

Solution. We have

$$\mathscr{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty f(x)dx$$

$$i.e., \quad \int_0^\infty e^{-st}\frac{f(t)}{t} dt = \int_s^\infty F(x)dx$$

Taking $s \to 0$, we get:

$$\int_0^\infty 1 \cdot \frac{f(t)}{t} dt = \int_0^\infty F(x)dx$$
$$i.e., \int_0^\infty \frac{f(t)}{t} dt = \int_0^\infty F(x)dx$$

Putting $f(t) = \sin t$ in the above equation, we get:

$$\int_0^\infty \frac{\sin t}{t} dt = \int_0^\infty \frac{1}{1+x^2} = \left[tan^{-1}(x)\right]_0^\infty = \pi/2$$

Example 78. Given $\mathscr{L}\left(2\sqrt{\left(\frac{t}{\pi}\right)}\right) = \frac{1}{s^{3/2}}$, show that $\mathscr{L}\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{s^{1/2}}$.

Solution. Let $f(t) = 2\sqrt{\frac{t}{\pi}}$. Then $f(0) = 2\sqrt{\frac{0}{\pi}} = 0$. Also

$$F'(t) = \frac{2}{2\sqrt{\pi t}} = \frac{1}{\sqrt{\pi t}}$$
$$\therefore \quad \mathscr{L}\{F'(t)\} = s\mathscr{L}\{f(t)\} - f(0) = sF(s) - 0$$
$$= s\frac{1}{s^{3/2}} = \frac{1}{\sqrt{s}}$$
$$\therefore \quad \mathscr{L}\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{s^{1/2}}.$$

Example 79. Evaluate $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} dt$

Solution. Let $f(t) = e^{-t} - e^{-3t}$. Then

$$\therefore \mathscr{L}\{f(t)\} = \mathscr{L}\{e^{-t} - e^{-3t}\}$$
$$= \frac{1}{s+1} - \frac{1}{s+3} = F(s)$$

We have

$$\mathscr{L}\left(\frac{f(t)}{t}\right) = \int_0^\infty F(s)ds \tag{3.11}$$

Substituting the value of $F(x)$ in equation (3.11), we get:

$$\mathscr{L}\left(\frac{f(t)}{t}\right) = \int_0^\infty \left(\frac{1}{s+1} - \frac{1}{s+3}\right) ds$$
$$= [\ln(s+1) - \ln(s+3)]_s^\infty$$
$$= \left[\ln\left(\frac{s+1}{s+3}\right)\right]_s^\infty = \left[\ln\left(\frac{1+1/s}{1+3/s}\right)\right]_s^\infty$$
$$= \ln(1) - \ln\left(\frac{1+1/s}{1+3/s}\right) = -\ln\left(\frac{s+1}{s+3}\right)$$

Example 80. Evaluate $\mathscr{L}\left(\dfrac{\cos at - \cos bt}{t}\right)$.

Solution.

$$\begin{aligned}
\mathscr{L}\left(\frac{\cos at - \cos bt}{t}\right) &= \int_s^\infty \mathscr{L}\{\cos at - \cos bt\} ds \\
&= \int_s^\infty \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right] ds \\
&= \frac{1}{2}\int_s^\infty \left[\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right] ds \\
&= \frac{1}{2}\left[\ln(s^2+a^2) - \ln s^2 + b^2\right]_s^\infty \\
&= \frac{1}{2}\left[\ln\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty \\
&= \frac{1}{2}\left[\ln\left(\frac{1+a^2/s^2}{1+b^2/s^2}\right)\right]_s^\infty \\
&= \frac{1}{2}\ln(1) - \ln\left(\frac{1+a^2/s^2}{1+b^2/s^2}\right) \\
&= -\ln\left(\frac{s^2+a^2}{s^2+b^2}\right)
\end{aligned}$$

Example 81. Evaluate $\mathscr{L}\{te^{at}\sin at\}$

Solution.

$$\begin{aligned}
\mathscr{L}\{te^{at}\sin at\} &= -\frac{d}{ds}\left[\mathscr{L}\{e^{at}\sin at\}\right] \\
&= -\frac{d}{ds}\left[\frac{a}{(s-a)^2+a^2}\right] \\
&= (-a)\left[\frac{-2(s-a)}{[(s-a)^2+a^2]^2}\right] \\
&= \frac{2a(s-a)}{[(s-a)^2+a^2]^2}
\end{aligned}$$

Example 82. Evaluate $\mathscr{L}\{t\sin^2 3t\}$

Solution.

$$\begin{aligned}
\mathscr{L}\{\sin^2 3t\} &= \mathscr{L}\left[\frac{1-\cos 6t}{2}\right] \\
&= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+36}\right]
\end{aligned}$$

$$\begin{aligned}
\therefore \quad \mathscr{L}\{t\sin^2 3t\} &= -\frac{d}{ds}\{\mathscr{L}\{\sin^2 3t\}\} \\
&= -\frac{d}{ds}\left[\frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+36}\right)\right] \\
&= \frac{1}{2s^2} + \frac{1}{2}\left[\frac{(s^2+36) - s(2s)}{(s^2+36)^2}\right] \\
&= \frac{1}{2}\left[\frac{1}{s^2} - \frac{s^2-36}{(s^2+36)^2}\right]
\end{aligned}$$

Example 83. Find the Laplace transform of $f(t) = \begin{cases} 0, & 0 < t < 1 \\ t, & 1 < t < 2 \\ 0, & t > 2 \end{cases}$

Solution.

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)\ dt \\
&= \left[\int_0^1 e^{-st} f(t)\ dt + \int_1^2 e^{-st} f(t)\ dt + \int_2^\infty e^{-st} f(t)\ dt\right] \\
&= \left[\int_0^1 e^{-st}\ 0\ dt + \int_1^2 e^{-st}\ t\ dt + \int_2^\infty e^{-st}\ 0\ dt\right] \\
&= \int_1^2 e^{-st}\ t\ dt \\
&= \left[t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right]_1^2 \\
&= \left[2\left(\frac{e^{-2s}}{-s}\right) - (1)\left(\frac{e^{-2s}}{s^2}\right)\right] - \left[\left(\frac{e^{-s}}{-s}\right) - (1)\left(\frac{e^{-s}}{s^2}\right)\right] \\
&= \frac{1}{s}(e^{-s} - 2e^{-2s}) - \frac{1}{s^2}(e^{-2s} - e^{-s})
\end{aligned}$$

Example 84. Find the Laplace transform of $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$ and $f(t)$ is periodic with period 2π.

Solution.

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \frac{1}{1-e^{2\pi s}} \int_0^{2\pi} e^{-st} f(t) dt \\
&= \frac{1}{1-e^{2\pi s}} \left[\int_0^\pi e^{-st} f(t) dt + \int_\pi^{2\pi} e^{-st} f(t) dt\right] \\
&= \frac{1}{1-e^{2\pi s}} \left[\int_0^\pi e^{-st} \sin t dt + \int_\pi^{2\pi} e^{-st}\ 0\ dt\right] \\
&= \frac{1}{1-e^{2\pi s}} \int_0^\pi e^{-st} \sin t\ dt
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1-e^{2\pi s}}\left[\frac{e^{-st}}{1+s^2}(-s\sin t-\cos t)\right]_0^{\pi}\\
&= \frac{1}{1-e^{2\pi s}}\left[\frac{e^{-s\pi}}{1+s^2}(-\cos\pi)+\frac{1}{1+s^2}\right]_0^{\pi}\\
&= \frac{1}{1-e^{2\pi s}}\left[\frac{e^{-s\pi}}{1+s^2}+\frac{1}{1+s^2}\right]_0^{\pi}\\
&= \frac{(1+e^{-s\pi})}{(1-e^{2\pi s})(1+s^2)}
\end{aligned}$$

Example 85. Find the laplace transform of $\int_0^t \left(\frac{1-e^{-2x}}{x}\right)\ dx$

Solution. We will find the Laplace transform of the given function using the following results:

(i) $\mathscr{L}\left\{\int_0^t F(x)dx\right\} = \frac{\mathscr{L}\{F(t)\}}{s}$

(ii) $\int_s^\infty F(s)ds = \mathscr{L}\left(\frac{f(t)}{t}\right)$

We have

$$\begin{aligned}
\mathscr{L}\{1-e^{-2t}\} &= \mathscr{L}\{1\} - \mathscr{L}\{e^{-2t}\}\\
&= \frac{1}{s}-\frac{1}{s+2}\\
\therefore\quad \mathscr{L}\left\{\frac{1-e^{-2t}}{t}\right\} &= \int_s^\infty \left(\frac{1}{s}-\frac{1}{s+2}\right)ds\\
&= [\ln(s)-\ln(s+2)]_s^\infty\\
&= -\left[\ln\left(\frac{s+2}{s}\right)\right]_s^\infty\\
&= -\left[\ln\left(1+\frac{2}{s}\right)\right]\\
&= [\ln(1)-\ln(1+2/s)] = \ln(1+2/s)\\
\therefore\quad \mathscr{L}\left[\int_0^t \left(\frac{1-e^{-2x}}{x}\right)\ dx\right] &= \frac{1}{s}\ln(1+2/s)
\end{aligned}$$

Example 86. Using Laplace transformation show that $\int_0^\infty te^{-3t}\sin t\ dt = \frac{3}{50}$.

Solution.

$$\begin{aligned}
\mathscr{L}\{\sin t\} &= \frac{1}{s^2+1}\\
\therefore\quad \mathscr{L}\{t\sin t\} &= (-1)\frac{d}{ds}\left[\frac{1}{s^2+1}\right]
\end{aligned}$$

$$= \frac{2s}{(s^2+1)^2}$$

$$i.e., \quad \int_0^\infty e^{-st} t \sin t \, dt = \frac{2s}{(s^2+1)^2}$$

Putting $p = 3$ in the above result, we get

$$\int_0^\infty e^{-3t} t \sin t \, dt = \frac{2 \cdot 3}{(3^2+1)^2} = \frac{3}{50}$$

Example 87. If $E(t) = \int_t^\infty \frac{e^{-x}}{x} \, dx$, show that $\mathscr{L}\{E(t)\} = \frac{\ln(s+1)}{s}$.

Solution. We have

$$E(t) = \int_t^\infty \frac{e^{-x}}{x} \, dx$$

$$= -\int_\infty^t \frac{e^{-x}}{x} \, dx$$

$$\therefore \quad E'(t) = -\frac{e^t}{t}$$

$$i.e., \quad tE(t) = -e^{-t}$$

$$\therefore \quad \mathscr{L}\{tE'(t)\} = -\mathscr{L}\{e^{-t}\}$$

$$i.e., \quad -\frac{d}{ds}\left[\mathscr{L}\{E'(t)\}\right] = -\frac{1}{s+1}$$

$$i.e., \quad -\frac{d}{ds}\left[s\mathscr{L}\{E(t)\} - E(0)\right] = -\frac{1}{s+1}$$

$$i.e., \quad -\frac{d}{ds}\left[s\mathscr{L}\{E(t)\}\right] = -\frac{1}{s+1} \quad (\because \ E(0) \text{ is a constant})$$

Integrating , we get:

$$s\mathscr{L}\{E(t)\} = \ln(s+1) + C \tag{3.12}$$

Taking $s \to 0$, we get:

$$\lim_{s\to 0} s\mathscr{L}\{E(t)\} = \lim_{s\to 0} \ln(s+1) + C$$

$$i.e., \quad \lim_{t\to\infty} E(t) = C$$

$$i.e., \quad 0 = C$$

Substituting the value of C in (3.12), we get:

$$s\mathscr{L}\{E(t)\} = \ln(s+1)$$

$$i.e., \quad \mathscr{L}\{E(t)\} = \frac{\ln(s+1)}{s}$$

Example 88. Find the Laplace transform of the function

$$f(t) = \begin{cases} \cos\left(t - \dfrac{2\pi}{3}\right), & t > \dfrac{2\pi}{3} \\ 0, & t < \dfrac{2\pi}{3} \end{cases}$$

Solution. We have,

$$\begin{aligned} \mathscr{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)\ dt \\ &= \int_0^{2\pi/3} e^{-st} f(t)\ dt + \int_{2\pi/3}^\infty e^{-st} f(t) dt \\ &= \int_0^{2\pi/3} e^{-st}\ 0\ \ dt + \int_{2\pi/3}^\infty e^{-st} \cos\left(t - \frac{2\pi}{3}\right) dt \\ &= \int_{2\pi/3}^\infty e^{-st} \cos\left(t - \frac{2\pi}{3}\right) dt \qquad (3.13) \end{aligned}$$

Putting $x = t - \frac{2\pi}{3}$ in equation (3.13). Then $dt = dx$. When $t = 2\pi/3$, $x = 0$ and when $t = \infty$, $x = \infty$. Hence

$$\begin{aligned} \mathscr{L}\{f(t)\} &= \int_0^\infty e^{-s(x+2\pi/3)} \cos x dx \\ &= e^{-2\pi s/3} \int_0^\infty e^{-sx} \cos x\ dx \\ &= e^{-2\pi s/3} \mathscr{L}\{\cos x\} = e^{-2\pi s/3} \frac{s}{s^2+1} \end{aligned}$$

Alitter

$$\begin{aligned} f(t) &= \cos(t - 2\pi/3) u(t - 2\pi/3) \\ \therefore \quad \mathscr{L}\{f(t)\} &= \mathscr{L}\{\cos(t - 2\pi/3) u(t - 2\pi/3)\} \\ &= e^{-2\pi/3} \mathscr{L}\{\cos t\} = \frac{e^{-2\pi/3}}{s^2+1} \end{aligned}$$

Example 89. Express the following function in terms of unit step functions and hence find its laplace transform.

$$f(t) = \begin{cases} k_1, & t < a \\ k_2, & t > a \end{cases}$$

Solution. We have

$$u((t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

$$\therefore \quad (k_2 - k_1)u((t-a) = \begin{cases} 0, & t < a \\ k_2 - k_1, & t > a \end{cases}$$

$$\therefore \quad k_1 + (k_2 - k_1)u((t-a) = \begin{cases} k_1, & t < a \\ k_2, & t > a \end{cases}$$

$$= f(t)$$

$$\therefore \quad \mathscr{L}\{f(t)\} = \mathscr{L}\{k_1 + (k_2 - k_1)u((t-a)\}$$

$$= \mathscr{L}\{k_1\} + (k_2 - k_1)\mathscr{L}\{u((t-a)\}$$

$$= \frac{k_1}{s} + (k_2 - k_1)\frac{e^{-as}}{s}$$

Example 90. Express the following function in terms of unit step functions and hence find its Laplace transform.

$$f(t) = \begin{cases} k_1, & a < t < b \\ k_2, & b < t < c \\ k_3, & c < t < d \end{cases}$$

Solution.

$$f(t) = k_1[u((t-a) - u((t-b)] + k_2[u((t-b) - u((t-c)] + k_3[u((t-c) - u((t-d)]$$

$$\therefore \quad \mathscr{L}\{f(t)\} = k_1\mathscr{L}[u((t-a) - u((t-b)] + k_2\mathscr{L}[u((t-b) - u((t-c)] + k_3\mathscr{L}[u((t-c) - u((t-d)]$$

$$= k_1\left[\frac{e^{-as}}{s} - \frac{e^{-bs}}{s}\right] + k_2\left[\frac{e^{-bs}}{s} - \frac{e^{-cs}}{c}\right] + k_3\left[\frac{e^{-cs}}{s} - \frac{e^{-ds}}{s}\right]$$

$$= \frac{1}{s}\left[k_e^{-as} + (k_2 - k_1)e^{-bs} + (k_3 - k_2)e^{-cs} - k_3e^{-ds}\right]$$

