MATHEMATICAL TOOLS FOR ECONOMICS - II

II Semester

COMPLEMENTARY COURSE FOR BA ECONOMICS

CUCBCSS

(2014 Admission)

UNIVERSITY OF CALICUT

SCHOOL OF DISTANCE EDUCATION

CALICUT UNIVERSITY P.O. MALAPPURAM, KERALA, INDIA - 673 635

971

UNIVERSITY OF CALICUT
SCHOOL OF DISTANCE EDUCATION

STUDY MATERIAL

(2014 Admission)

II SEMESTER

COMPLEMENTARY COURSE FOR BA ECONOMICS

MATHEMATICAL TOOLS FOR ECONOMICS - II

Parepared by:

Module I — Dr. Shibi M. Thomas
Associate Professor,
Department of Economics,
St. Joseph's College, Devagiri
Calicut, Kerala

Module II & III: — Dr. Chacko Jose P.
Associate Professor of Economics,
Sacred Heart College, Chalakudy
Thrissur, Kerala

Edited and Compiled by: — Dr. P. P. Yusuf Ali
Associate Professor
Postgraduate Department of Economics
Farook College, Calicut, Kerala
[Chairman, Board of Studies in Economics (UG)]

***LAYOUT & SETTINGS*:** COMPUTER CELL, SDE

CONTENTS

MODULE – I

THEORY OF SETS

Set

A set is a collection of definite and well distinguished objects. One way of representing a set is complete enumeration of all the objects or elements and putting them in brace brackets { }

Eg : a group of three students, a deck of 52 cards,a collection of telephones in a city.

Elements of set

Each object which is a member of the set is called the element of a set. For example, consider a set S = $\{1,2,3\}$. Here the set contains 3 elements,1,2 and 3 as its elements

Null set

A set with no elements is called a null, empty, void or zero set and is denoted by $\emptyset$,0 or{ }

Equal sets or identical sets

Two sets A and B are considered equal if each element of A is an element of B and each element of B is an element of A. In this case , we write A=B

Finite set

A set is finite, if it contains finite number of elements. Such a set consists of specific number of different elements.

Eg: A={a/a:students in my class}

Infinite set

If the number of elements is very large and infinite, then the set is infinite set.

Eg: A={x:x is a point on the line AB}

Unit set or singleton set

A set containing only one element is termed as unit set or singleton set

Eg: A={a} is a unit set, A=set of minimum possible marks in a test={0}, it is to be noted that the set contain one element 0, hence, it is not a null set.

Universal set

If all the sets under consideration are subsets of fixed set, say, **U,** then this set **U** is called universal set. That is any set under discussion can be treated as a subset of a big set. This big set is the universe of the set under consideration.

Consider sets, A , B, C, D ……etc, then the totality of all possible elements out of which the elements of the sets A, B, C,D…….. are drawn is called the universal set with reference to this particular discussion. The universal set is denoted by Ω.

Eg: if A = {2, 4, 6, 8}

B = {8,12,16.20………}

C= {a: a=2npositive integer}. Here all the sets are formed out of set of positive even integers, The set of all positive even integers is the universal set.

Equivalent sets

Two sets are equivalent if there is one to one correspondence between elements of the two sets. Equivalent sets have the same number of distinct elements but not the same elements

Eg: A = {p,l,a,n}

B = {1,2.,3.4} are equivalent sets, written as

$A \equiv B$ or $A \leftrightarrow B$

Subsets

A subset of a set S is a set S_i such that it consists only of some or all of the elements of S- every element of S_i is also an element of S.

Eg: Let S = {1,2,3}. Then the subsets will be S_0 = {0}, S_1 = {1}, S_2 = {2}, S_3 = {3}, S_4 = {1,2}, S_5 = {1,3}, S_6 = {2,3}, S_7 = {1,2,3}

We write $S1 \subseteq S$ and reads: S_1 is a subset of S

Proper subset

When $S_1 \subseteq S$,, when S contains at least one element not in S_1, S_1 is called a proper subset of S. In the above example, S_1i=0,1,2…….,6) are proper subsets.

Power of the set

The set of all subsets of set S is called the power of the set

Eg: suppose R = {S_i: $S_i \subseteq S$}, This R will be called the power of the set S and will be denoted by R(S).

Ordered pair of elements

A grouping of two elements in definite order is called ordered pair elements

Eg: (a,b), here the order of elements is important

Set Operations

Basic set operations are:

1. Union of two sets
2. Intersection of two sets
3. Difference of two sets
4. Complement of a set

i) Union of two sets

Union ot two sets A and B is a set of all those elements which belong to A or B or to both. Union of A and B is written a $A \cup B$ and is read as 'A union B'

Eg:- If A = {1, 2, 3, 4} and B = {2, 4, 7, 9}, thjen $A \cup B$ = {1, 2, 3, 4, 7, 9}

ii) Intersection of two sets

Intersection of two sets A and B is a set of all those elements which belong to both A and B. The intersection of A and B is written as $A \cap B$ and is read as A intersection B. It contains elements common to A and B.

Example: A = {2, 3, 4} and B = {2, 4, 10}, then $A \cap B$ = {2, 4}

iii) Difference of two sets

Difference ot two sets A and B is a set of all those elements which belong to A but not B. The difference of A and B is written as A – B.