Example 91. Express the following function in terms of unit step functions and hence find its Laplace transform.

$$f(t) = \begin{cases} k_1, & t < a \\ k_2, & a < t < b \\ k_3, & t > b \end{cases}$$

Solution.

$$f(t) = k_1 - k_1u((t-a) + k_2[u((t-a) - u((t-b)] + k_3u((t-b)$$

$$\therefore \quad \mathscr{L}\{f(t)\} = \mathscr{L}\{k_1\} - k_1\mathscr{L}\{u((t-a)\} + k_2\mathscr{L}[u((t-a) - u((t-b)] + k_3\mathscr{L}\{u((t-b)\}$$

$$= \frac{k_1}{s} - k_1\frac{e^{-as}}{s} + k_2\left[\frac{e^{-as}}{s} - \frac{e^{-bs}}{s}\right] + k_3\frac{e^{-bs}}{s}$$

$$= \frac{1}{s}\left[k_1 + (k_2 - k_1)e^{-as} + (k_3 - k_2)e^{-bs}\right]$$

Example 92. Express the following function in terms of unit step functions and hence find its Laplace transform.

$$f(t) = \begin{cases} k_1, & a < t < b \\ k_2, & b < t < c \\ k_3, & t > c \end{cases}$$

Solution.

$$\begin{aligned} f(t) &= k_1[u((t-a) - u((t-b)] + k_2[u((t-b) - u((t-c)] + k_3u((t-c) \\ \therefore \quad \mathscr{L}\{f(t)\} &= k_1\mathscr{L}[u((t-a) - u((t-b)] + k_2\mathscr{L}[u((t-b) - u((t-c)] + k_3\mathscr{L}\{u((t-c)\} \\ &= k_1\left[\frac{e^{-as}}{s} - \frac{e^{-bs}}{s}\right] + k_2\left[e^{-bs}s - \frac{e^{-cs}}{s}\right] + k_3e^{-cs}s \\ &= \frac{1}{s}\left[k_1e^{-as} + (k_2 - k_1)e^{-bs} + (k_3 - k_2)e^{-cs}\right] \end{aligned}$$

Example 93. Express the following function in terms of unit step functions and hence find its Laplace transform.

$$f(t) = \begin{cases} k_1, & t < a \\ k_2, & a < t < b \\ k_3, & b < t < c \end{cases}$$

Solution.

$$\begin{aligned} f(t) &= k_1 + (k_2 - k_1)[u((t-a) - u((t-b)] + k_3[u((t-b) - u((t-c)] \\ \therefore \quad \mathscr{L}\{f(t)\} &= \mathscr{L}\{k_1\} + (k_2 - k_1)\mathscr{L}[u((t-a) - u((t-b)] + k_3\mathscr{L}[u((t-b) - u((t-c)] \\ &= \frac{k_1}{s} + (k_2 - k_1)\left[\frac{e^{-as}}{s} - \frac{e^{-bs}}{s}\right] + k_3\left[\frac{e^{-bs}}{s} - \frac{e^{-cs}}{s}\right] \\ &= \frac{k-1}{s} + (k_2 - k_1)\frac{e^{-as}}{s} + (k_3 - k_2 + k_1)\frac{e^{-bs}}{s} - k_3\frac{e^{-cs}}{s} \\ &= \frac{1}{s}\left[k - 1 + (k_2 - k_1)e^{-as} + (k_3 - k_2 + k_1)e^{-bs} - k_3e^{-cs}\right] \end{aligned}$$

Example 94. Using unit step function find the laplace transform of

$$f(t) = \begin{cases} t^2, & 0 < t < 2 \\ t - 1, & 2 < t < 3 \\ 7, & t > 3 \end{cases}$$

Solution. Using the unit step functions $u((t-0), u((t-2)$ and $u((t-3)$, we can write $f(t)$ in the following form :

$$f(t) = t^2[u((t-0) - u((t-2)] + (1-t)[u((t-2) - u((t-3)] + 7[u((t-3)]$$

$$\therefore \quad \mathscr{L}\{f(t)\} = \mathscr{L}[(u((t-0) - u((t-2))t^2] + \mathscr{L}[(u((t-2) - u((t-3))(1-t)] + 7\mathscr{L}[u((t-3)]$$

$$= t^2 u(t - (t^2 - t + 1)u((t-2) - (t-8)u((t-3) \tag{3.14}$$

Let

$$t^2 - 2t + 1 = A(t-2)^2 + B(t-2) + C$$
$$= At^2 + (-4A + B)t + (4A - 2B + C)$$

Equating the coefficients of like terms, we get:

$$A = 1, -4A + B = -1, 4A - 2B + C = 1$$

Solving for A, B and C,

$$A = 1, B = 3, C = 3$$

Hence

$$t^2 - t + 1 = (t-2)^2 + 3(t-2) + 3$$

Therefore equation (3.14) becomes:

$$f(t) = t^2 u(t - \left[(t-2)^2 - 3(t-2) + 3\right] u((t-2) - [(t-3) - 5]u((t-3)$$
$$= t^2 u(t - (t-2)^2 u((t-2) - 3(t-2)u((t-2) + 3u_{(t-2)} - (t-3)u_{(t-3)} - 5u_{(t-3)}$$

$$\therefore \quad \mathscr{L}\{f(t)\} = \mathscr{L}\{t^2 u_t\} - \mathscr{L}\{(t-2)^2 u_{(t-2)}\}$$
$$- 3\mathscr{L}\{(t-2)u_{(t-2)}\} + 3\mathscr{L}\{u_{(t-2)}\} - \mathscr{L}\{(t-3)u_{(t-3)}\} - 5\mathscr{L}\{u_{(t-3)}\}$$
$$= \frac{2}{s^3} - \frac{2e^{-2s}}{s^3} - \frac{3e^{-2s}}{s^2} - \frac{3e^{-2s}}{s} - \frac{e^{-3s}}{s^2} + \frac{5e^{-3s}}{s}$$

Example 95. Using Laplace transformation prove that $\int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$

Solution.

$$\mathscr{L}\left(\frac{\sin^2 t}{t^2}\right) = \mathscr{L}\left(\frac{1 - \cos 2t}{2t^2}\right)$$
$$= \frac{1}{2}\mathscr{L}\left(\frac{1 - \cos 2t}{t^2}\right)$$

$$= \frac{1}{2}\mathscr{L}\left[\frac{\left(\frac{1-\cos t}{t}\right)}{t}\right]$$
$$= \frac{1}{2}\int_s^\infty \mathscr{L}\left\{\frac{1-\cos 2t}{t}\right\} ds$$
$$= \frac{1}{2}\int_s^\infty \left[\int_s^\infty \mathscr{L}\{1-\cos 2t\} ds\right] ds$$
$$= \frac{1}{2}\int_s^\infty \left[\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds\right] ds$$
$$= \frac{1}{2}\int_s^\infty \left(ln(s) - \frac{1}{2}\ln(s^2+4)\right)_s^\infty ds$$
$$= \frac{1}{2}\int_s^\infty \left(\ln\left(\frac{s}{\sqrt{s^2+4}}\right)\right)_s^\infty ds$$
$$= \frac{1}{2}\int_s^\infty \left(0 - \ln\left(\frac{s}{\sqrt{s^2+4}}\right)\right) ds \qquad \left(\because \lim_{s\to\infty} \frac{s}{\sqrt{s^2+4}} = 0\right)$$
$$= -\frac{1}{2}\int_s^\infty \ln\left(\frac{s}{\sqrt{s^2+4}}\right) ds$$
$$= -\frac{1}{2}\int_s^\infty \ln\left(\frac{s}{\sqrt{s^2+4}}\right) \cdot 1\ ds$$
$$= -\frac{1}{2}\left[\left(s\ln\left(\frac{s}{\sqrt{s^2+4}}\right)\right)_s^\infty - \int_s^\infty s\left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds\right]$$

(Using integration by parts)

$$= -\frac{1}{2}\left[0 - s\ln\left(\frac{s}{\sqrt{s^2+4}}\right) - \int_s^\infty \frac{4}{s^2+4} ds\right]$$
$$= \frac{s}{2}\ln\left(\frac{s}{\sqrt{s^2+4}}\right) + \left[\tan^{-1}(s/2)\right]_s^\infty$$
$$= \frac{s}{2}\ln\left(\frac{s}{\sqrt{s^2+4}}\right) + \frac{\pi}{2} - tan^{-1}(s/2)$$
$$= -\frac{s}{2}\ln\left(\frac{\sqrt{s^2+4}}{s^2}\right) + \frac{\pi}{2} - tan^{-1}(s/2)$$
$$= -\frac{s}{2}\ln\left(\sqrt{1+4/s^2}\right) + \frac{\pi}{2} - tan^{-1}(s/2)$$
$$= -\frac{s}{4}\ln\left(1+4/s^2\right) + \frac{\pi}{2} - tan^{-1}(s/2)$$
$$= -\frac{s}{4}\left[\frac{4}{s^2} - \frac{16}{2s^4} + \cdots\right] + \frac{\pi}{2} - tan^{-1}(s/2)$$

$$i.e., \int_0^\infty e^{-st}\left(\frac{\sin^2 t}{t^2}\right) dt = \left[\frac{1}{s} - \frac{2}{s^3} + \cdots\right] +] + \frac{\pi}{2} - tan^{-1}(s/2)$$

Taking $s \to 0$ on both sides:

$$\int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$$

$$\lim_{s\to\infty} \ln\left(\frac{s}{\sqrt{s^2+4}}\right) = -\lim_{s\to\infty} \ln\left(\frac{\sqrt{s^2+4}}{s}\right) = -\lim_{s\to\infty} \ln\left(\sqrt{1+4/s^2}\right)$$
$$= -\lim_{s\to\infty} \frac{1}{2}\ln(1+4/s^2) = 0$$

Again

$$\lim_{s\to\infty} s\ln\left(\frac{s}{\sqrt{s^2+4}}\right) = -s\lim_{s\to\infty} \ln\left(\frac{\sqrt{s^2+4}}{s}\right) = -s\lim_{s\to\infty} \ln\left(\sqrt{1+4/s^2}\right)$$
$$= -\lim_{s\to\infty} \frac{s}{2}\ln(1+4/s^2) = -\lim_{s\to\infty}(s/2)\left[\frac{1}{s^2} - \frac{16}{s^4} + \cdots\right]$$
$$= \lim_{s\to\infty}\left[\frac{1}{s} - \frac{8}{s^3} + \cdots\right] = 0$$

Example 96. Let $\varphi(s)$ be the laplace transform of $f(t)$ and $\varphi^r(p)$ be the laplace transform of $f^{(r)}(t)$, show that

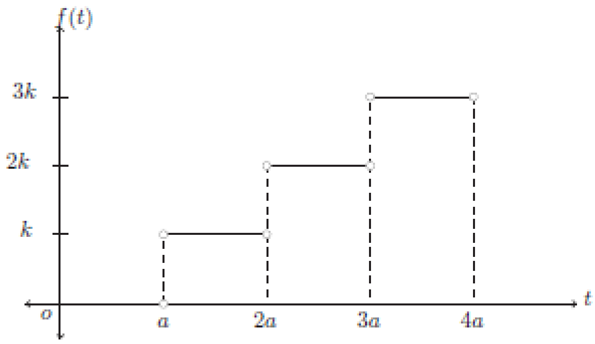
$$\varphi^r(s) = p^r\varphi(s)$$

if $f(t) = f^{(n)}(t) = 0$ at $t = 0$ for $n = 1, 2, \cdots, r-1$

Solution. We have

$$\mathscr{L}\{f^{(r)}\} = s^r\mathscr{L}\{f(t)\} - s^{r-1}f(0) - s^{r-2}f'(0) - \cdots - f^{(r-1)}(0)$$
$$= s^r\varphi(s) \quad (\because f(t) = f^{(n)}(t) = 0 \text{ at } t = 0 \text{ for } n = 1, 2, \cdots, r-1)$$

Example 97. Find the Laplace transform the function whose graph is given below:



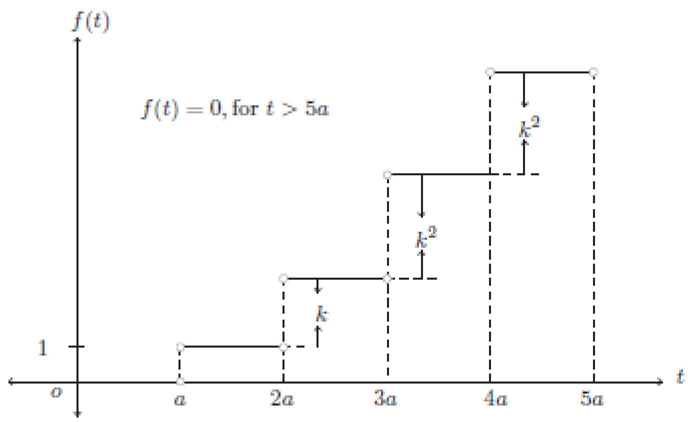
Solution. The equation of the graph in the interval $[0, \infty)$ is

$$f(t) = \begin{cases} 0, & 0 < t < a \\ k, & a < t < 2a \\ 2k, & 2a < t < 3a \\ 3k, & 3a < t < 4a \\ 0, & t > 4a \end{cases}$$

Using unit step function we can rewrite $f(t)$ as follows:

$$\begin{aligned} f(t) &= 0[u(t) - u(t-a)] + k[u(t-a) - u(t-2a)] \\ &\quad + 2k[u(t-2a) - u(t-3a)] + 3k[u(t-3a) - u(t-4a)] \\ &= ku(t-a) + ku(t-2a) + ku(t-3a) - 3ku(t-4a) \\ \therefore \quad \mathscr{L}\{f(t)\} &= k\mathscr{L}\{u(t-a)\} + k\mathscr{L}\{u(t-2a)\} + k\mathscr{L}\{u(t-3a)\} - 3k\mathscr{L}\{u(t-4a)\} \\ &= k\frac{e^{-as}}{s} + k\frac{e^{-2as}}{s} + k\frac{e^{-3as}}{s} - 3k\frac{e^{-4as}}{s} \\ &= \frac{e^{-as}k}{s}[1 + e^{-as} + e^{-2as} - 3e^{-4as}] \end{aligned}$$

Example 98. Find the Laplace transform the function whose graph is given below:



Solution. From the figure we find that

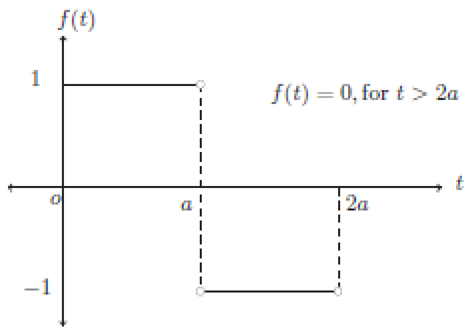
$$f(t)=\begin{cases}0, & 0\le t<a\\ 1, & a\le t<2a\\ 1+k, & 2a\le t<3a\\ 1+k+k^2, & 3a\le t<4a\\ 1+k+k^2+k^3, & 4a\le t<5a\\ 0, & t>5a\end{cases}$$

In terms of unit step function we can write $f(t)$ in the following form:

$$\begin{aligned}f(t)&=[u(t-a)-u(t-2a)]+(1+k)[u(t-2a)-u(t-3a)]+\\&\quad[u(t-3a)-u(t-4a)]+(1+k+k^2+k^3)[u(y-4a)-u(t-5a)]\\&=u(t-a)+ku(t-2a)+k^2u(t-3a)+k^3u(t-4a)\\&\quad-(1+k+k^2+k^3)u(t-5a)\end{aligned}$$

$$\begin{aligned}\therefore\quad \mathscr{L}\{f(t)\}&=\mathscr{L}\{u(t-a)\}+k\mathscr{L}\{u(t-2a)\}+k^2\mathscr{L}\{u(t-3a)\}\\&\quad+k^3\mathscr{L}\{u(t-4a)\}-(1+k+k^2+k^3)\mathscr{L}\{u(t-5a)\}\\&=\frac{e^{-as}}{s}+k\frac{e^{-2as}}{s}+k^2\frac{e^{-3as}}{s}+k^3\frac{e^{-4as}}{s}-(1+k+k^2+k^3)\frac{e^{-5as}}{s}\\&=\frac{e^{-as}}{s}\left[1+ke^{-as}+k^2e^{-2as}+k^3e^{-3as}-(1+k+k^2+k^3)e^{-4as}\right]\end{aligned}$$

Example 99. Find the Laplace transform the function whose graph is given below:



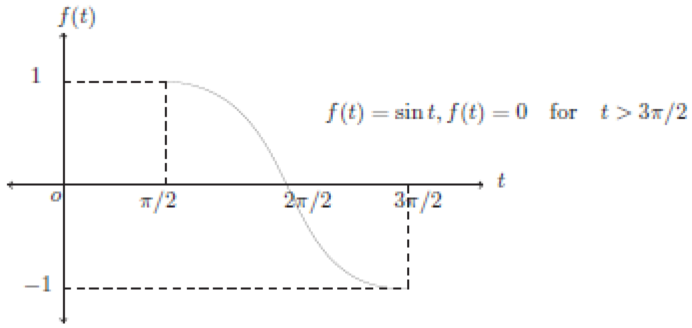
Solution. From the figure we have,

$$f(t)=\begin{cases}1, & 0\le t<a\\ -1, & a<t<2a\\ 0, & t>2a\end{cases}$$

Using unit step function we can write $f(t)$ in the following form:

$$\begin{aligned}
f(t) &= 1[u(t) - u(t-a)] + (-1)[u(t-a) - u(t-2a)] \\
&= u(t) - 2u(t-a) + u(t-2a) \\
\therefore \quad \mathscr{L}\{f(t)\} &= \mathscr{L}\{u(t)\} - 2\mathscr{L}\{u(t-a)\} + \mathscr{L}\{u(t-2a)\} \\
&= \frac{1}{s} - 2\frac{e^{-as}}{s} + \frac{e^{-2as}}{s} = \frac{1}{s}\left[1 - 2e^{-as} + e^{-2as}\right] \\
&= \frac{1}{s^2}[1 - e^{-as}]^2
\end{aligned}$$

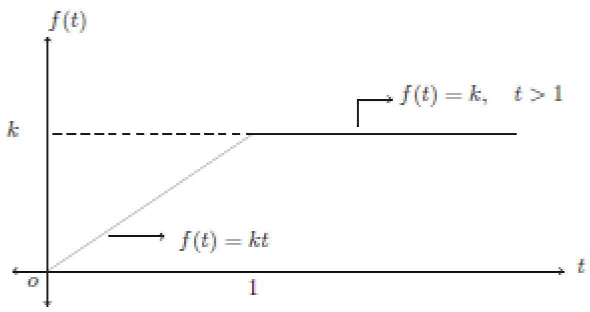
Example 100. Find the Laplace transform the function whose graph is given below:



Solution.

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)\ dt \\
&= \int_0^{\pi/2} e^{-st} f(t)\ dt + \int_{\pi/2}^{3\pi/2} e^{-st} f(t)\ dt + \int_{3\pi/2}^{\infty} e^{-st} f(t)\ dt \\
&= \int_0^{\pi/2} e^{-st}(0)\ dt + \int_{\pi/2}^{3\pi/2} e^{-st} \sin t\ dt + \int_{3\pi/2}^{\infty} e^{-st}(0)\ dt \\
&= \int_{\pi/2}^{3\pi/2} e^{-st} \sin t\ dt \\
&= \left[\frac{e^{-st}}{1+s^2}\{(-s)\sin t - \cos t\}\right]_{\pi/2}^{3\pi/2} \\
&= \frac{1}{1+s^2}\left[-se^{-3\pi s/2}\{\sin(3\pi/2) - \cos(3\pi/2)\} + e^{-\pi/s}\{s\sin(\pi/2) + \cos(\pi/2)\}\right] \\
&= \frac{1}{s^2+1}\left[se^{-3\pi s/2} + se^{-\pi s/2}\right] \\
&= \frac{se^{-\pi s/2}}{s^2+1}(e^{-\pi s} + 1)
\end{aligned}$$

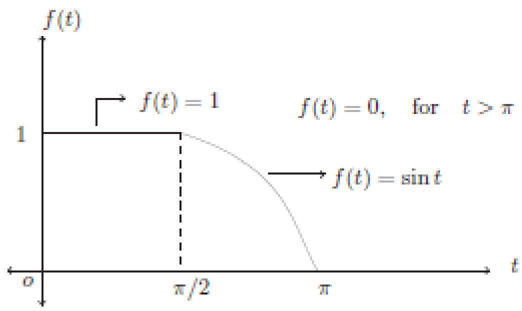
Example 101. Find the Laplace transform the function whose graph is given below:



Solution.

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)\ dt \\
&= \int_0^1 e^{-st} f(t)\ dt + \int_1^\infty e^{-st} f(t)\ dt \\
&= \int_0^1 e^{-st} kt\ dt + \int_1^\infty e^{-st} k\ dt \\
&= k\left[t\frac{e^{-st}}{(-s)} - (1)\frac{e^{-st}}{(-s)^2}\right]_0^1 + k\left[\frac{e^{-st}}{(-s)}\right]_1^{t\to\infty} \\
&= k\left[-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right] + k\left[0 + \frac{1}{s}\right] \\
&= \frac{k}{s^2}(1 - e^{-s})
\end{aligned}$$

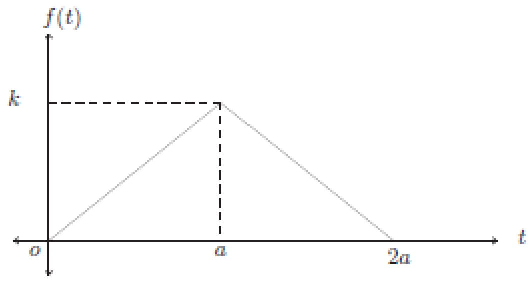
Example 102. Find the Laplace transform the function whose graph is given below:



Solution.

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)\ dt \\
&= \int_0^{\pi/2} e^{-st} f(t)\ dt + \int_{\pi/2}^{\pi} e^{-st} f(t)\ dt + \int_{\pi}^{\infty} e^{-st} f(t)\ dt \\
&= \int_0^{\pi/2} e^{-st}(1)\ dt + \int_{\pi/2}^{\pi} e^{-st} \sin t\ dt + \int_{\pi}^{\infty} e^{-st}(0)\ dt \\
&= \int_0^{\pi/2} e^{-st}(1)\ dt + \int_{\pi/2}^{\pi} e^{-st} \sin t\ dt \\
&= \left[\frac{e^{-st}}{(-s)}\right]_0^{\pi/2} + \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\right]_{\pi/2}^{\pi} \\
&= -\frac{e^{-\pi s/2}}{s} + \frac{1}{s} + \frac{e^{-s\pi}}{s^1+1} + \frac{se^{-\pi s/2}}{s^2+1} \\
&= \frac{1-e^{-\pi s/2}}{s} + \frac{e^{-s\pi} + se^{-\pi s/2}}{s^2+1}
\end{aligned}$$

Example 103. Find the Laplace transform the function whose graph is given below:



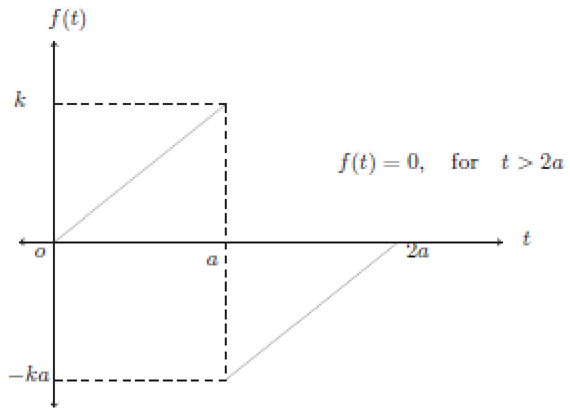
Solution. The equation of the given line in the interval $[0, a]$ is $y = \frac{k}{a}t$. Similarly equation of the line in the interval $[a, 2a]$ is $\frac{k}{a}(2a - t)$. Hence the function $f(t)$ can be written as :

$$f(t) = \begin{cases} k/at & 0 \le t \le a \\ k/a(2a-t) & a \le t \le 2a \\ 0, & t > 2a \end{cases}$$

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)\ dt \\
&= \int_0^{a} e^{-st} f(t)\ dt + \int_{a}^{2a} e^{-st} f(t)\ dt + \int_{2a}^{\infty} e^{-st} f(t)\ dt
\end{aligned}$$

$$= \int_0^a e^{-st}(kt/a)\ dt + \int_a^{2a} e^{-st}k/a(2a-t)\ dt + \int_{2a}^{\infty} e^{-st}(0)\ dt$$
$$= \int_0^a e^{-st}(kt/a)\ dt + \int_a^{2a} e^{-st}k/a(2a-t)\ dt$$
$$= \frac{k}{a}\left[t\frac{e^{-st}}{(-s)} - \frac{e^{-st}}{(-s)^2}\right]_0^a + \frac{k}{a}\left[(2a-t)\frac{e^{-st}}{(-s)} - (-1)\frac{e^{-st}}{(-s)^2}\right]_a^{2a}$$
$$= \frac{k}{a}\left[\frac{-ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \frac{ae^{-sa}}{s} - \frac{e^{-as}}{s^2}\right]$$
$$= \frac{k}{as^2}\left[1 + e^{-2as} - 2e^{-as}\right]$$

Example 104. Find the Laplace transform the function whose graph is given below:



Solution. The equation of the line in the interval $[0, a]$ is $y = kt$. Similarly equation of the line in the interval $[a, 2a]$ is $k(t - 2a)$. Hence the function is

$$f(t) = \begin{cases} kt, & 0 \le t < a \\ k(t-2a), & a < t \le 2a \\ 0, & t > 2a \end{cases}$$

$$\mathscr{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\ dt$$
$$= \int_0^a e^{-st} f(t)\ dt + \int_a^{2a} e^{-st} f(t)\ dt + \int_{2a}^{\infty} e^{-st} f(t)\ dt$$
$$= \int_0^a e^{-st}(kt)\ dt + \int_a^{2a} e^{-st}k(t-2a)\ dt + \int_{2a}^{\infty} e^{-st}(0)\ dt$$
$$= \int_0^a e^{-st}(kt)\ dt + \int_a^{2a} e^{-st}k(t-2a)\ dt$$

$$
\begin{aligned}
&= k\int_0^a e^{-st} t\ dt + k\int_a^{2a} e^{-st}(t-2a)\ dt \\
&= k\left[\left\{t\frac{e^{-st}}{(-s)} - \frac{e^{-st}}{(-s)^2}\right\}_0^a + \left\{(t-2a)\frac{e^{-st}}{(-s)} - (1)\frac{e^{-st}}{(-s0^2}\right\}_a^{2a}\right] \\
&= k\left[-\frac{e^{-as}}{s} + \frac{e^{-as}}{s^2} + \frac{1}{s^2} - \frac{e^{-2as}}{s^2} - \frac{ae^{-as}}{s} + \frac{e^{-as}}{s^2}\right] \\
&= d\left[-\frac{e^{-as}}{s} + \frac{2e^{-as}}{s^2} - \frac{e^{-2as}}{s^2} + \frac{1}{s^2}\right] \\
&= \frac{k}{s^2}\left[1 + 2e^{-as} - e^{-2as} - 2ae^{-as}\right]
\end{aligned}
$$

3.16 Inverse Laplace transforms and solution of differential equations

If $F(s)$ is the Laplace transform of a function $f(t)$, that is $\mathscr{L}\{f(t)\} = F(s)$, then $f(t)$ is called the inverse transform of $F(s)$ and is denoted:

$$f(t) = \mathscr{L}^{-1}\{F(s)\}$$

Crudely, we may think of $\mathscr{L}^{-1}$ as 'undoing' the Laplace transform operation.

Theorem 33 (Linearity Property)**.** If $F(s)$ and $G(s)$ are Laplace transforms of $f(t)$ and $g(t)$, then

$$\mathscr{L}^{-1}\{c_1F(s) + c_2G(s)\} = c_1\mathscr{L}^{-1}\{F(s)\} + c_2\mathscr{L}^{-1}\{G(s)\}$$

Proof. Since the Laplace transform operator $\mathscr{L}$ is linear, we have

$$
\begin{aligned}
\mathscr{L}\{c_1f(t) + c_2g(t)\} &= c_1\mathscr{L}\{f(t)\} + c_2\mathscr{L}\{g(t)\} \\
&= c_1F(s) + c_2G(s)
\end{aligned}
$$

Therefore by the definition of $\mathscr{L}^{-1}$, we have

$$
\begin{aligned}
c_1f(t) + c_2g(t) &= \mathscr{L}^{-1}\{c_1F(s) + c_2G(s)\} \\
\textit{i.e.,}\quad c_1\mathscr{L}^{-1}\{F(s)\} + c_2\mathscr{L}^{-1}\{G(s)\} &= \mathscr{L}^{-1}\{c_1F(s) + c_2G(s)\}
\end{aligned}
$$

□

Theorem 34 (First shifting property)**.** If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}F(s)$$

Proof. First shifting property of $\mathscr{L}$ is

$$\mathscr{L}\{e^{at}f(t)\} = F(s-a)$$

Therefore by the definition of $\mathscr{L}^{-1}$, we have

$$\begin{aligned} e^{at}f(t) &= \mathscr{L}^{-1}\{F(s-a)\} \\ i.e., \quad e^{at}\mathscr{L}^{-1}\{F(s)\} &= \mathscr{L}^{-1}\{F(s-a)\} \end{aligned}$$

□

Remark. If $F(s)$ is the laplace transform of $f(t)$, then

$$\mathscr{L}^{-1}\{F(s+a)\} = e^{-at}\mathscr{L}^{-1}\{F(s)\}$$

Theorem 35. If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$$

Proof. The second shifting property of $\mathscr{L}$ is

$$\mathscr{L}\{e^{-as}F(s) = f(t-a)u(t-a)$$

Therefore by the definition of $\mathscr{L}^{-1}$, we have

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$$

□

Theorem 36. If $\mathscr{L}\{f(t)\} = F(s)$, then $\mathscr{L}^{-1}\{F(as)\} = (1/a)f(t/a)$.