Example: If A = {2, 3, 4, 5, 6} and B = {2, 4, 5, 8, 9}, then A – B = {3, 6}

That is we remove from A the elements common to A and B.

iv) Complement of a set

The complement of a set A is the set of all those elements belonging to the universal set but not belonging to A. Therefore complement of A^c or A' is $\Omega - A$

Example: $\Omega =$ {2, 3, 4, 5, 6} and A = {2, 4, 5,} , then A' = {3, 6}

Laws of Set Operation

1. **Commutative Law**
   a) $A \cup B = B \cup A$
   b) $A \cap B = B \cap A$
2. **Associative Law**
   a) $A \cup (B \cup C) = (A \cup B) \cup C$
   c) $A \cap (B \cap C) = (A \cap B) \cap C$
3. **Distributive Law**
   a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4. **De Morgan's Law**
   a) $(A \cup B)' = A' \cap B'$
   b) $(A \cap B)' = A' \cup B'$

Examples

1. If A = {2, 3, 5, 8}, B = {5, 9, 12, 15}and C = {2, 5, 9, 15}
   Find $A \cup B$, $A \cap B$ and A – B, (A – B) - C

Ans: $A \cup B$ = {2, 3, 5, 8} ∪ {5, 9, 12, 15} = {2, 3, 5, 8, 9, 12, 15}

$A \cap B$ = {2, 3, 5, 8} ∩ {5, 9, 12, 15} = {5}

A – B = {2, 3, 5, 8} – {5, 9, 12, 15} = {2, 3, 8}

(A – B) – C = {2, 3, 8} - {2, 5, 9, 15} = {3, 8}

2. If A = {1, 2, 3, 4}, B = {2, 4, 6, 8} and C = {3, 4, 5, 6}
   Verify that $A \cap (B \cap C) = (A \cap B) \cap C$.

Ans: $B \cap C$ = {2, 4, 6, 8} ∩ {3, 4, 5, 6} = {4, 6}

A ∩ $(B \cap C)$ = {1, 2, 3, 4} ∩ {4, 6} = {4} …………….. (1)

$(A \cap B)$ = {1, 2, 3, 4} ∩ {2, 4, 6, 8} = {2, 4}

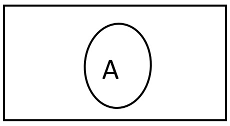
$(A \cap B) \cap C$ = {2, 4} ∩ {3, 4, 5, 6} = {4} ………………(2)

Hence from (1) and (2) it is clear that $A \cap (B \cap C) = (A \cap B) \cap C$.

Venn diagrams

Sets and Set operations can be represented by drawing diagrams termed as Venn diagrams. The universal set Ω is represented by points within a rectangle and set A, B, C are shown by points within circles or ovals inside the rectangle.

Eg.



Now let consider a set made up of our ten friends:
{alex, blair, casey, drew, erin, francis, glen, hunter, ira, jade}
Each friend is an "element" (or "member") of the set (it is normal to use **lowercase letters** for them.)
Now let's say that alex, casey, drew and hunter play **Soccer**:

Soccer = {alex, casey, drew, hunter}

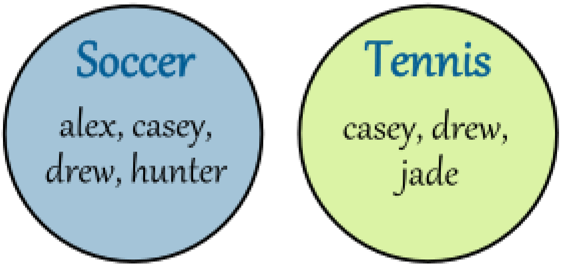
(The Set "Soccer" is made up of the elements alex, casey, drew and hunter).

And casey, drew and jade play **Tennis**:

Tennis = {casey, drew, jade}

(The Set "Tennis" is made up of the elements casey, drew and jade).

We could put their names in two separate circles:



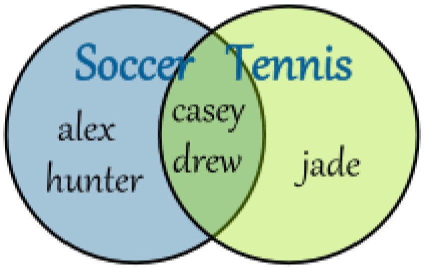
Union of two sets

We can now list our friends that play **Soccer OR Tennis**. This is called a "Union" of sets and has the special symbol ∪:

Soccer ∪ Tennis = {alex, casey, drew, hunter, jade}

Not everyone is in that set ... only your friends that play Soccer or Tennis (or both).

We can also put it in a "Venn Diagram":



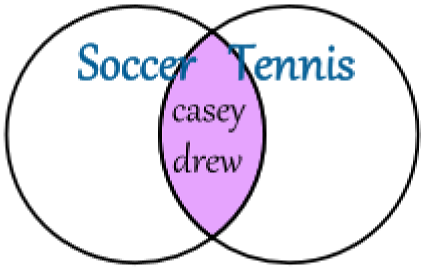
Venn Diagram: Union of two sets

- We can see that alex, casey, drew and hunter are in the "Soccer" set?
- And that casey, drew and jade are in the "Tennis" set?
- And here is the clever thing: **casey and drew are in BOTH sets!**

Intersection of two sets

"Intersection" is when you have to be in BOTH sets. In our case that means **they play both Soccer AND Tennis** ... which is casey and drew. The special symbol for Intersection is an upside down "U" like this: ∩. And this is how we write it down:

Soccer ∩ Tennis = {casey, drew}   In a Venn Diagram:

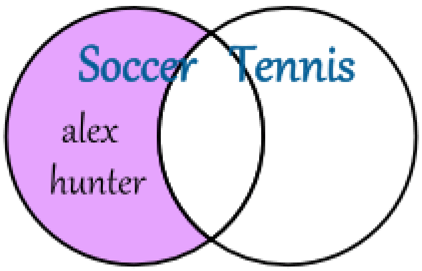


Venn Diagram: Intersection of two sets

Difference of two sets

We can also "subtract" one set from another. For example, taking Soccer and subtracting Tennis means people that **play Soccer but NOT Tennis** ... which is alex and hunter. And this is how we write it down:

Soccer − Tennis = {alex, hunter}   In a Venn Diagram:



Venn Diagram: Difference of two sets

Three Sets

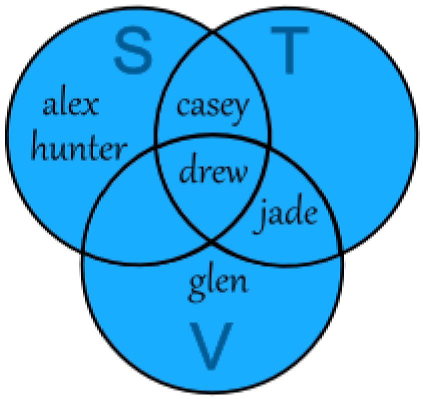
We can also use Venn Diagrams for 3 sets. Let us say the third set is "Volleyball", which drew, glen and jade play:

Volleyball = {drew, glen, jade}

But let's be more "mathematical" and use a Capital Letter for each set:

- **S** means the set of Soccer players
- **T** means the set of Tennis players

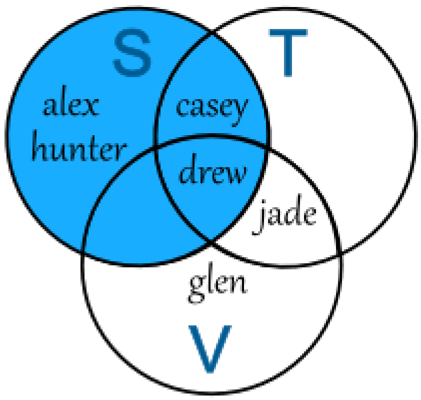
- **V** means the set of Volleyball players The Venn Diagram is now like this:



Union of 3 Sets: S ∪ T ∪ V

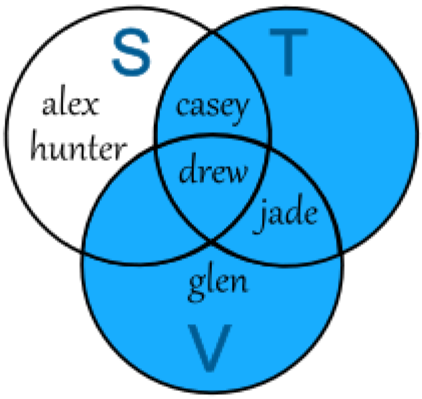
You can see (for example) that:

- drew plays Soccer, Tennis **and** Volleyball
- jade plays Tennis and Volleyball
- alex and hunter play Soccer, but don't play Tennis or Volleyball
- no-one plays **only** Tennis



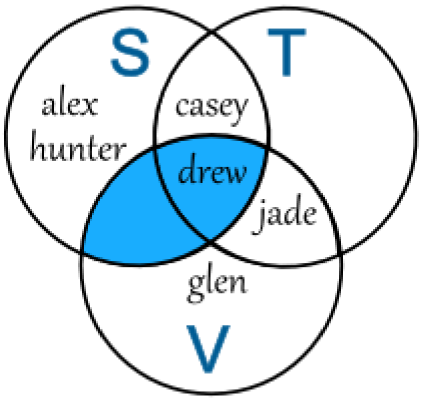
We can now have some fun with Unions and Intersections...

This is just the set S S = {alex, casey, drew, hunter}



This is the Union of Sets T and V

T ∪ V = {casey, drew, jade, glen}

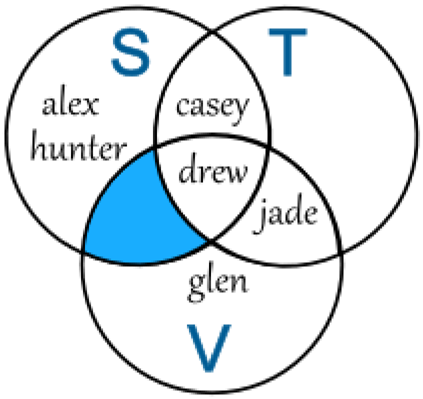


This is the **Intersection** of Sets S and V

S ∩ V = {drew}

And how about this ...

- take the **previous set** S ∩ V
- then **subtract T**:



This is the Intersection of Sets S and V **minus** Set T

$(S \cap V) - T = \{\}$

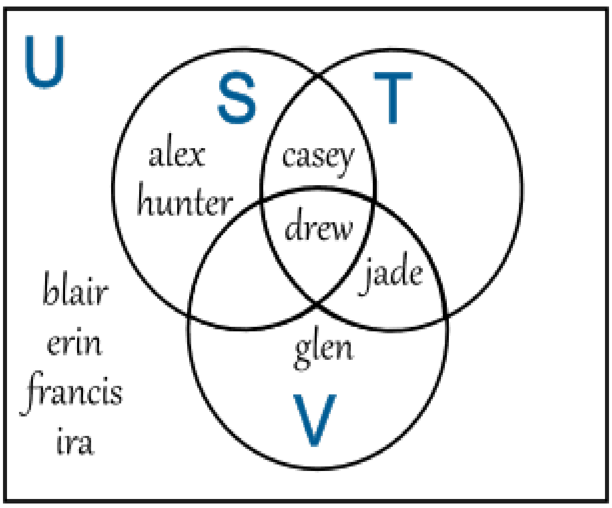
Hey, there is nothing there! That is OK, it is just the "Empty Set". It is still a set, so we use the curly brackets with nothing inside: {}

Universal Set

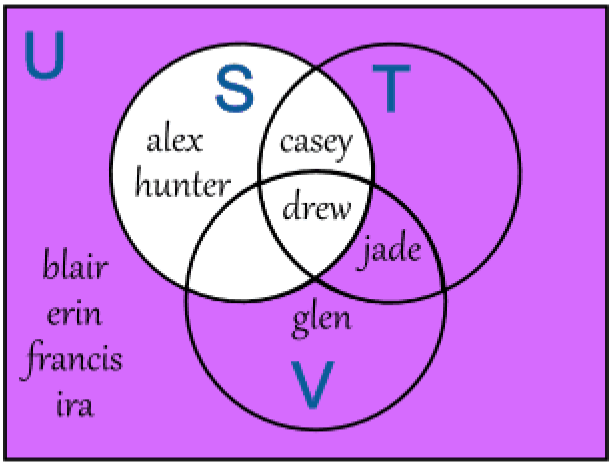
The **Universal Set** is the set that contains everything. Well, not *exactly*everything. **Everything that we are interested in now.**

Sadly, the symbol is the letter "U" ... which is easy to confuse with the ∪ for Union. You just have to be careful, OK? In our case the Universal Set is our Ten Best Friends.