Proof. We have

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st}f(t)\ dt \\ \therefore \quad F(as) &= \int_0^\infty e^{-ast}f(t)\ dt \end{aligned}$$

Setting $x = at$. Then $dx = adt$. Also limits remains unchanged. Hence

$$F(as) = \frac{1}{a}\int_0^\infty e^{-xt}f(x/a)\ dx$$

$$= \frac{1}{a}\int_0^\infty e^{-st} f(t/a)\ dt \quad \left(\because \int_a^b f(x)dx = \int_a^b f(t)dt\right)$$
$$= \frac{1}{a}\mathscr{L}\{f(t/a)\} = \mathscr{L}\{(1/a)f(t/a)\}$$
$$\therefore \quad \mathscr{L}^{-1}\{F(as)\} = (1/a)f(t/a)$$

Theorem 37. If n is a positive integer and $F^{(n)}$ denotes the n^{th}derivative of $F(s)$, then

$$\mathscr{L}^{-1}\{F^{(n)}(s)\} = (-1)^n \mathscr{L}^{-1}\{t^n f(t)\}$$

We have

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\left[\mathscr{L}\{f(t)\}\right], \quad \text{for} \quad n = 1, 2, 3, \cdots$$
$$= (-1)^n \frac{d^n}{ds^n} F(s)$$

Then by the definition of $\mathscr{L}^{-1}$,we have

$$t^n f(t) = \mathscr{L}^{-1}\left\{(-1)^n F^{(n)}\right\}$$
$$= (-1)^n \mathscr{L}^{-1}\{F^n(s)\} \quad \left(\because \mathscr{L}^{-1} \quad \text{is linear}\right)$$
$$\therefore \quad \mathscr{L}^{-1}\{F^{(n)}(s)\} = \frac{1}{(-1)^n} t^n f(t) = \frac{(-1)^n}{(-1)^{2n}} t^n f(t) = (-1)^n t^n f(t)$$

□

Summary

(1)	$\mathscr{L}^{-1}\{c_1 F(s) + c_2 G(s)\} = c_1 \mathscr{L}^{-1}\{f(t)\} + c_2 \mathscr{L}^{-1}\{g(t)\}$
(2)	$\mathscr{L}^{-1}\{F(s \mp a)\} = e^{\pm at} \mathscr{L}^{-1}\{F(s)\}$
(3)	$\mathscr{L}^{-1}\{e^{-as} F(s)\} = f(t-a)u(t-a)$
(4)	$\mathscr{L}^{-1}\{F(as)\} = (1/a)f(t/a)$

3.17 Inverse transforms of simple functions

Example 105. Find the following inverse Laplace transforms.

$$(a)\mathscr{L}^{-1}\left\{\frac{1}{s^2+9}\right\} \qquad (b)\mathscr{L}^{-1}\left\{\frac{5}{3s-1}\right\}$$

Solution. (a) We have

$$\mathscr{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = a\mathscr{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \sin at$$

$$\therefore \quad \mathscr{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$$

Hence $\mathscr{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\} = \frac{\sin 3t}{3}$

(b) $\mathscr{L}^{-1}\left\{\frac{5}{3s-1}\right\} = \mathscr{L}^{-1}\left\{\frac{5}{3\,(s-1/3)}\right\} = \frac{5}{3}\mathscr{L}^{-1}\left\{\frac{1}{(s-1/3)}\right\} = \frac{5}{3}e^{1/3t}$

Example 106. Find the following inverse Laplace transforms:

$$(a)\mathscr{L}^{-1}\left\{\frac{6}{s^3}\right\} \qquad (b)\mathscr{L}^{-1}\left\{\frac{3}{s^4}\right\}$$

Solution. (a) We have $\mathscr{L}^{-1}\left\{\frac{2}{s^3}\right\} = t^2$. Hence

$$\mathscr{L}^{-1}\left\{\frac{6}{s^3}\right\} = 3\mathscr{L}^{-1}\left\{\frac{2}{s^3}\right\} = 3t^2$$

(b) We have $\mathscr{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \mathscr{L}^{-1}\left\{\frac{6}{s^4}\right\} = t^3$.

Thus $\mathscr{L}^{-1}\left\{\frac{3}{s^4}\right\} = \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{6}{s^4}\right\} = \frac{t^3}{2}$.

Example 107. Determine (a) $\mathscr{L}^{-1}\left\{\frac{3}{s^2-4s+13}\right\}$ (b)$\mathscr{L}^{-1}\left\{\frac{2(s+1)}{s^2+2s+10}\right\}$.

Solution.

$$\mathscr{L}^{-1}\left\{\frac{3}{s^2-4s+13}\right\} = \mathscr{L}^{-1}\left\{\frac{3}{(s-2)^2+3^2}\right\} = e^{2t}\sin 3t$$

$$\mathscr{L}^{-1}\left\{\frac{2(s+1)}{s^2+2s+10}\right\} = \mathscr{L}^{-1}\left\{\frac{2(s+1)}{(s+1)^2+3^2}\right\}$$

$$= 2\mathscr{L}^{-1}\left\{\frac{(s+1)}{(s+1)^2+3^2}\right\} = 2e^{-t}\cos 3t$$

Example 108. Determine (a) $\mathscr{L}^{-1}\left\{\frac{5}{s^2+2s-3}\right\}$ (b)$\mathscr{L}^{-1}\left\{\frac{4s-3}{s^2-4s-5}\right\}$

Solution.

$$(a) \qquad \mathscr{L}^{-1}\left\{\frac{5}{s^2+2s-3}\right\} = \mathscr{L}^{-1}\left\{\frac{5}{(s+1)^2-2^2}\right\}$$
$$= \frac{5}{2}\mathscr{L}^{-1}\left\{\frac{2}{(s+1)^2-2^2}\right\} = \frac{5}{2}e^{-t}\sinh 2t$$

$$(b) \qquad \mathscr{L}^{-1}\left\{\frac{4s-3}{s^2-4s-5}\right\} = \mathscr{L}^{-1}\left\{\frac{4s-3}{(s-2)^2-3^2}\right\}$$
$$= \mathscr{L}^{-1}\left\{\frac{4(s-2)+5}{(s-2)^2-3^2}\right\}$$
$$= 4\mathscr{L}^{-1}\left\{\frac{s-2}{(s-2)^2-3^2}\right\} + 5\mathscr{L}^{-1}\left\{\frac{1}{(s-2)^2-3^2}\right\}$$
$$= 4e^{2t}\cosh 3t + \frac{5}{3}\mathscr{L}^{-1}\left\{\frac{3}{(s-2)^2-3^2}\right\}$$
$$= 4e^{2t}\cosh 3t + \frac{5}{3}e^{2t}\sinh 3t$$

3.18 Inverse Laplace transform using partial fractions

Some times the function whose inverse is required is not in the standard type. In such cases we may split the fraction into several fractions, by using *partial fractions*. These simpler fractions can be easily inverted.

Partial fractions are discussed in the appendix of this book .

Example 109. Determine $\mathscr{L}^{-1}\left\{\frac{4s-5}{s^2-s-2}\right\}$.

Solution. We first resolve $\frac{4s-5}{s^2-s-2}$ into partial fractions. By cover up rule, we have

$$\frac{4s-5}{s^2-s-2} = \frac{4s-5}{(s-2)(s+1)} = \frac{1}{(s-2)} + \frac{3}{(s+1)}$$

Hence

$$\mathscr{L}^{-1}\left\{\frac{4s-5}{s^2-s-2}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{(s-2)} + \frac{3}{(s+1)}\right\}$$
$$= \mathscr{L}^{-1}\left\{\frac{1}{(s-2)}\right\} + \mathscr{L}^{-1}\left\{\frac{3}{(s+1)}\right\}$$
$$= e^{2t} + 3e^{-t}$$

Example 110. Determine $\mathscr{L}^{-1}\left\{\frac{9s^2+4s-10}{s(s-1)(s+2)}\right\}$.

Solution. By cover up rule,

$$\left\{\frac{9s^2+4s-10}{s(s-1)(s+2)}\right\} = \frac{5}{s} + \frac{1}{(s-1)} + \frac{3}{(s-2)}$$

$$\therefore \quad \mathscr{L}^{-1}\left\{\frac{9s^2+4s-10}{s(s-1)(s+2)}\right\} = \mathscr{L}^{-1}\left\{\frac{5}{s}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{(s-1)}\right\} + \mathscr{L}^{-1}\left\{\frac{3}{(s-2)}\right\}$$

$$= 5 + e^t + 3e^{-2t}$$

Example 111. Determine $\mathscr{L}^{-1}\left\{\frac{3s^3+s^2+12s+2}{(s-3)(s+1)^2}\right\}$.

Solution. We first resolve the given fraction into partial fractions:

$$\left\{\frac{3s^3+s^2+12s+2}{(s-3)(s+1)^2}\right\} = \frac{A}{s-3} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3}$$

$$= \frac{2}{s-3} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{3}{(s+1)^3} \quad \text{(by cover up rule)}$$

Multiplying both sides by $(s-3)(s+1)^3$, we get:

$$3s^3+s^2+12s+2 = 2(s+1)^3 + B(s-3)(s+1)^2 + c(s-3)(s+1) + 3(s-3)$$

Equating s^3 terms gives : $3 = 2 + B$, from which, $B = 1$

Equating constant term gives: $2 = 2 - 3B - 3C - 9$, from which $C = -4$

Hence

$$\left\{\frac{3s^3+s^2+12s+2}{(s-3)(s+1)^2}\right\} = \frac{2}{s-3} + \frac{1}{s+1} + \frac{-4}{(s+1)^2} + \frac{3}{(s+1)^3}$$

$$\therefore \mathscr{L}^{-1}\left\{\frac{3s^3+s^2+12s+2}{(s-3)(s+1)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{2}{s-3}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$+ \mathscr{L}^{-1}\left\{\frac{-4}{(s+1)^2}\right\} + \mathscr{L}^{-1}\left\{\frac{3}{(s+1)^3}\right\}$$

$$= 2e^{3t} + e^{-t} - 4te^{-t} + \frac{3}{2}t^2e^{-t}$$

Example 112. Determine $\mathscr{L}^{-1}\left\{\frac{5s^2+8s-1}{(s+3)(s^2+1)}\right\}$.

Solution.

$$\left\{\frac{5s^2+8s-1}{(s+3)(s^2+1)}\right\} = \frac{2}{s+3} + \frac{Bs+C}{s^2+1}$$

Multiplying both sides by $(s+3)(s^2+1)$, we get:

$$5s^2+8s-1 = 2(s^2+1) + (Bs+C)(s+3)$$

When $s = 0$, $-1 = 2 + 3C$, from which, $C = -1$

Equating s^2 terms gives: $5 = 2 + B$, from which, $B = 3$

Hence

$$\begin{aligned}
\left\{\frac{5s^2+8s-1}{(s+3)(s^2+1)}\right\} &= \frac{2}{s+3} + \frac{3s-1}{s^2+1} \\
&= \frac{2}{s+3} + \frac{3s}{s^2+1} - \frac{1}{s^2+1} \\
\therefore \quad \mathscr{L}^{-1}\left\{\frac{5s^2+8s-1}{(s+3)(s^2+1)}\right\} &= \mathscr{L}^{-1}\left\{\frac{2}{s+3}\right\} + \mathscr{L}^{-1}\left\{\frac{3s}{s^2+1}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\
&= 2e^{-3t} + 3\cos t - \sin t
\end{aligned}$$

Example 113. Determine $\mathscr{L}^{-1}\left\{\frac{7s+13}{s(s^2+4s+13)}\right\}$.

Solution.

$$\frac{7s+13}{s(s^2+4s+13)} = \frac{1}{s} + \frac{Bs+C}{s^2+4s+13}\text{(by cover up rule)}$$

Hence

$$7s + 13 = (s^2 + 4s + 13) + Bs + Cs$$

Equating s^2 terms gives: $0 = 1 + B$, from which $B = -1$

Equating s terms gives: $7 = 4 + C$, from which $C = 3$

Hence

$$\begin{aligned}
\frac{7s+13}{s(s^2+4s+13)} &= \frac{1}{s} + \frac{-s+3}{s^2+4s+13} \\
&= \frac{1}{s} + \frac{-(s-2)+5}{(s+2)^2+3^2)} \\
&= \frac{1}{s} - \frac{(s-2)}{(s+2)^2+3^2)} + \frac{5}{(s+2)^2+3^2)} \\
\therefore \quad \mathscr{L}^{-1}\left\{\frac{7s+13}{s(s^2+4s+13)}\right\} &= \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \mathscr{L}^{-1}\left\{\frac{(s-2)}{(s+2)^2+3^2)}\right\} \\
&\quad + \mathscr{L}^{-1}\left\{\frac{5}{(s+2)^2+3^2)}\right\} \\
&= 1 - e^{-2t}\cos 3t + \frac{5}{3}e^{-2t}\sin 3t
\end{aligned}$$

Example 114. Prove that $\int_0^\infty \frac{\cos xt}{1+t^2}dt = \frac{\pi}{2}e^{-x}$

Solution. Let $C = \int_0^\infty \frac{\cos xt}{1+t^2} dt$ and $S = \int_0^\infty \frac{\sin xt}{1+t^2} dt$. Then

$$\begin{aligned}
C + iS &= \int_0^\infty \frac{\cos xt + i \sin xt}{1+t^2} dt \\
&= \int_0^\infty \frac{e^{ixt}}{1+t^2} dt \\
\therefore \quad \mathscr{L}\{C+iS\} &= \int_0^\infty e^{-sx} \left\{ \int_0^\infty \frac{e^{ixt}}{1+t^2} dt \right\} dx \\
&= \int_{x=0}^\infty \int_{t=0}^\infty \frac{e^{-sx+ixt}}{1+t^2} dt dx \\
&= \int_{t=0}^\infty \left[\int_{x=0}^\infty \frac{e^{-sx+ixt}}{1+t^2} dx \right] dt \\
&= \int_{t=0}^\infty \frac{1}{1+t^2} \left[\frac{e^{-s+it}}{-s+it} \right]_{x=0}^\infty dt \\
&= \int_{t=0}^\infty \frac{1}{1+t^2} \frac{1}{s-it} dt \\
&= \int_{t=0}^\infty \frac{s+it}{(1+t^2)(s^2+t^2)} dt
\end{aligned}$$

Equating real parts, we get:

$$\begin{aligned}
\mathscr{L}\{C\} &= \int_0^\infty \frac{s}{(1+t^2)(s^2+t^2)} dt \\
&= s \int_0^\infty \frac{1}{(1+t^2)(s^2+t^2)} dt \\
&= s \int_0^\infty \left[\frac{1/(s^2-1)}{1+t^2} + \frac{1/(1-s^2)}{s^2+t^2} \right] dt \\
&= \frac{s}{s^2-1} \int_0^\infty \left[\frac{1}{1+t^2} - \frac{1}{t^2+s^2} \right] dt \\
&= \frac{s}{s^2-1} \left[tan^{-1} t - \frac{1}{s} tan^{-1}(t/s) \right]_0^\infty \\
&= \frac{s}{s^2-1} \left[\frac{\pi}{2} - \frac{1}{s} \frac{\pi}{2} \right] \\
&= \frac{s}{s^2-1} \frac{(s-1)\pi}{2s} = \frac{1}{s+1} \frac{\pi}{2} \\
\therefore \quad C &= \mathscr{L}^{-1} \left[\frac{1}{s+1} \frac{\pi}{2} \right] = \frac{\pi}{2} \mathscr{L}^{-1} \left(\frac{1}{s+1} \right) = \frac{\pi}{2} e^{-x}
\end{aligned}$$

3.18.1 Convolution operation

Definition. Let the functions $f(t)$ and $g(t)$ be defined for $t \geq 0$. Then the *convolution* of the functions f and g is denoted by $(f * g)(t)$ and defined as

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\ d\tau$$

Theorem 38. Convolution operation is commutative, that is, $(f * g) = (g * f)$.

Proof. By definition,

$$\begin{aligned}(f * g)(t) &= \int_0^t f(\tau)g(t-\tau)\ d\tau \\ &= \int_0^t f(t-\tau)g[t-(t-\tau)]d\tau \quad (\because \int_0^a f(t)dt = \int_0^a f(a-t)dt) \\ &= \int_0^t f(t-\tau)g(\tau)d\tau = (g * f)(t)\end{aligned}$$

$therefore \qquad (f * g) = (g * f)$

□

Example 115. Find $(t^2 * \cos t)$.