U = {alex, blair, casey, drew, erin, francis, glen, hunter, ira, jade}We can show the Universal Set in a Venn Diagram by putting a box around the whole thing:



Now you can see ALL your ten best friends, neatly sorted into what sport they play (or not!).And then we can do interesting things like take the whole set and **subtract the ones who play Soccer**:

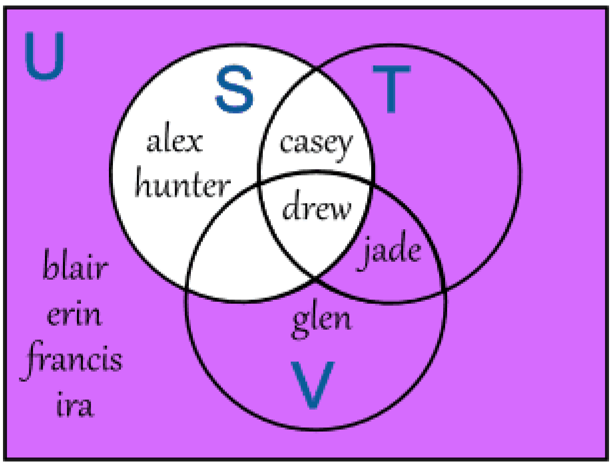


We write it this way: U − S = {blair, erin, francis, glen, ira, jade}

Which says "The Universal Set minus the Soccer Set is the Set {blair, erin, francis, glen, ira, jade}" In other words "everyone who does **not** play Soccer".

Complement of a set

And there is a special way of saying "everything that is **not**", and it is called ***"complement"***.
We show it by writing a little "C" like this: S^c
Which means "everything that is NOT in S", like this:



S^c = {blair, erin, francis, glen, ira, jade}
(just like the **U − C** example from above)

Cartesian Product (Cartesian Set)

The set of all ordered pairs (x,y) that can be obtained from two sets A and B is called the Cartesian product of A and B and is denoted by A x B. It can be extended to more than two elements depends on the number of sets and then it can be ordered triples, ordered quadruples etc.

Eg:- A = {1,2} and B = {3,4}, then A x B = {(1,3), (1,4), (2,3), (2,4)}

If A is a set with m elements and B is a set with n elements, then A x B ≠ B x A. But if A = B, then A x B = B x A

Relations

Given a set A and set B, a relation R is a subset of the Cartesian product A x B. The elements of the subset R which are ordered pairs (x,y) are such that $x \in A, y \in B$.

Domain of the relation (function f)

The set of x elements that are paired with the y elements in (x, y) which belong to the relation (function f) R.

For example in the relation R = {(1, 1), (4,2)}, then Domain is {1, 4}

The range of the Relation (function f)

The set of Y elements that are paired with x elements in (x,y) which belong to the relation(function f) R.

For example in the relation R = {(1, 1), (4,2)}, Range is {1, 2}

Functions

A function is a relation f (ie .., a subset of ordered pairs) such that for each element $x \in X$ there is a unique element $y \in Y$.

Model Questions

1. **Express each of the following sets in descriptive phrase form or rule form**
   a) The set of all odd integers
   b) Set of all integers divisible by 4
   c) Set of all women presidents of India
   d) The set of squares of integers from 1 to 5
   e) The set of numbers such that x = x -1

Ans: a) A={x:x odd integers} b) A={x/x is integer divisible by 4}

c) A={x/x is women president of India }={1} d) A={x^2/1$\leq x \leq 5$}

e)A={x/x=-1}=∅

2. **Describe each of the following sets in words**:

a) A={1,8,27,64}
b) A={x:15$\leq x < 60\ years\ of\ age$}
c) X={x/x^2- 7x+12=0}
d) A={x/x^2}

Ans:

a) Set of cubes of positive integers from 1to4
b) Set of all persons aged from 15 years to less than 60 years
c) Set of root of x^2-7x=12=0
d) Set of perfect squares

3. **Which of the following sets is finite or infinite**?

a) A= {x:x isa multiple of 3} - infinite

b) B={x:x is a person in age group 15-60 years in India } - finite

c) C= {x:x is mango trees in India} - finite

d) D= {x:x is a ray emanating from sun at any moment of time} – infinite

4. **Which of the following is correct or incorrect?**

a) {a,b,c}$\equiv \{\alpha, \beta, \gamma\}$- correct

b) {a,b,c}=$\{\alpha, \beta, \gamma\}$- incorrect

c) {0,2,6,4}={4,6,0.2}-correct

d) ∅={0}- incorrect

c) If A = {1,3,5}

B = {7,9,11}

C = {1,5,9,13…….}

Then Ω {x:x = all positive integers} - incorrect

5. **Suppose there is a Set S, which is the largest possible subset of S**

(a) $S_{1,}$ (b) S (c) ∅, (d) None of these . Ans:(b)

6. **Suppose there is a set S, which is the smallest possible subset of S**

(a) S_1 (b) S (c) ∅ (d) None of these Ans(c)

7. **If set has n elements, the total number subsets will be**

(a) 2^n (b) 2n (c)2/n (d) None of these Ans: (a)

8. **Whether the statements are true or false**:

The null set is a subset of Set S – True

Set S is a subset of set S - True

9. If a set S_1 has n elements and S_2 has m elements, how many ordered pairs can form in general

(a) m x n (b) m + n (c) m/n, (d) none of these: Ans(a)……. The question may be asked by taking numerical values also.

Short Answer Questions

1. **Explain the different properties of the set inclusion and set equivalence**

Inclusion

A$\subseteq A; ie\ set\ inclusion\ is\ reflexive$.

A$\subset B\ and\ B \subset A\ cannot\ hold\ simultanously\ ie\ set\ inclusion\ is\ anti-symmetric$

A$\subset B$,B⊂C, then A$\subset C: ie\ set\ inclusion\ is\ transitive$

Equivalence

A=A – set equivalence is reflexive

A=B, then, B=A; set equivalence is symmetric

A=B:B=C,then,A=C:- set equivalence is transitive

2. Prove that null set is unique

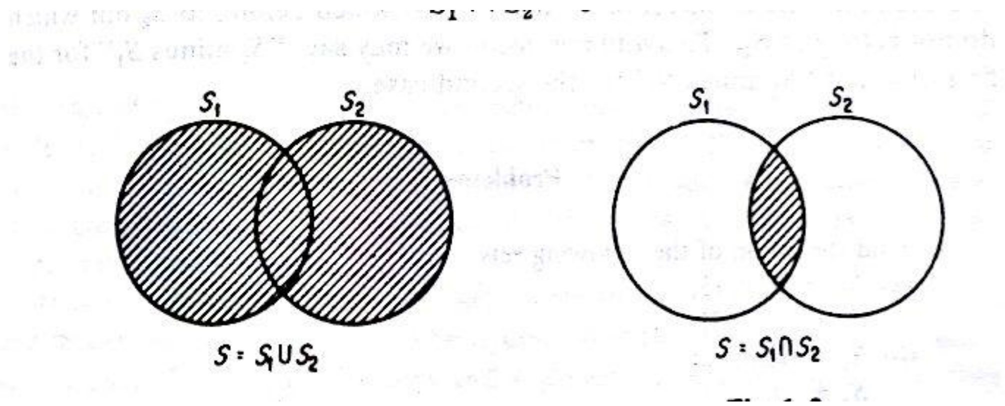
Let 0 and Δ be two null sets, then by definition equality , 0⊆Δ and Δ⊆ 0. *this imples that* 0 =Δ. Thus 0 is unique.

3. Explain the concept of Union and intersection of sets

Let S_1 = {a,b,c,2} and S_2 = {1,2,3}. Then the union (or sum) of S_1 and S_2 will be the set S
S = $S_1 \cup S_2$= {a,b,c,1,2,3}

$S_1 \cup S_2$ is the set of elements that belonging to both S_1 or S_2 or both
The intersection (or product of) of $S_1$and S_2 is the set S=$S_1 \cap S_2$={2},

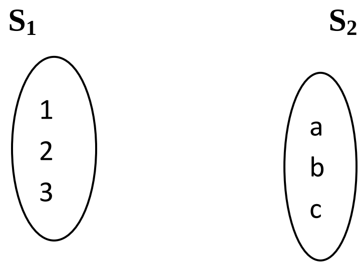
Here it is the set of elements belong to both S_1 and S_2.



4. **Explain the concept of disjoint and complement of a set , explain with Venn diagrams also** .

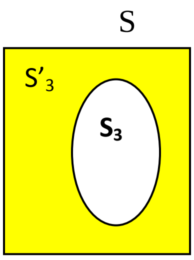
If $S_1 \cap S_2$ =0 , then these sets are called disjoint (or non overlapping, or mutually exclusive). Let the universal set in the given discussion is S= { a,b,c,1,2,3} and let S_3={a,b, 1,2} be a subset of S . The complement of S_3 with respect to the S is the set S'_3={c,3} that is , it will be those elements of universe that are not elements of S_3.

Disjoint sets



Here S_1 and S_2 are disjoint sets .

Complement of sets



5. **Verify that (AUB)'=A'∩ B' by taking concrete examples**

Let A={1,2,3,4,5}, B={4,5,6,7} and Ω={0,1,2,3,4,5,6,7,8,9}, then AUB={1,2,3,4,5,6,7,8,9}

And (AUB)'={8,9,0}= {elements of Ω not in AUB}…………………………..(1)

Now A'={elements of Ω not in A}={0,6,7,8,9}

B'={0,1,2,3,8,9}

∴A'∩ $B' = \{0,8,9\}$ ……………………………………………….(2)

Hence from(1) and (2) we have (AUB)'=A'∩ B'.

6. **Verify that (A∩B)'=A'UB' by taking concrete examples.**

From the above example, it can be seen that A∩B={4,5}

(A∩B)'={0,1,2,3,6,7,8,9} …….1 and again A'={0,6,7,8,9} and B'={0,1,2,3,8,9}, then A'UB'={0,1,2,3,6,7,8,9}………2 . from (1) and (2), we verify the result.

7. **Explain the laws of set operations (unions and intersections) .**

Suppose A,B,C are three sets

1. AUB = BUA and A∩ $B = B \cap A$ ––––commutative law
2. $AU(BUC) = (AUB)UC$
   $A \cap (B \cap C) = (A \cap B) \cap C$ -------------- } Associative law
3. AU$(B \cap C)$ = (AUB) ∩(AUC)
   A∩(BUC) = $(A \cap B)$U(A∩C)……………………… distributive law

8. Explain the concept of Cartesian product with an example

Suppose we have S_1={1,2,3}, S_2={4,5} , from these two sets we can get three choices for the first place and two choices for the second place, and there are 3x2=6 ordered pairs .

The ordered pairs are(1.4)(1,5) (2.4),(2,5) (3,4),(3,5). In general, if S_1 has n elements and S_2 has m elements , we may form mxn ordered pairs. The set of all ordered pairs derived from two or more sets are called Cartesian product of that sets .

9. Verify the distributive law, given A = {4,5}, B = {3,6,7} and C = {2,3}

To verify the first part of the law, we find the left –and right –hand expressions separately:

Left: AU(B$\cap C$)={4,5}U{3}={3,4,5}

Right: (AUB) ∩ (AUC)={3,4,5,6,7} $\cap\cap \{2,3,4,5\} = \{3,4,5\}$

Since the two sides yield the same result, the law is verified. Repeating the procedure for the second part of the law , we have

Left: A∩(BUC)={4,5}$\cap \{2,3,6,7\} = \emptyset$

Right: (A$\cap B)U(A \cap C) = \emptyset \cap \emptyset = \emptyset$, Thus the law again verified.

10. Explain the different type of relations

A relation from a set A to set B is nothing but a subset of the cartesian product of A and B which is denoted by AXB. The types of relations are nothing but their properties. There are different types of relations namely **reflexive, symmetric, transitive and anti symmetric** which are defined and explained as follows.

Reflexive relation:

A relation *R* is said to be reflexive over a set *A* if (a,a) € *R* for every *a* € *R*.

Example-1:

If A is the set of all males in a family, then the relation "is brother of" is not reflexive over A. Because any person from the set A cannot be brother of himself.

Example-2:

The relation R = {(1,1)(2,2)(3,3)} is reflexive over the set A = {1,2,3}.

Symmetric relation:

A relation *R* is said to be symmetric if (a,b) € R => (b,a) € R

Example-1:

If A is the set of all males in a family, then the relation "is brother of" is symmetric over A.

Because if *a* is brother of *b* then *b* is brother of *a*.