Solution. We have

$$\begin{aligned}(t^2 * \cos t) &= \int_0^t \tau^2 \cos(t-\tau)d\tau \\ &= \int_0^t \tau^2[\cos t \cos\tau + \sin t \sin\tau]d\tau \\ &= \cos t \int_0^t \tau^2 \cos\tau d\tau + \sin t \int_0^t \tau^2 \sin\tau d\tau \\ &= \cos t \left[(\tau^2)(\sin\tau) - (2\tau)(-\cos\tau) + (2)(-\sin\tau)\right]_0^t + \\ &\qquad \sin t \left[(\tau^2)(-\cos\tau) - (2\tau)(-\sin\tau) + (2)(\cos\tau)\right]_0^t \\ &= 2(t - \sin t)\end{aligned}$$

Theorem 39 (Convolution theorem)**.** Let $\mathscr{L}\{f(t)\} = F(s)$ and $\mathscr{L}\{g(t)\} = G(s)$. Then

$$\mathscr{L}\{(f * g)(t)\} = F(s)G(s)$$

or, equivalently,

$$\mathscr{L}\left\{\int_0^t f(\tau)g(t-\tau)d\tau\right\} = F(s)G(s)$$

Conversely,

$$\mathscr{L}^{-1}\{F(s)G(s)\} = \int_0^t f(\tau)g(t-\tau)d\tau$$

Proof. From the definition of the Laplace transform,

$$\begin{aligned}
\mathscr{L}\{(f*g)(t)\} &= \int_0^\infty e^{-st}(f*g)(t)dt \\
&= \int_0^\infty e^{-st}\left[\int_0^t f(\tau)g(t-\tau)\, d\tau\right] dt \\
&= \int_0^\infty f(\tau)\left[\int_{t=\tau}^\infty e^{-st}g(t-\tau)dt\right] d\tau \\
&\quad \text{(interchanging the order of integration)} \\
&= \int_0^\infty G(s)e^{-s\tau}f(\tau)d\tau \quad \text{(by second shifting theorem)} \\
&= G(s)\int_0^\infty e^{-s\tau}f(\tau)d\tau = G(s)F(s)
\end{aligned}$$

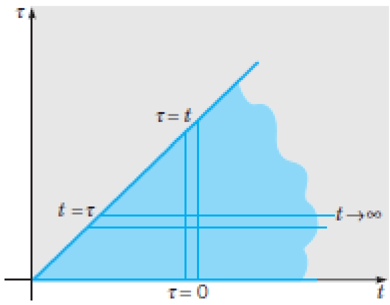


Figure 3.12: Region of integration for convolution theorem

□

3.18.2 Inverse transforms using convolution theorem

Example 116. Using convolution find $\mathscr{L}\{t^2 * \cos t\}$

Solution. We have $\mathscr{L}\{t^2\} = 2/s^2$ and $\mathscr{L}\{\cos t\} = s/(s^2+1)$. Therefore by convolution theorem

$$\mathscr{L}\{t^2 * \cos t\} = \mathscr{L}\{t^2\}\mathscr{L}\{\cos t\} = \frac{2s}{(s^2+1)}$$

Example 117. Using convolution theorem evaluate $\mathscr{L}^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)$

Solution. Writing

$$\frac{s}{(s^2+a^2)^2} = \frac{1}{(s^2+a^2)}\frac{s}{(s^2+a^2)}$$

Take $F(s) = \dfrac{1}{(s^2+a^2)}$ and $G(s) = \dfrac{s}{(s^2+a^2)}$. But then $\mathscr{L}^{-1}\{F(s)\} = (1/a)\sin at = f(t)$ and $\mathscr{L}^{-1}\{G(s)\} = \cos at = g(t)$. Then by convolution theorem,

$$\begin{aligned}
\mathscr{L}^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) &= (f*g)(t) \\
&= (1/a)\sin at * \cos at \\
&= \int_0^t f(\tau)g(t-\tau)d\tau \\
&= \int_0^t (1/a)\sin a\tau \cos a(t-\tau)d\tau \\
&= (1/a)\int_0^t \frac{1}{2}\left[\sin(a\tau+at-a\tau)+\sin(a\tau-at+a\tau)\right]d\tau \\
&= (1/a)\int_0^t \frac{1}{2}\left[\sin(at)+\sin(2a\tau-at)\right]d\tau \\
&= \frac{1}{2a}\left[\tau\sin(at)-\frac{\cos(2a\tau-at)}{2a}\right]_0^t \\
&= \frac{1}{2a}\left[t\sin(at)-\frac{\cos(2at-at)}{2a}+\frac{\cos at}{2a}\right] \\
&= \frac{1}{2a}t\sin(at)
\end{aligned}$$

Example 118. Using convolution theorem find the inverse of $\dfrac{1}{s^2(s^2-a^2)}$.

Solution. Writing

$$\frac{1}{s^2(s^2-a^2)} = \frac{1}{s^2}\frac{1}{s^2-a^2}$$

Take $F(s) = \dfrac{1}{s^2}$ and $G(s) = \dfrac{1}{s^2-a^2}$. Then $\mathscr{L}^{-1}\{F(s)\} = \mathscr{L}^{-1}(1/s^2) = t = f(t)$ and $\mathscr{L}^{-1}\{G(s)\} = \mathscr{L}^{-1}\left(\dfrac{1}{s^2-a^2}\right) = \dfrac{\sinh at}{a} = g(t)$. Then by convolution theorem

$$\begin{aligned}
\mathscr{L}^{-1}\left(\frac{1}{s^2(s^2-a^2)}\right) &= \int_0^t f(\tau)g(t-\tau)d\tau \\
&= \int_0^t \tau\frac{1}{a}\sinh a(t-\tau)d\tau \\
&= \frac{1}{a}\int_0^t \tau\sinh(at-a\tau)d\tau \\
&= \frac{1}{a}\left[(\tau)\left(-\frac{\cosh(at-a\tau)}{a}\right)-\frac{\sinh(at-a\tau)}{a^2}\right]_0^t
\end{aligned}$$

$$= \frac{1}{a}\left[-\frac{t}{a} + \frac{1}{a^2}\sinh at\right]$$
$$= \frac{1}{a^2}(-at + \sinh at)$$

Example 119. Apply convolution theorem to prove that

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(n+m)}$$

Solution. Let $f(t) = t^{m-1}$ and $g(t) = t^{m-1}$. Then $\mathscr{L}\{f(t)\} = \dfrac{\Gamma(n)}{s^n} = F(s)$ and $\mathscr{L}\{g(t)\} = \dfrac{\Gamma(m)}{s^m} = G(s)$.Therefore by convolution theorem,

$$F(s)G(s) = \mathscr{L}\left\{\int_0^t f(\tau)g(t-\tau)d\tau\right\}$$
$$i.e., \frac{\Gamma(n)}{s^n}\frac{\Gamma(m)}{s^m} = \mathscr{L}\left\{\int_0^t \tau^{n-1}(t-\tau)^{m-1}d\tau\right\}$$
$$\frac{\Gamma(n)\Gamma(m)}{s^{m+n}} = \mathscr{L}\left\{\int_0^t \tau^{n-1}(t-\tau)^{m-1}d\tau\right\}$$
$$\therefore \quad \Gamma(n)\Gamma(m)\mathscr{L}^{-1}\left(\frac{1}{s^{m+n}}\right) = \int_0^t \tau^{n-1}(t-\tau)^{m-1}d\tau$$
$$i.e., \quad \Gamma(n)\Gamma(m)\frac{t^{m+n-1}}{\Gamma(m+n)} = \int_0^t \tau^{n-1}(t-\tau)^{m-1}d\tau$$

Setting $t = 1$, we get,

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(n+m)} = \int_0^1 \tau^{n-1}(1-\tau)^{m-1}d\tau = \int_0^1 x^{n-1}(1-x)^{m-1}dx$$

Example 120. Use Convolution theorem to find $\mathscr{L}^{-1}\left\{\dfrac{s^2}{(s^2+4)^2}\right\}$

Solution. Writing

$$\frac{s^2}{(s^2+4)^2} = \frac{s}{(s^2+4)}\frac{s}{(s^2+4)}$$

Take $F(s) = \dfrac{s}{(s^2+4)}$. Then $\mathscr{L}^{-1}\{F(s\} = \cos 2t$. Therefore by convolution theorem

$$\mathscr{L}^{-1}\{F(s)F(s)\} = \int_0^t f(\tau)f(t-\tau)d\tau$$
$$= \int_0^t \cos 2\tau \cos 2(t-\tau)d\tau$$
$$= \frac{1}{2}\int_0^t \left[\cos 2t + \cos 2(t-2\tau)\right]d\tau$$
$$= \frac{1}{2}\left[\tau\cos 2t - \frac{1}{4}\cos 2(t-2\tau)\right]_{\tau=0}^{t}$$

$$= \frac{1}{2}\left[t\cos 2t - \frac{1}{4}\left(\sin(-2t) - \sin(2t)\right)\right]$$
$$= \frac{1}{2}\left[t\cos 2t + \frac{1}{2}\sin 2t\right]$$

Example 121. Apply convolution theorem to evaluate $\mathscr{L}^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$

Solution. Writing

$$\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{s^2}{(s^2+a^2)}\frac{s^2}{(s^2+b^2)}$$

Take $F(s) = \frac{s^2}{(s^2+a^2)(s^2+b^2)}$ and $G(s) = \frac{s^2}{(s^2+b^2)}$. Then

$$\mathscr{L}^{-1}\{F(s)\} = \mathscr{L}^{-1}\left[\frac{s^2}{(s^2+a^2)}\right] = \cos at = f(t)$$
$$\mathscr{L}^{-1}\{G(s)\} = \mathscr{L}^{-1}\left[\frac{s^2}{(s^2+b^2)}\right] = \cos bt = g(t)$$

Therefore by convolution theorem

$$\begin{aligned}
\mathscr{L}^{-1}\left[F(s)G(s)\right] &= \int_0^t f(\tau)g(t-\tau)d\tau \\
&= \int_0^t \cos a\tau\cos(t-\tau)d\tau \\
&= \frac{1}{2}\int_0^t [\cos((a-b)\tau+bt) + \cos(a+b)\tau - bt]d\tau \\
&= \frac{1}{2}\left[\frac{\sin((a-b)\tau+bt)}{a-b} + \frac{\sin(a+b)\tau - bt}{a+b}\right]_0^t \\
&= \frac{1}{2}\left[\frac{\sin((a-b)t+bt)}{a-b} + \frac{\sin(a+b)t - bt}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right] \\
&= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right] \\
&= \frac{1}{2}\left[\frac{\sin at - \sin bt}{a-b} + \frac{\sin at + \sin bt}{a+b}\right] \\
&= \frac{1}{2}\left[\frac{(a+b)(\sin at - \sin bt) + (a-b)(\sin at + \sin bt)}{a^2-b^2}\right] \\
&= \frac{1}{2}\left[\frac{(a+b+a-b)\sin at + (a-b-a-b)\sin bt}{a^2-b^2}\right] \\
&= \frac{1}{2}\left[\frac{a\sin at - b\sin bt}{a^2-b^2}\right]
\end{aligned}$$

3.19 Use of Laplace transform to the solution second order differential equations

Rule to solve differential equations by using Laplace transforms

1. Take the laplace transform of both sides of the differential equation.
2. Then apply the following formulae:

$$\mathscr{L}\{y'\} = s\mathscr{L}\{y\} - y(0)$$
$$\mathscr{L}\{y''\} = s^2\mathscr{L}\{y\} - sy(0) - y'(0)$$

3. Apply the given initial conditions, that is, $y(0)$ and $y'(0)$.
4. Rearrange the equation and solve for $\mathscr{L}\{y\}$
5. Apply partial fractions, if necessary
6. Take the inverse transform

Example 122. Use Laplace transformation to solve the differential equation

$$2y'' + 5y' - 3y = 0$$

given that $y(0) = 4$ and $y'(0) = 9$.

Solution. Taking the laplace transform of the differential equation we have:

$$\begin{aligned} &2\mathscr{L}\{y''\} + 5\mathscr{L}\{y'\} - 3\mathscr{L}\{y\} = \mathscr{L}\{0\} \\ &2[s^2\mathscr{L}\{y\} - sy(0) - y'(0)] + 5[s\mathscr{L}\{y\} - y(0)] - 3\mathscr{L}\{y\} = 0 \qquad (3.15) \end{aligned}$$

Putting the initial conditions $y(0) = 4$ and $y'(0) = 9$, we get:

$$2[s^2\mathscr{L}\{y\} - 4s - 9] + 5[s\mathscr{L}\{y\} - 4] - 3\mathscr{L}\{y\} = 0$$

Rearranging gives:

$$\begin{aligned} (2s^2 + 5s - 3)\mathscr{L}\{y\} &= 8s + 38 \\ \therefore \quad \mathscr{L}\{y\} &= \frac{8s + 38}{2s^2 + 5s - 3} = \frac{8s + 38}{(2s - 1)(s + 3)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{12}{2s-1} + \frac{-2}{(s+3)} \quad \text{(by cover up rule)} \\
\therefore \quad y &= 6\mathscr{L}^{-1}\left\{\frac{1}{(s-1/2)}\right\} -\} - 2\mathscr{L}^{-1}\left\{\frac{1}{(s+3)}\right\} \\
&= 6e^{x/2} - 2e^{-3x}
\end{aligned}
$$

Example 123. Solve the initial value problem

$$y'' + 3y' + 2y = \sin 2x \qquad y(0) = 2 \quad \text{and} \quad y'(0) = -1$$

Solution. Taking the Laplace of the differential equation we have:

$$
\begin{aligned}
&\mathscr{L}\{y''\} + 3\mathscr{L}\{y'\} + 2\mathscr{L}\{y\} = \mathscr{L}\{\sin 2x\} \\
&\left[s^2\mathscr{L}\{y\} - sy(0) - y'(0)\right] + 3[s\mathscr{L}\{y\} - y(0)] + 2\mathscr{L}\{y\} = \frac{2}{s^2+4}
\end{aligned}
$$

Putting $y(0) = 2$and $y'(0) = -1$, we get:

$$[s^2\mathscr{L}\{y\} - 2s + 1] + 3[s\mathscr{L}\{y\} - 2] + 2\mathscr{L}\{y\} = \frac{2}{s^2+4}$$

Rearranging gives:

$$
\begin{aligned}
(s^2+3s+2)\mathscr{L}\{y\} &= \frac{2s^3+5s^2+8s+22}{s^2+4} \\
\therefore \quad \mathscr{L}\{y\} &= \frac{2s^3+5s^2+8s+22}{(s^2+4)(s^2+3s+2)} \\
&= \frac{2s^3+5s^2+8s+22}{(s^2+4)(s+1)(s+2)} \\
\therefore \quad y &= \mathscr{L}^{-1}\left\{\frac{2s^3+5s^2+8s+22}{(s^2+4)(s+1)(s+2)}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{2s^3+5s^2+8s+22}{(s^2+4)(s+1)(s+2)} &= \frac{-5/4}{s+2} + \frac{17/5}{s+1} + \frac{As+B}{s^2+4} \\
\therefore \quad 2s^3+5s^2+8s+22 &= -\frac{5}{4}(s+1)(s^2+4) + \frac{17}{5}(s+2)(s^2+4) \\
&\quad + (As+B)(s+1)(s+2)
\end{aligned}
$$