Example-2:

If A is the set of mothers and B is the set of children in a family then a relation R on AxB is not symmetric because if *a* is mother of *b* then *b* cannot be mother of *a*.

Transitive relation:

A relation *R* is said to be symmetric if (a,b) € R, (b,c) € R => (a,c) € R.

Example-1:

If A is the set of all males in a family, then the relation "is brother of" is transitive over A.

Because if *a* is brother of *b* and *b* is brother of *c* then *a* is brother of *c*.

Example-2:

The relation R = {(1,1)(2,2)(3,3)(1,2)(2,3)} is not transitive over the set A = {1,2,3} because thought (1,2), (2,3) € R , (1,3) is not in R.

MODULE II

Fundamental of Linear Algebra – Matrices

Linear Algebra

The subject of linear algebra can be partially explained by the meaning of the two terms comprising the title.“Linear” is a term which mean anything that is “straight” or “flat.”“Algebra” means, roughly, “relationships”. Then, “Linear Algebra” means, roughly, “line-like relationships”.

The matrix and determinant are extremely useful tools of linear algebra. One central problem of linear algebra is the solution of the matrix equationAx=b for x. (which we will see in the next chapter).

Matrix

A Matrix is a rectangular arrangement of numbers in rows and columns. We usually enclose the numbers with brackets. . These numbers can be anynumbers we choose - positive, negative, zero, fractions, decimals, and so on. The numbers in a matrix is called elements of a matrix. Matrices is the plural of the term matrix.

Any information can be arranged in rows and columns and represented as a matrix. For example the marks scored by three students Anu, Binu and Cinu in Maths, Science and English are given. Anuscored 25 in maths, 19 in science and 18 in English. Binu scored 24, 23and 20.Cinu scored, 21, 22 and 17. This information can be represented for better understanding in theform of a table.

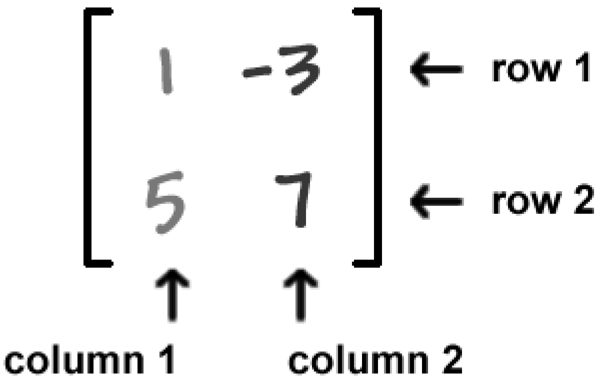
Subject → Students ↓	Maths	Science	English
Anu	25	19	18
Binu	24	23	20
Cinu	21	22	17

This information can be also represented as a matrix as follows.

$$\begin{bmatrix} 25 & 19 & 18 \\ 24 & 23 & 20 \\ 21 & 22 & 17 \end{bmatrix}$$

Thus any information can be represented in the form of a matrix.

We stated in the definition of a matrix that the information in a matrix is arranged in rows and columns.



In the above matrix 1 is an element appearing in the first row and first column. To represent an element we us the term a_{ij} where i stands for column and j stands for row. So in the above matrix, to represent 1 which is an element appearing in the first row and first column, we can write a_{11}. Similarly to represent −3 which is in the first row and second column, we can write a_{12}. To represent 5 which is in the second row and first column, we can write a_{21}. To represent 7 which is in the second row and second column, we can write a_{22}.

In general form a matrix may be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

To refer briefly to a specific matrix we might label it, usually with a capital letter.

For example

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 2.3 \\ 6.5 \\ 4.1 \end{bmatrix}$$

$$C = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

The order of a matrix or the dimension of a matrix

All the above matrices have different sizes. To distinguish different matrices, we use the number of rows and columns in it. When we want to refer to the size of a matrix westate its number of rows and number of columns, in that order.The number of rows and columns that a matrix has is called its order or its dimension. By convention, rows are listed first; and columns, second. If we use 'm' to represent rows and 'n' to represent columns, the order of the matrix is written as $m \times n$. But note that$m \times n$ is read as m by n (and not m into n). Thus, we would say that the order (or dimension) of the matrix A above is 1 x 3 (read as 1 by 3), meaning that it has 1 row and 3 columns. Similarly the order (or dimension) of the matrix B above is 3 x 1 (read as 3 by 1), meaning that it has 3 rows and 1 column. The order (or dimension) of the matrix C above is 1 x 2 (read as 1 by 2), meaning that it has 1 row and 3 columns.

Question: Define matrix. What do you mean by order / dimension of a matrix?

A matrix is a rectangular array of numbers arranged in rows and columns, Following is an example of a matrix.

$$A = \begin{bmatrix} 1 & -5 & 0.9 \\ 6 & 7 & 0 \\ -9 & 4 & 1 \end{bmatrix}$$

The number of rows and columns that a matrix has is called its dimension or its order. By convention, rows are listed first; and columns, second. Thus, we would say that the dimension (or order) of the above matrix is 3 x 3, meaning that it has 3 rows and 3 columns.

Problem

1. Form a matrix with the information given here.

$$a_{11} = 5, a_{12} = 6, a_{13} = 8, a_{21} = 1, a_{22} = 2, a_{23} = 3,$$

Solution

$$D = \begin{bmatrix} 5 & 6 & 8 \\ 1 & 2 & 3 \end{bmatrix}$$

2. Complete the following matrix with the given information

$$A = \begin{bmatrix} - & 5 & - \\ 2 & - & 0 \\ - & 4 & - \end{bmatrix}$$

$$a_{11} = 6, a_{13} = 8, a_{22} = 7, a_{31} = -3, a_{33} = 9$$

Solution

$$A = \begin{bmatrix} 6 & 5 & 8 \\ 2 & 7 & 0 \\ -3 & 4 & 9 \end{bmatrix}$$

3. The order of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$is (a) 1×1 (b) 1×3 (c) 3×1 (d) 3×3

Correct answer is (b) 1×3 as the matrix has 1 row and 3 columns.

Such a matrix with one row is called a row matrix or row vector. (Note that a vector is a matrix having either a single row or a single column.)

4. The order of the matrix $A = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$is (a) 1×1 (b) 1×3 (c) 3×1 (d) 3×3

Correct answer is (c) 3×1 as the matrix has 3 rows and 1 column.

Such a matrix with one column is called a column matrix or column vector.

Square Matrix

Question: What is a square matrix.

A square matrix is a matrix that has the same number of rows and columns. For a square matrix m = n.

Examples

$A = [1]$, Matrix A is a square matrix of order 1. (we need not say 1×1 as the number of rows and the number of columns are the same)

$B = \begin{bmatrix} 5 & 3 \\ 8 & 1 \end{bmatrix}$, Matrix B is a square matrix of order 2.

$C = \begin{bmatrix} 6 & 5 & 8 \\ 2 & 7 & 0 \\ -3 & 4 & 9 \end{bmatrix}$Matrix C is a square matrix of order 3.

Diagonal matrix

Question: What is a diagonal matrix.

A diagonal matrix is a square matrix with zeros everywhere except on the diagonal whichruns from the top left to the bottom right. This diagonal is called the leading diagonal or principal diagonal.

Examples

$D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

$$E = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$F = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}$$

Identity Matrix

Question: What is an identity matrix.

An identity matrixis a diagonal matrix with all its diagonal elementsequal to 1. An identity matrix is sometimes called a unit matrix, considering the fact that an identity matrix serves the purpose of unity (1) in matrix algebra. Normally an identity matrix is represented by I.

Examples

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

When we have several identity matrices in the same context, it will be difficult to distinguish the as we use the same letter I to denote an identity matrix. So to distinguish between different identity matrices, we use a subscript to indicate the size of the particularidentity matrix. So we might write I_2 to represent an identity matrix of order 2.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transpose of a matrix

Transpose of a matrix is a new matrix obtained by inter changing the rows and columns. Thus the transpose of matrix A is obtained by creating a new matrix called A transpose (A^T) in which the rows of A will become columns or the columns of A will become rows.

Examples

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$$

To get its transpose we write all the rows of matrix A as columns in the new matrix that we are forming.

So the elements of first row of A namely 0, 1, 2 will become the first column in the new matrix.

The elements of second row of A namely 3, 4, 5 will become the second column in the new matrix.

The elements of third row of A namely 6, 7, 8 will become the third column in the new matrix.

$$A^T = \begin{bmatrix} 0 & 3 & 6 \\ 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}$$

Note that transpose of a transpose gives the original matrix. That is, given a matrix A,

$$(A^T)^T = A$$

Question: Define transpose of a matrix.

The transpose of a matrix is a new matrix whose rows are the columns of the original (which makes its columns the rows of the original).

Problem 1: Find the transpose of

$$D = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$D^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Problem 2: Find the transpose of

$$E = \begin{bmatrix} 5 & -1 & 7 \\ -9 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$E^T = \begin{bmatrix} 5 & -9 & 0 \\ -1 & 5 & 0 \\ 7 & 0 & 4 \end{bmatrix}$$

Problem 3: Find the transpose of

$$F = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}$$

$$F^T = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}$$

Not that here F = FT. A matrix for which the transpose is the same as the original matrix is called a symmetric matrix. (We will discuss it later)

Problem 4: Find the transpose of

$$G = \begin{bmatrix} 5 & 6 & 8 \\ 1 & 2 & 3 \end{bmatrix}$$

$$G^T = \begin{bmatrix} 5 & 1 \\ 6 & 2 \\ 8 & 3 \end{bmatrix}$$

Note that matrix G is a 2×3 matrix, but its transpose is GT is a 3×2 matrix.

Problem 5: Find the transpose of

$$H = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$$

$$H^T = \begin{bmatrix} 2 & 5 & 9 \end{bmatrix}$$

Here the column vector H becomes a row vector as we take the transpose.

Problem 6: Find the transpose of

$$J = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$J^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Here the row vector J becomes a column vector as we take the transpose.

Problem 7: Given the following matrix, prove that
$(A^T)^T = A$

$$A = \begin{bmatrix} 5 & -1 & 7 \\ -9 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Find A^T

$$A^T = \begin{bmatrix} 5 & -9 & 0 \\ -1 & 5 & 0 \\ 7 & 0 & 4 \end{bmatrix}$$

Now find transpose again.

$$A = \begin{bmatrix} 5 & -1 & 7 \\ -9 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Negative Matrices

Matrices of all elements replaced by the additive matrices inverse. Thus a matrix A is negative expressed by –A.

$$A = \begin{bmatrix} 5 & -1 & 7 \\ -9 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$-A = \begin{bmatrix} -5 & 1 & -7 \\ 9 & -5 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

Different types of matrices

Question: What are the different types of matrices?

Null Matrix

Null or Zero Matrixis any matrix in whichall the elements have zero values. It isusually denoted as 0.

$A = [0]$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Symmetric Matrix

A symmetric matrix is a square matrix which is symmetric about its leading diagonal. In other words, A symmetric matrix is a square matrix in which $a_{ij} = a_{ji}$.

Example

$$A = \begin{bmatrix} 1 & 2 & \mathbf{3} \\ 2 & 4 & 5 \\ \mathbf{3} & 5 & 6 \end{bmatrix}$$

Matrix A is a symmetric matrix since the $a_{13} = a_{31}$

$$B = \begin{bmatrix} 2 & 7 & 3 \\ 7 & 9 & 4 \\ 3 & 4 & 7 \end{bmatrix}$$

Matrix B is a symmetric matrix since the $a_{13} = a_{31}$

$$C = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}$$

Matrix C is a symmetric matrix since the $a_{12} = a_{21}$

A much easier way to define a symmetric matrix is to use the concept of transpose of a matrix. A square matrix A is called a symmetric matrix if the transpose of A is the same as the original matrix A.

A symmetric matrix is a square matrix A that satisfies $A^T = A$, where A^T denotes the transpose,so $a_{ij} = a_{ji}$.

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$D^T = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$E = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$E^T = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The transpose of this identity matrix is the same.

$$I^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Antisymmetric Matrix

An antisymmetric matrix is a square matrix that verifies $A = -A^T$

Skew-symmetric matrix

A Skew-symmetric matrix is a square matrix in which corresponding elements with respect to the diagonal are negatives of each other.