Equating s^3 terms gives: $2 = -5/4 + 17/5 + A$, from which $A = -3/20$

Equating constant terms: $22 = -5 + 136/5 + 2B$, from which $B = -1/10$

Hence

$$y = \mathscr{L}^{-1}\left\{\frac{2s^3+5s^2+8s+22}{(s^2+4)(s+1)(s+2)}\right\}$$

$$= \mathscr{L}^{-1}\left[\frac{-5/4}{s+2} + \frac{17/5}{s+1} + \frac{(-3/20)s + (-1/10)}{s^2+4}\right]$$
$$= -\frac{5}{4}\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{17}{5}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}$$
$$- \frac{3}{20}\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{1}{20}\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$
$$= -\frac{5}{4}e^{-2x} + \frac{17}{5}e^{-x} - \frac{1}{20}\sin 2x - \frac{3}{20}\cos 2x$$

Example 124. Solve the initial value problem

$$y'' - 7y' + 10y = e^{2x} + 20 \qquad y(0) = 0 \quad \text{and} \quad y'(0) = -1/3$$

Solution. Taking Laplace transforms on both sides of the given equation, we get

$$\mathscr{L}\{y''\} - 7\mathscr{L}\{y'\} + 10\mathscr{L}\{y\} = \mathscr{L}\{e^{2x}\} + \mathscr{L}\{20\}$$
$$\textit{i.e.,} \quad [s^2\mathscr{L}\{y\} - sy(0) - y'(0)] - 7[\mathscr{L}\{y\} - y(0)] + 10\mathscr{L}\{y\} = \frac{1}{s-2} + \frac{20}{s}$$

Applying the initial conditions $y(0) = 0 \text{and} y'(0) = -1/3$, we get:

$$[s^2\mathscr{L}\{y\} + \frac{1}{3}] - 7\mathscr{L}\{y\} + 10\mathscr{L}\{y\} = \frac{21s-40}{s(s-2)}$$

Rearranging gives:

$$(s^2 - 7s + 10)\mathscr{L}\{y\} = \frac{21s-40}{s(s-2)} - \frac{1}{3}$$
$$= \frac{-s^2 + 65s - 120}{3s(s-2)}$$
$$\therefore \quad \mathscr{L}\{y\} = \frac{-s^2 + 65s - 120}{3s(s-2)(s^2 - 7s + 10)} = \frac{1}{3}\left[\frac{-s^2 + 65s - 120}{s(s-5)(s-2)^2}\right]$$
$$\therefore \quad y = \frac{1}{3}\mathscr{L}^{-1}\left[\frac{-s^2 + 65s - 120}{s(s-5)(s-2)^2}\right]$$

$$\frac{-s^2 + 65s - 120}{s(s-5)(s-2)^2} = \frac{6}{s} + \frac{4}{s-5} + \frac{C}{s-2} + \frac{-1}{(s-2)^2}$$

Hence

$$-s^2 + 65s - 120 = 6(s-5)(s-2)^2 + 4(s)(s-2)^2 + C(s)(s-5)(s-2) - 1(s)(s-5)$$

Equating s^3 terms gives: $0 = 6 + 4 + C$, from which $C = -10$

Hence

$$y = \frac{1}{3}\mathscr{L}^{-1}\left[\frac{-s^2 + 65s - 120}{s(s-5)(s-2)^2}\right]$$

$$\begin{aligned} y &= \frac{1}{3}\mathscr{L}^{-1}\left[\frac{6}{s} + \frac{4}{s-5} - \frac{10}{s-2} + \frac{-1}{(s-2)^2}\right] \\ &= \frac{1}{3}[6 + 4e^{5x} - 10e^{2x} - xe^{2x}] \\ &= 2 + \frac{4}{3}e^{5x} - \frac{10}{3}e^{2x} - \frac{x}{3}e^{2x} \end{aligned}$$

Example 125. Use Laplace transforms to solve the differential equation

$$y'' - 3y' = 9$$

given that $y(0) = 0$ and $y'(0) = 0$.

Solution. Taking Laplace transforms on both sides of the given equation, we get:

$$\mathscr{L}\{y''\} - 3\mathscr{L}\{y'\} = \mathscr{L}\{9\}$$
$$i.e., [s^2\mathscr{L}\{y\} - sy(0) - y'(0)] - 3[s\mathscr{L}\{y\} - y(0)] = \frac{9}{s}$$

Applying the initial conditions $y(0) = 0$ and $y'(0) = 0$, we get

$$s^2\mathscr{L}\{y\} - 3s\mathscr{L}\{y\} = \frac{9}{s}$$

Rearranging gives:

$$\begin{aligned} (s^2 - 3s)\mathscr{L}\{y\} &= \frac{9}{s} \\ \therefore \quad \mathscr{L}\{y\} &= \frac{9}{s(s^2-3s)} = \frac{9}{s^2(s-3)} \\ \therefore y &= 9\mathscr{L}^{-1}\left\{\frac{1}{s^2(s-3)}\right\} \end{aligned}$$

$$\frac{1}{s^2(s-3)} = \frac{-1}{s} + \frac{B}{s^2} + \frac{1}{s-3}$$

Multiplying both sides by $s^2(s-3)$, we get:

$$9 = -(s)(s-3) + B(s-3) + s^2$$

When $s = 0$, $9 = -3B$, from which $B = -3$

Hence

$$\begin{aligned} y &= 9\mathscr{L}^{-1}\left\{\frac{1}{s^2(s-3)}\right\} \\ &= 9\mathscr{L}^{-1}\left\{-\frac{1}{s} - \frac{3}{s^2} + \frac{1}{s-3}\right\} \\ &= -1 - 3x + e^{3x} \end{aligned}$$

Example 126. Use Laplace transforms to solve the differential equation:

$$y'' + 6y' + 13y = 0$$

given that $y(0) = 3$ and $y'(0) = 7$.

Solution. Taking Laplace transforms on both sides, we get:

$$\mathscr{L}\{y''\} + 6\mathscr{L}\{y'\} + 13\mathscr{L}\{y\} = \mathscr{L}\{0\}$$
$$i.e.,[s^2\mathscr{L}\{y\} - sy(0) - y'(0)] + 6[s\mathscr{L}\{y\} - y(0)] + 13\mathscr{L}\{y\} = 0$$

Putting $y(0) = 3$ and $y'(0) = 7$, we get

$$[s^2\mathscr{L}\{y\} - 3s - 7] + 6[s\mathscr{L}\{y\} - 3] + 13\mathscr{L}\{y\} = 0$$

Rearranging gives:

$$(s^2 + 6s + 13)\mathscr{L}\{y\} = 3s + 25$$
$$\therefore \quad \mathscr{L}\{y\} = \frac{3s + 25}{s^2 + 6s + 13}$$
$$\therefore \quad y = \mathscr{L}^{-1}\left[\frac{3s + 25}{s^2 + 6s + 13}\right] = \mathscr{L}^{-1}\left[\frac{3(s + 3) + 16}{(s + 3)^2 + 2^2}\right]$$
$$= 3\mathscr{L}^{-1}\left[\frac{(s + 3)}{(s + 3)^2 + 2^2}\right] + 8\mathscr{L}^{-1}\left[\frac{2}{(s + 3)^2 + 2^2}\right]$$
$$= 3e^{-3x}\cos 2x + 8e^{-3x}\sin 2x = e^{-3x}(3\cos 2x + 8\sin 2x)$$

Example 127. The current flowing in an electrical circuit is given by the differential equation

$$Ri + L\left(\frac{di}{dt}\right) = E,$$

where E, L and R are constants. Use Laplace transforms to solve the equation for current i given that when $t = 0$, $i = 0$.

Solution. Taking Laplace transforms on both sides of the equation, we get:

$$R\mathscr{L}\{i\} + L\mathscr{L}\left(\frac{di}{dt}\right) = \mathscr{L}\{E\}$$
$$i.e.,R\mathscr{L}\{i\} + L\mathscr{L}\{s\mathscr{L}\{i\} - i(0)\} = \mathscr{L}\{E\}$$

Putting $i(0) = 0$, we get:

$$
\begin{aligned}
R\mathscr{L}\{i\} + L\mathscr{L}\{s\mathscr{L}\{i\}\} &= \mathscr{L}\{E\} \\
i.e., (R + Ls)\mathscr{L}\{i\} &= \frac{E}{s} \\
\therefore \quad \mathscr{L}\{i\} &= \frac{E}{s(R + Ls)} \\
&= \frac{E/R}{s} + \frac{-EL/R}{R + Ls} \quad \text{(by cover up ryule)} \\
\therefore \quad i &= \mathscr{L}^{-1}\left[\frac{E/R}{s} + \frac{-EL/R}{R + Ls}\right] \\
&= \frac{E}{R}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{EL}{R}\mathscr{L}^{-1}\left\{\frac{1}{(R + Ls)}\right\} \\
&= \frac{E}{R}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{E}{R}\mathscr{L}^{-1}\left\{\frac{1}{(S + R/L)}\right\} \\
&= \frac{E}{R}(1) - \frac{E}{R}e^{-(R/L)t} = \frac{E}{R}\left(1 - e^{-(R/L)t}\right)
\end{aligned}
$$

CHAPTER 4

Partial Differential Equations and Fourier Series

In many important physical problems there are two or more independent variables, so the corresponding mathematical models involve partial, rather than ordinary, differential equations. This chapter treats one important method for solving partial differential equations, a method known as separation of variables. Its essential feature is the replacement of the partial differential equation by a set of ordinary differential equations, which must be solved subject to given initial or boundary conditions. The first section of this chapter deals with some basic properties of boundary value problems for ordinary differential equations. The desired solution of the partial differential equation is then expressed as a sum, usually an infinite series, formed from solutions of the ordinary differential equations. In many cases we ultimately need to deal with a series of sines and/or cosines, so part of the chapter is devoted to a discussion of such series, which are known as Fourier series.

4.1 Two Boundary Value Problems

Consider a second order differential equation

$$f(y, t, y'(t), y''(t)) = 0$$

with the initial conditions $y(t_0) = y_0, y'(t_0) = y'_0.$. Note that the conditions are specified at the same point. A differential equation with suitable initial conditions form an initial value problem.

$$\text{Initial Value Problem}(IVP)\begin{cases} f(y,t,y'(t),y''(t)) & = 0 \\ y(t_0)=y_0,\ y'(t_0) & = y_0'. \end{cases}$$

If the value of the dependent variable y or its derivative is specified at two different points, such conditions are called boundary conditions. A differential equation with suitable boundary conditions form a boundary value problem. A typical example is the differential equation

$$f(y,t,y'(t),y''(t))$$

with the boundary conditions

$$y(\alpha)=y_0, y(\beta)=y_1.$$

$$\text{Boundary Value Problem}(BVP)\begin{cases} f(y,t,y'(t),y''(t))=0 \\ y(\alpha)=y_0, y(\beta)=y_1. \end{cases}$$

Example 128. Solve the boundary value problem

$$y''+2y=0, \quad y(0)=1, y(\pi)=0.$$

Solution. The characteristic equation of the given differential equation is

$$r^2+2=0 \Rightarrow r=\pm i\sqrt{2}.$$

The general solution of the differential equation is given by

$$y=c_1\cos\sqrt{2}x+c_2\sin\sqrt{2}x.$$

The first boundary condition requires that $c_1=0$, and the second boundary condition leads to $c_2\sin\sqrt{2}\pi=0$. Since $\sin\sqrt{2}\pi\neq 0$, it follows that $c_2=0$. Consequently, $y=0$ for all x is the only solution of the problem . This example illustrates that a homogeneous boundary value problem may have only the trivial solution $y=0$.

Example 129. Solve the boundary value problem

$$y''+y=0, \quad y(0)=0, y(\pi)=0.$$

Solution. The general solution is given by

$$y = c_1 \cos x + c_2 \sin x,$$

and the first boundary condition requires that $c_1 = 0$. Since $\sin \pi = 0$, the second boundary condition is also satisfied when $c_1 = 0$, regardless of the value of c_2. Thus the solution of the problem is $y = c_2 \sin x$, where c_2 remains arbitrary. This example illustrates that a homogeneous boundary value problem may have infinitely many solutions.

Example 130. Solve the boundary value problem

$$y'' + 2y = 0, y(0) = 1, y(\pi) = 0.$$

Solution. The general solution of the differential equation is

$$y = c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x.$$

The first boundary condition requires that $c_1 = 1$. The second boundary condition implies that $c_1 \cos \sqrt{2}\pi + c_2 \sin \sqrt{2}\pi = 0$, so $c_2 = -\cot \sqrt{2}\pi$ Thus the solution of the boundary value problem is

$$y = \cos \sqrt{2}x - \cot \sqrt{2}\pi \sin \sqrt{2}x$$

This example illustrates the case of a nonhomogeneous boundary value problem with a unique solution.

Definition. Consider the problem consisting of the differential equation

$$y'' + \lambda y = 0, \tag{4.1}$$

together with the boundary conditions

$$y(0) = 0, y(L) = 0. \tag{4.2}$$

The values of λ for which nontrivial solutions of (4.1), (4.2) occur are called eigenvalues, and the nontrivial solutions themselves are called eigenfunctions.

Example 131. Find the eigenvalues and the corresponding eigenfunctions of the boundary value problem:

$$y'' + \lambda y = 0, \quad y(0) = 0, y(\pi) = 0.$$

Solution. We need to consider separately the cases $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$, since the form of the solution of the given equation is different in each of these cases.

Case 1: Suppose first that $\lambda > 0$. To avoid the frequent appearance of radical signs, it is convenient to let $\lambda = \mu^2$ and to write the given equation as

$$y'' + \mu^2 y = 0. \tag{4.3}$$

The characteristic polynomial equation is $r^2 + \mu^2 = 0$ with roots $r = \pm i\mu$, so the general solution is

$$y = c_1 \cos \mu x + c_2 \sin \mu x. \tag{4.4}$$

Note that μ is nonzero (since $\lambda > 0$) and there is no loss of generality if we also assume that μ is positive. The first boundary condition requires that $c_1 = 0$, and then the second boundary condition reduces to

$$c_2 \sin \mu\pi = 0.$$

We are seeking nontrivial solutions so we must require that $c_2 \neq 0$. Consequently, $\sin \mu\pi$ must be zero. We know that $\sin \mu\pi = 0$ if and only if $\mu\pi = n\pi, \quad n = 0, 1, 2, \ldots$. This implies that $\mu = n$. This implies that

$$\lambda = n^2, n = 1, 2, \ldots (\because \lambda \neq 0)$$

Therefore the eigenvalues are

$$\lambda_1 = 1, \lambda_2 = 4, \lambda_3 = 9, \ldots.$$

The corresponding eigenfunctions are given by:

$$y = c_2 \sin nx, \ n = 1, 2, \ldots. \tag{4.5}$$

We will usually choose the multiplicative constant to be 1 and write the eigenfunctions as

$$y_1(x) = \sin x, \ y_2(x) = \sin 2x, \ \ldots, \ y_n(x) = \sin nx, \ldots,$$

remembering that multiples of these functions are also eigenfunctions.

Case 2: Now let us suppose that $\lambda < 0$. If we let $\lambda = -\mu^2$, then the differential equation becomes

$$y'' - \mu^2 y = 0. \tag{4.6}$$

The characteristic equation for(4.6) is

$$r^2 - \mu^2 = 0$$

with roots $r = \pm\mu$, so its general solution can be written as

$$y = c_1 e^{\mu x} + c_2 e^{-\mu x}.$$

The constant c_1 can be determined from the boundary condition $y(0) = 0$;

$$c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

The constant c_2 can be determined from the boundary condition $y(\pi) = 0$;

$$c_1 e^{\mu\pi} + c_2 e^{-\mu\pi} = 0 \Rightarrow c_1(e^{\mu\pi} - e^{-\mu\pi}) = 0 \Rightarrow c_1 \sinh \mu\pi = 0 \Rightarrow c_1 = 0$$

This implies that $c_2 = 0$. Consequently, $y = 0$ and there are no nontrivial solutions for $\lambda < 0$. In other words, the given boundary value problem has no negative eigenvalues.

Case 3: Finally, consider the possibility that $\lambda = 0$. Then given equation becomes

$$y'' = 0$$

and its general solution is

$$y = c_1 x + c_2.$$

Applying the boundary conditions, we get $c_1 = c_2 = 0$. Hence $y = 0$. This implies that $\lambda = 0$ is not an eigenvalue.

Example 132. Find the eigenvalues and the corresponding eigenfunction of the boundary value problem:

$$y'' + \lambda y = 0, \quad y(0) = 0, y(L) = 0.$$

Solution. The solution process is exactly the same as example 131. The eigenvalues and eigenvectors are given by

$$\lambda_n = \frac{n\pi x}{L}, \quad y_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots$$

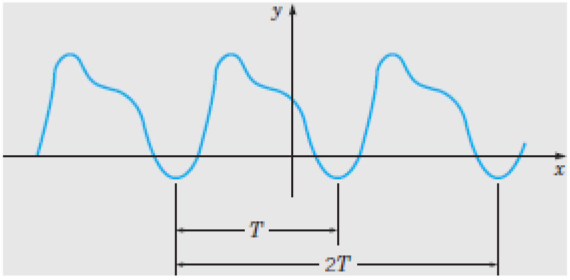


Figure 4.1: A periodic function.

4.2 Fourier Series

Definition. Consider a series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + a_n \sin\left(\frac{n\pi x}{L}\right)\right) \tag{4.7}$$

On the set of points where the series (4.7) converges, it defines a function f, whose value at each point is the sum of the series for that value of x. In this case the series (4.7) is said to be the Fourier series for f.

Definition. A function f is said to be periodic with period $T > 0$ if the domain of f contains $x + T$ whenever it contains x, and if

$$f(x + T) = f(x) \tag{4.8}$$

for every value of x.

An example of a periodic function is shown in Figure 4.1. It follows immediately from the definition that if T is a period of f , then $2T$ is also a period, and so indeed is any integral multiple of T. The smallest value of T for which equation (4.8) holds is called the fundamental period of f . A constant function is a periodic function with an arbitrary period but no fundamental period.

Remarks.

1. If f and g are any two periodic functions with common period T, then their product fg and any linear combination $c_1f + c_2g$ are also periodic with period T.

2. The sum of any finite number, or even the sum of a convergent infinite series, of functions of period T is also periodic with period T.

3. If the period of $f(x)$ is T, then the period of $f(ax)$ is T/a.

4.2.1 Orthogonality of the Sine and Cosine Functions

Definition. The standard inner product $< u, v >$ of two real-valued functions u and v on the interval $[\alpha, \beta]$ is defined by

$$< u, v >= \int_{\alpha}^{\beta} u(x)v(x)dx$$

The functions u and v are said to be orthogonal on $[\alpha, \beta]$ if their inner product is zero, that is, if

$$\int_{\alpha}^{\beta} u(x)v(x)dx = 0$$

A set of functions is said to be mutually orthogonal if each distinct pair of functions in the set is orthogonal.

The functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L), m = 1, 2, \dots$ form a mutually orthogonal set of functions on the interval $-L < x < L$. In fact, they satisfy the following orthogonality relations:

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if} m = n \end{cases} \tag{4.9}$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases} \tag{4.10}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) = 0, \text{for all } m, n \tag{4.11}$$

4.2.2 The Euler Fourier Formulas

Now let us suppose that a series of the form (4.7) converges, and let us call its sum $f(x)$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + a_n \sin\left(\frac{n\pi x}{L}\right)\right) \tag{4.12}$$

The coefficients a_n and b_n can be related to $f(x)$ as a consequence of the orthogonality conditions.

To determine a_n, taking the inner product with respect to $\cos\left(\frac{n\pi x}{L}\right)$t:

$$\left\langle f, \cos\left(\frac{n\pi x}{L}\right)\right\rangle = a_n L$$

Therefore

$$
\begin{aligned}
a_n &= \frac{1}{L}\left\langle f(x), \cos\left(\frac{n\pi x}{L}\right)\right\rangle \\
&= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, n = 1, 2, 3, \ldots
\end{aligned}
$$

Hence

$$
a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, n = 1, 2, 3, \ldots \tag{4.13}
$$

To determine a_0, take the inner product $< f(x), 1 >$:

$$
< f(x), 1 >= \langle a_0/2, 1\rangle
$$

That is,

$$
\int_{-L}^{L} f(x)dx = a_0 L
$$

Therefore

$$
a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx \tag{4.14}
$$

To determine b_n, the take inner product $\left\langle f(x), \sin\left(\frac{n\pi x}{L}\right)\right\rangle$:

$$
\begin{aligned}
b_n &= \frac{1}{L}\left\langle f(x), \sin\left(\frac{n\pi x}{L}\right)\right\rangle \\
&= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, n = 1, 2, 3, \ldots
\end{aligned}
$$

Hence

$$
b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, n = 1, 2, 3, \ldots \tag{4.15}
$$

4.2.3 Even and Odd Functions

Definition. A function f is an even function if its domain contains the point $-x$ whenever it contains the point x, and if

$$
f(-x) = f(x)
$$

for each x in the domain of f . Similarly, f is an odd function if its domain contains $-$x whenever it contains x, and if

$$
f(-x) = -f(x)
$$

for each x in the domain of f .

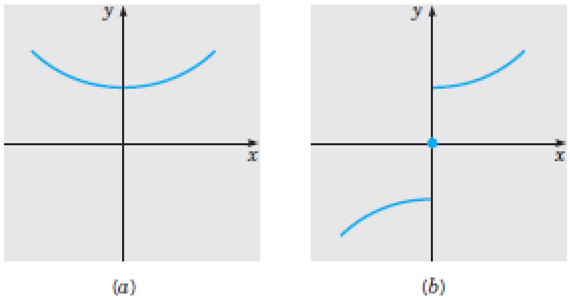


Figure 4.2: (a) An even function. (b) An odd function.

Remarks.

1. Even functions are symmetrical about the y-axis.
2. Odd functions are symmetrical about the origin.
3. The sum (difference) and product (quotient) of two even functions are even.
4. The sum (difference) of two odd functions is odd; the product (quotient) of two odd functions is even.
5. The sum (difference) of an odd function and an even function is neither even nor odd; the product (quotient) of two such functions is odd.
6. If f is an even function, then
$$\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx.$$
7. If f is an odd function, then
$$\int_{-L}^{L} f(x)dx = 0$$

Example 133. Assume that there is a Fourier series converging to the function f defined by

$$f(x) = \begin{cases} -x & -2 \leq x < 0, \\ x & 0 \leq x < 2 \end{cases} \tag{4.16}$$

$$f(x+4) = f(x).$$

Find the coefficients in the Fourier series for f .

Solution. Here $L = 2$, and the Fourier series has the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{2}\right) + a_n \sin\left(\frac{n\pi x}{2}\right)\right) \tag{4.17}$$

Note that the given function can be written as:

$$f(x) = |x|, \quad , -2 \leq x \leq 2, f(x+4) = f(x).$$

Clearly $f(x)$ is an even function.

The Fourier Coefficient a_0

$$\begin{aligned} a_0 &= \frac{1}{2}\int_{-2}^{2} f(x)dx \\ &= \frac{1}{2}\int_{-2}^{2} |x|dx \\ &= \int_{0}^{2} xdx = 2. \end{aligned}$$

The Fourier Coefficient a_n

$$\begin{aligned} a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx \\ &= \frac{1}{2}\int_{-2}^{2} |x|\cos\left(\frac{n\pi x}{2}\right)dx \\ &= \int_{0}^{2} x\cos\left(\frac{n\pi x}{2}\right)dx = \\ &= \left[x\left(\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right) - \left(\frac{-\cos\left(\frac{n\pi x}{2}\right)}{(\frac{n\pi}{2})^2}\right)\right]_0^2 \\ &= -\frac{4}{n^2\pi^2}[(-1)^n - 1] \\ &= \begin{cases} -\frac{8}{n^2\pi^2} & \text{if } n \text{ odd} \\ & 0 \text{ if } n \text{ even} \end{cases} \end{aligned}$$

The Fourier Coefficient b_n

$$\begin{aligned} a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx \\ &= 0(\because \text{the integrand is odd}) \end{aligned}$$

Figure 4.3: A piecewise continuous function

By substituting the Fourier coefficients in the series (4.17), we obtain the Fourier series for f :

$$f(x) = 1 - \frac{8}{\pi^2}\left[\cos\left(\frac{\pi x}{2}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{2}\right) + \cdots\right]$$

4.2.4 The Fourier Convergence Theorem

Before stating a convergence theorem for Fourier series, we define a term that appears in the theorem.

Definition. A function f is said to be piecewise continuous on an interval $a < x < b$ if the interval can be partitioned by a finite number of points

$$a = x_0 < x_1 < \cdots < x_n = b$$

so that

1. f is continuous on each open subinterval $x_{i-1} < x < x_i$.
2. f approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval.

The notation $f(c+)$ is used to denote the limit of $f(x)$ as $x \to c$ from the right; similarly, $f(c-)$ denotes the limit of $f(x)$ as x approaches c from the left.

Theorem 40. Suppose that f and f' are piecewise continuous on the interval $-L < x < L$. Further, suppose that f is defined outside the interval $-L < x < L$ so that it is periodic with period $2L$. Then f has a Fourier series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_n \cos\left(\frac{m\pi x}{L}\right) + a_n \sin\left(\frac{m\pi x}{L}\right)\right) \tag{4.18}$$

whose coefficients are given by:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx, n = 0, 1, 2, 3, \ldots$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx, n = 1, 2, 3, \ldots$$

The Fourier series converges to $f(x)$ at all points where f is continuous, and to $[f(x+) + f(x-)]/2$ at all points where f is discontinuous.

4.2.5 Fourier Sine and Cosine Series

Suppose that f and f' are piecewise continuous on $-L < x < L$ and that f is an odd periodic function of period $2L$. Then it follows that $f(x)\cos(n\pi x/L)$ is odd and $f(x)\sin(n\pi x/L)$ is even. In this case the Fourier coefficients of f are

$$a_n = 0, n = 0, 1, 2, 3, \ldots$$
$$b_n = \frac{2}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, n = 1, 2, 3, \ldots$$

and the Fourier series for f is of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{4.19}$$

Thus the Fourier series for any odd function consists only of the odd trigonometric functions $\sin(n\pi x/L)$; such a series is called a Fourier sine series. Again observe that only half of the coefficients need to be calculated by integration, since each a_n, for $n = 0, 1, 2, \ldots$, is zero for any odd function.

Suppose that f and f' are piecewise continuous on $-L < x < L$ and that f is an even periodic function with period $2L$. Then it follows that $f(x)cos(n\pi x/L)$ is even and $f(x)sin(n\pi x/L)$ is odd. In this case the Fourier coefficients of f are

$$a_n = \frac{2}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, n = 0, 1, 2, 3, \ldots$$
$$b_n = 0, n = 1, 2, 3, \ldots$$

and the Fourier series for f is of the form

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \tag{4.20}$$

In other words, the Fourier series of any even function consists only of the even trigonometric functions $cos(n\pi x/L)$ and the constant term; it is natural to call such a series a Fourier cosine series. From a computational point of view, observe that only. the coefficients a_n, for $n = 0, 1, 2, \dots$, need to be calculated from the integral formula. Each of the b_n, for $n = 1, 2, \dots$, is automatically zero for any even function and so does not need to be calculated by integration.

4.3 Partial Differential Equations

Definition. If the dependent varaible u is a function of more one independent variable, say $x_1, x_2, \dots, x_n$, an equation involving the variables $x_1, x_2, \dots, x_n, u$ and the partial derivatives of u with respect to $x_1, x_2, \dots, x_n$ is called a Partial Differential equation(PDE).

For example

$$u_{xx} + u_{yy} = 0,$$
$$xu_x + yu_y = 2u,$$
$$u_{xy} + u_x + u_y = e^{x+y}$$

4.4 Method of Separation of Variables

The method of separation of variables is illustrated through a few examples.

Example 134. Solve $xu_x - yu_y + 2u = 0$.

Solution. The solution of the partial differential equation $u(x, y)$ is a function of x and y. Apply the method of separation of variables by assuming

$$u(x, y) = X(x)Y(y),$$

i.e., consider $u(x, y)$ as the product of a function of x and a function of y, hence the name "separation of variables". Substituting into the differential equation yields

$$xX'(x)Y(y) - yX(x)Y'(y) + 2X(x)Y(y) = 0$$

Dividing the equation XY leads to:

$$x\frac{X'Y}{XY} - y\frac{XY'}{XY} + 2\frac{XY}{XY} = 0 \Rightarrow \underbrace{\frac{x}{X}X' + 2}_{\text{A function of } x \text{ only}} = \underbrace{\frac{y}{Y}Y'}_{\text{A function of } y \text{ only}}$$

For a function of x only to be equal to a function of y only, they must be equal to the same constant k, i.e.,

$$\frac{x}{X}X' + 2 = \frac{y}{Y}Y' = k$$

The $X-$ equation gives:

$$\frac{x}{X}X' + 2 = k$$

The solution is given by

$$\int \frac{1}{X}dX = \int \frac{k-2}{x}dx + c \Rightarrow \ln X = (k-2)\ln x + \ln C \Rightarrow X(x) = Cx^{k-2}$$

The Y- equation yields

$$\frac{y}{Y}Y' = k$$

The solution is easily obtained as

$$\int \frac{1}{Y}dY = \int \frac{k}{y}dy + D \Rightarrow \ln Y = k\ln y + \ln D \Rightarrow Y(y) = Dy \Rightarrow^k$$

The solution of the differential equation is given by:

$$u(x,y) = X(x)Y(y) = Cx^{k-2}Dy^k = Ax^{k-2}y^k, A = CD.$$

4.5 Heat Conduction in a Rod

Let us now consider a heat conduction problem for a straight bar of uniform cross section and homogeneous material. Let the x-axis be chosen to lie along the axis of the bar, and let $x = 0$ and $x = L$ denote the ends of the bar (see Figure 4.5). Suppose further that the sides of the bar are perfectly insulated so that no heat passes through them. We also assume that the cross-sectional dimensions are so small that the temperature u can be considered constant on any given cross section. Then u is a function only of the axial coordinate x and the time t. The variation of temperature in the bar is governed by a partial differential equation:

$$\alpha^2 u_{xx} = u_t, 0 < x < L, t > 0, \tag{4.21}$$

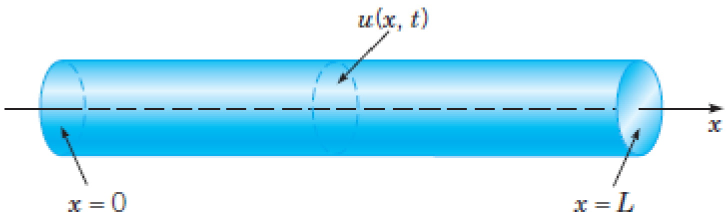


Figure 4.4: A heat-conducting solid bar.

where α^2 is a constant known as the thermal diffusivity. The parameter α^2 depends only on the material from which the bar is made and is defined by

$$\alpha^2 = \kappa/\rho s,$$

where κ is the thermal conductivity, ρ is the density, and s is the specific heat of the material in the bar. In addition, we assume that the initial temperature distribution in the bar is given; thus

$$u(x,0) = f(x), 0 < x < L, \tag{4.22}$$

where f is a given function. Moreover, assume that u is always zero when $x = 0$ or $x = L$:

$$u(0,t) = 0, u(L,t) = 0, t > 0. \tag{4.23}$$

The fundamental problem of heat conduction is to find $u(x,t)$ that satisfies the differential equation (4.21) for $0 < x < L$ and for $t > 0$, the initial condition (4.22) when $t = 0$, and the boundary conditions (4.23) at $x = 0$ and $x = L$.
Assume that $u(x,t)$ is a product of two functions, one depending only on x and the other depending only on t; thus

$$u(x,t) = X(x)T(t). \tag{4.24}$$

Substituting from equation (4.24) for u in the differential equation (4.21) yields

$$\alpha^2 X''T = XT', \tag{4.25}$$

where primes refer to ordinary differentiation with respect to the independent variable, whether x or t. Equation (4.25) is equivalent to

$$\frac{X''}{X} = \frac{1}{\alpha^2}T'T, \tag{4.26}$$

in which the variables are separated; that is, the left side depends only on x and the right side only on t.

It is now crucial to realize that for equation (4.26) to be valid for $0 < x < L, t > 0$, it is necessary that both sides of equation (4.26) must be equal to the same constant. Otherwise, if one independent variable (say, x) were kept fixed and the other were allowed to vary, one side (the left in this case) of equation (4.26) would remain unchanged while the other varied, thus violating the equality.

If we call this separation constant $-\lambda$, then equation (4.26) becomes

$$\frac{X''}{X} = \frac{1}{\alpha^2} T'T = -\lambda, \tag{4.27}$$

Hence we obtain the following two ordinary differential equations for $X(x)$ and $T(t)$:

$$X'' + \lambda X = 0, \tag{4.28}$$

$$T' + \alpha^2 \lambda T = 0. \tag{4.29}$$

The assumption (4.24) has led to the replacement of the partial differential equation (4.21) by the two ordinary differential equations (4.28) and (4.29). Each of these equations is linear and homogeneous, with constant coefficients, and so can be readily solved for any value of λ. The product of two solutions of equations (4.28) and (4.29), respectively, provides a solution of the partial differential equation (4.21). However, we are interested only in those solutions of equation (4.21) that also satisfy the boundary conditions 4.23. As we now show, this severely restricts the possible values of λ. Substituting for $u(x, t)$ from (4.23) in the boundary condition at $x = 0$, we obtain

$$u(0, t) = X(0)T(t) = 0. \tag{4.30}$$

If equation (**??**) is satisfied by choosing $T(t)$ to be zero for all t, then $u(x, t)$ is zero for all x and t, and we have already rejected this possibility. Therefore equation (4.30) must be satisfied by requiring that

$$X(0) = 0. \tag{4.31}$$

Similarly, the boundary condition at $x = L$ requires that

$$X(L) = 0. \tag{4.32}$$

We now want to consider equation (4.28) subject to the boundary conditions (4.31) and (4.32). Thus we have the eigenvalue problem:

$$X'' + \lambda X = 0, X(0) = 0, X(L) = 0 \tag{4.33}$$

The eigenvalues this problem are given by:

$$\lambda_n = n^2\pi^2/L^2, n = 1, 2, 3, \ldots \tag{4.34}$$

The corresponding eigenfunction are given by:

$$X_n(x) = sin(n\pi x/L), n = 1, 2, 3, \ldots \tag{4.35}$$

Putting the value $\lambda(= \lambda_n)$ in equation (4.29) we get:

$$T' + (n^2\pi^2\alpha^2/L^2)T = 0. \tag{4.36}$$

The characteristic equation of the above differential equation is

$$r + n^2\pi^2\alpha^2/L^2 = 0$$

Thus the general solution of equation (4.36) is

$$T(t) = c_n e^{n^2\pi^2\alpha^2/L^2} t \tag{4.37}$$

Hence from equations (4.36) and (4.37) it follows that

$$u_n(x, t) = c_n e^{(n^2\pi^2\alpha^2 t)/L^2} \sin(n\pi x/L) \tag{4.38}$$

are solutions of the equation (4.21) for $n = 1, 2, \ldots$. Since the differential equation (4.21) is linear,

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{(n^2\pi^2\alpha^2 t)/L^2} \sin(n\pi x/L) \tag{4.39}$$

is again a solution to (4.21). Applying the initial condition $u(x, 0) = f(x)$, we get

$$f(x) = u(x, 0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L) \tag{4.40}$$

The series in (4.40) is just the Fourier sine series for f ;its coefficients are given by

$$c_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)dx$$

Hence the solution of the heat conduction problem:

$$\alpha^2 u_{xx} = u_t, 0 < x < L, t > 0,$$

$$u(x,0) = f(x), 0 < x < L,$$

$$u(0,t) = 0, u(L,t) = 0, t > 0.$$

is given by

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{(n^2\pi^2\alpha^2 t)/L^2} \sin(n\pi x/L)$$

where

$$c_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)dx$$

Example 135. Find the solution of the heat conduction problem:

$$\alpha^2 u_{xx} = u_t,\ 0 < x < 50, t > 0,$$

$$u(x,0) = 20,\quad 0 < x < 50,$$

$$u(0,t) = 0,\ u(50,t) = 0,\ t > 0.$$

Solution. The solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{n^2\pi^2\alpha^2 t/2500} \sin(n\pi x/50) \tag{4.41}$$

where

$$\begin{aligned} c_n &= \frac{2}{50}\int_0^{50} 20\sin(n\pi x/50)dx \\ &= \frac{40}{n\pi}(1-\cos n\pi) \\ &= \begin{cases} 80/n\pi & \text{if} \quad n \text{ odd} \\ 0 & \text{if} \quad n \text{ even} \end{cases} \end{aligned}$$

Finally, by substituting for c_n in equation (4.41), we obtain

$$u(x,t) = \frac{80}{\pi}\sum_{n=1,3,5,\ldots} e^{-n^2\pi^2\alpha^2 t/2500}\sin\left(\frac{n\pi x}{50}\right)$$

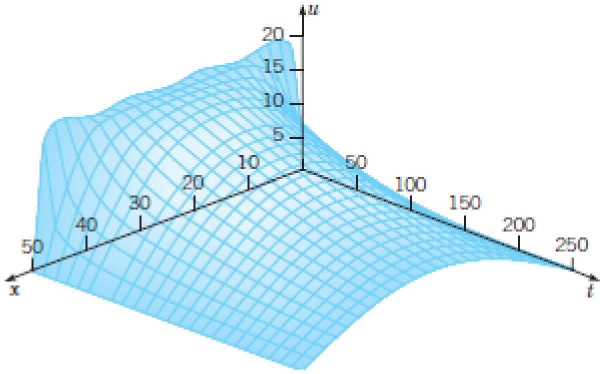


Figure 4.5: Plot of temperature u versus x and t for the heat conduction problem of Example 135

4.6 The Wave Equation: Vibrations of an Elastic String

Suppose that the string is set in motion (by plucking, for example) so that it vibrates in a vertical plane, and let $u(x, t)$ denote the vertical displacement experienced by the string at the point x at time t. If damping effects, such as air resistance, are neglected, and if the amplitude of the motion is not too large, then $u(x, t)$ satisfies the partial differential equation

$$a^2 u_{xx} = u_{tt} \tag{4.42}$$

in the domain $0 < x < L, t > 0$. Equation (4.42) is known as the *one-dimensional wave equation*. The constant coefficient a^2 appearing in equation (4.42) is given by

$$a^2 = T/\rho,$$

where T is the tension (force) in the string, and ρ is the mass per unit length of the string material.

To describe the motion of the string completely, it is necessary also to specify suitable initial and boundary conditions for the displacement $u(x, t)$. The ends are assumed to remain fixed, and therefore the boundary conditions are

$$u(0, t) = 0, u(L, t) = 0, t > 0. \tag{4.43}$$

Since the differential equation (4.42) is of second order with respect to t, it is plausible to prescribe two initial conditions. These are the initial position of the string

$$u(x, 0) = f(x), 0 \le x <\le L, \tag{4.44}$$

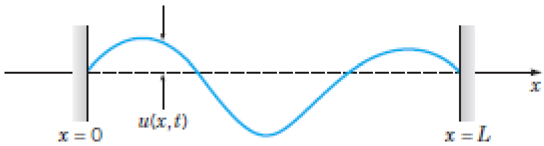


Figure 4.6: A vibrating string.

and its initial velocity

$$u_t(x,0) = g(x), 0 \le x \le L, \tag{4.45}$$

where f and g are given functions.

Case 1: First, assume that $u_t(x,0) = 0, 0 \le x \le L$.

The method of separation of variables can be used to obtain the solution of equations (4.42), (4.43), and (4.44). Assuming that

$$u(x,t) = X(x)T(t) \tag{4.46}$$

and substituting for u in (4.42), we obtain

$$\frac{X''}{X} = \frac{1}{a^2}\frac{T''}{T} = -\lambda \tag{4.47}$$

where λ is a separation constant. Thus we find that $X(x)$ and $T(t)$ satisfy the ordinary differential equations

$$X'' + \lambda X = 0, \tag{4.48}$$

$$T'' + a^2\lambda T = 0. \tag{4.49}$$

Applying the initial condition $u(0,t) = 0$, we get:

$$0 = X(0)T(t) \Rightarrow X(0) = 0$$

Applying the initial condition $u(L,t) = 0$, we get:

$$0 = X(L)T(t) \Rightarrow X(L) = 0$$

Applying the initial condition $u_t(x,0) = 0$, we get

$$X(x)T'(0) = u_t(x,0) = 0 \Rightarrow T'(0) = 0 \tag{4.50}$$

We now want to consider equation (4.48) subject to the boundary conditions $X(0) = X(L) = 0$. This is an eigenvalue problem:

$$X'' + \lambda X = 0, \quad X(0) = 0, X(L) = 0 \tag{4.51}$$

The eigenvalues of (4.51) are given by:

$$\lambda = \frac{n^2\pi^2}{L}, \quad n = 1, 2, 3, \ldots \tag{4.52}$$

The corresponding eigenfunctions are given by:

$$X(x) = c_n \sin\left(\frac{n\pi x}{L}\right), n = 1, 2, 3, \ldots \tag{4.53}$$

Using the values of λ given by equation (4.52) in equation (4.49), we obtain

$$T'' + \frac{a^2 n^2 \pi^2}{L} T = 0 \tag{4.54}$$

The characteristic equation of equation (4.54) is given by:

$$r^2 + \frac{a^2 n^2 \pi^2}{L^2} = 0 \tag{4.55}$$

The roots are $r = \pm i(an\pi/2)$. Therefore

$$T(t) = k_1 \cos\left(\frac{an\pi t}{L}\right) + k_2 \sin\left(\frac{an\pi t}{L}\right) \tag{4.56}$$

Applying the condition $T'(0) = 0$, we get $k_2 = 0$. Hence equation (4.57) becomes:

$$T(t) = k_1 \cos\left(\frac{an\pi t}{L}\right) \tag{4.57}$$

Thus

$$u_n(x,t) = a_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{an\pi t}{L}\right) \tag{4.58}$$

are solutions of the partial differential equation (4.42), the boundary conditions (4.43), and the second initial condition (4.44). Since the one dimensional wave equation is linear,

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{an\pi t}{L}\right) \tag{4.59}$$

is again a solution of the differential equation. The initial condition $u(x,0) = f(x)$ requires that

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \tag{4.60}$$

Consequently, the coefficients c_n must be the coefficients in the Fourier sine series of period $2L$ for f ; hence

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)$$

Case 2: Nest, assume that $u_t(x,0) = g(x), 0 \le x \le L$.

We have to solve the problem:

$$\begin{array}{lll} PDE & a^2u_{xx} = u_{tt} & 0 \le x \le L \\ BCs & \begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases} & \\ ICs & \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} & 0 \le x \le L \end{array}$$

The solution to the boundary value problem:

$$\begin{array}{lll} PDE & a^2u_{xx} = u_{tt} & 0 \le x \le L \\ BCs & \begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases} & \end{array}$$

is given by:

$$u(x,t) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[a_n\cos\left(\frac{an\pi t}{L}\right) + b_n\sin\left(\frac{an\pi t}{L}\right)\right] \tag{4.61}$$

Therefore

$$u_t(x,t) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[-a_n(an\pi/L)\sin\left(\frac{an\pi t}{L}\right) + b_n(an\pi/L)\cos\left(\frac{an\pi t}{L}\right)\right] \tag{4.62}$$

Applying the initial conditions

$$\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \quad 0 \le x \le L$$

gives the two equations:

$$f(x) = \sum_{n=1}^{\infty} a_n\sin\left(\frac{n\pi x}{L}\right) \tag{4.63}$$

$$g(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) b_n(an\pi/L)\cos\left(\frac{an\pi t}{L}\right) \tag{4.64}$$

The constants a_n and b_n are given by

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right),$$

$$b_n = \frac{2}{n\pi a}\int_0^L g(x)\cos\left(\frac{n\pi x}{L}\right).$$

Note that Case 1 is a special case of Case 2.

4.6.1 Exercise

1. Find the eigenvalues and eigenfunctions of the given boundary value problem. Assume that all eigenvalues are real.

   (a) $y'' + \lambda y = 0, y(0) = 0, y'(\pi) = 0.$

   (b) $y'' + \lambda y = 0, y'(0) = 0, y(\pi) = 0.$

   (c) $y'' + \lambda y = 0, y'(0) = 0, y'(\pi) = 0.$

   (d) $y'' + \lambda y = 0, y'(0) = 0, y'(L) = 0.$

2. In each of Problems (a) through (f) determine whether the given function is periodic. If so, find its fundamental period.

   (a) $\sin 5x$

   (b) $\cos 2\pi x$

   (c) $\sinh 2x$

   (d) $\sin(\pi x/L)$

   (e) $\tan \pi x$

   (f) x^2

   (g) $f(x) = \begin{cases} 0, & 2n-1 \le x < 2n, \\ 1, & 2n \le x < 2n+1; \end{cases} \quad n = 0, \pm 1, \pm 2, \ldots$

3. In each of Problems (a) through (e):

   (1) Sketch the graph of the given function for three periods.

(2) Find the Fourier series for the given function.

(a) $f(x) = -x, -L \leq x < L; f(x+2L) = f(x)$

(b) $f(x) = \begin{cases} 1, & -L \leq x < 0, \\ 0, & 0 \leq x < L; \end{cases} \quad f(x+2L) = f(x)$

(c) $f(x) = \begin{cases} x, & -\pi \leq x < 0, \\ 0, & 0 \leq x < \pi; \end{cases} \quad f(x+2\pi) = f(x)$

(d) $f(x) = \begin{cases} x+1, & -1 \leq x < 0, \\ 1-x, & 0 \leq x < 1; \end{cases} \quad f(x+2) = f(x)$

(e) $f(x) = \begin{cases} 0, & -2 \leq x < -1, \\ x, & -1 \leq x < 1; \\ 0, & 1 \leq x < 2; \end{cases} \quad f(x+4) = f(x)$

4. Find the solution of the heat conduction problem $u_{xx} = 9u_t$, with the initial condition $u(x,0) = 2\sin\left(\frac{3\pi x}{L}\right)$ and the boundary condition $u(0,t) = 0, u(L,t) = 0$ for $t > 0$

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