In other words, A skew-symmetric matrix is square matrix with all values on the principal diagonal equal to zero and with off-diagonal values given such that $a_{ij} = -a_{ji}$.

Example

$$A = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -5 \\ -3 & 5 & 0 \end{bmatrix}$$

Note that in the upper part of the principal diagonal (comprising of 0,0,0) we have −2, 3 and −5 and in the lower part of the principal diagonal we have the same numbers with opposite signs 2, −3 and 5.

Another example

$$B = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}$$

Diagonal matrix

A square matrix in which all of the elements are zero except for the principal diagonal elements. It is often written as $D = \text{diag}(a_{11}, a_{22}, a_{33}, \ldots, a_{nn})$

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Upper and Lower Triangular Matrices

In Upper Triangular Matrices all the matrix elements below the diagonal are zero.

That is $A_{ij} = 0 \; for \; i > j$

Example

$$A = \begin{bmatrix} 5 & 2 & -1 \\ 0 & 7 & 5 \\ 0 & 0 & 9 \end{bmatrix}$$

Note that in the above matrix all elements below the principal diagonal comprising of 5,7,9 are zeros.

In Lower Triangular Matrices all the matrix elements above the diagonal are zero.

That is $A_{ij} = 0 \; for \; i < j$

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 4 & 7 & 0 \\ 1 & 2 & 9 \end{bmatrix}$$

Note that in the above matrix all elements above the principal diagonal comprising of 5,7,9 are zeros.

Scalar matrix

A scalar matrix is a diagonal matrix in which all of the diagonal elements are equal to some constant "k"

$$A = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$$

In other words, a scalar matrix is a diagonal matrix whose diagonal elements all contain the same scalar k. A scalar matrix is therefore equivalent to kI, where I is the identity matrix.

We will see the following types of matrices also, but to understand the you should have an understanding of matrix multiplication and concept like inverse of a matrix which we will study later. Please re read these matrix types after you have finished studying those portions.

Regular Matrix

A regular matrix is a square matrix that has an inverse.

Singular Matrix

A singular matrix is a square matrix that has no inverse.

Idempotent Matrix

The matrix A is idempotent if:

$A^2 = A$.

Involutive Matrix

The matrix A is involutiveif$A^2 = I$, where I is an identity matrix

An involutory matrix is a matrix that is its own inverse. That is, multiplication by matrix A is an involution if and only if $A^2 = I$

Commutative and anti-commutative matrices

If A and B are square matrices such that AB = BA, then A and B are called commutative or are said to commute. If AB = -BA, the matrices are said to anti-commute.

Periodic matrix

A matrix A for which $\mathbf{A^{k+1} = A}$, where k is a positive integer. If k is the least positive integer for which $A^{k+1} = A$, then A is said to be of **period k**.

If k = 1, so that $A^2 = A$, then A is called **idempotent**.

Equality of two matrices

For two matrices to be equal, they must have

1. The same dimensions (order).
2. Corresponding elements must be equal.

Examples: Matrices A and B are equal since both of them are of the same order (dimension) 2×3 and corresponding elements are equal.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Matrices A and C are not equal since both of them are of different order. A is of order 2×3 and C is of order 3×2.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad D = \begin{bmatrix} -1 & -2 & 3 \\ 4 & 5 & -6 \end{bmatrix}$$

Matrices A and D are not equal since the corresponding elements are not the same.

Matrix equality can also be interpreted as follows.

$$\begin{bmatrix} 5x + 2y \\ x - 3y \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

This implies that

$5x + 2y = 7$

$x - 3y = 1$

Question 1: Given that the following matrices are equal, find the values of x, y and z.

$$\begin{bmatrix} x+3 & -1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 6 & y \\ z-3 & 5 \end{bmatrix}$$

Solution: Equate the corresponding elements and solve for the variables.

$x + 3 = 6$

$x = 3$

$y = -1$

$z - 3 = 4$

$z = 7$

Question 2: Solve for a_{11}, a_{12}, a_{21}, and a_{22} in the following matrix equation.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}$$

Solution:

Equate the corresponding elements and solve for the variables.

$a_{11} = 2$ $a_{12} = -1$ $a_{21} = -3$ $a_{22} = 0$

Operations on Matrices

Under this section we study the arithmetic operations in matrix algebra namely (a) Addition and Subtraction of Matrices (b) Scalar Multiplication (b) Vector Multiplication (d) Multiplication of Matrices. There is no division in matrix algebra, insteadthere is matrix inversion, which we will study later.

Addition and Subtraction of Matrices

Addition and Subtraction of matrices is very easy but can be doneonly when all the matrices have the same dimension.

Two matrices are said to be compatible when they have the same size, that is, the same number ofrows and the same number of columns. When two matrices are compatible they can be added (orsubtracted).When two matrices are compatible we add (or subtract) them by adding (or subtracting) theelements in corresponding positions.

Adding two matrices

Matrix addition is defined as adding corresponding elements.

Consider the following two matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

To add the matrix A to B, add the corresponding values. Add a_{11} of A to a_{11} of B, a_{12} of A to a_{12} of Band so on.

$$A + B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

Question 1: Find A+ B

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -6 \\ 7 & 8 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1 & -4 \\ 10 & 4 \end{bmatrix}$$

Question 2: Find A + B

$$A = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix}$$

Question 3: Find A + B

$$A = \begin{bmatrix} 1 & 0 & 6 \end{bmatrix}, B = \begin{bmatrix} 5 & 4 & 3 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 6 & 4 & 9 \end{bmatrix}$$

Question 4: Find A + B

$$A = \begin{bmatrix} 1 & -5 & 0 \\ 8 & 4 & -3 \\ -4 & 1 & 8 \end{bmatrix} \; B = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & -4 \\ 2 & -1 & 6 \end{bmatrix}$$

$$A + b = \begin{bmatrix} 4 & -5 & 2 \\ 8 & 5 & -7 \\ -2 & 0 & 14 \end{bmatrix}$$

Question 5: Find A + I_3

$$A = \begin{bmatrix} 1 & -5 & 0 \\ 8 & 4 & -3 \\ -4 & 1 & 8 \end{bmatrix}$$

Note that here I3 represents an identity matrix of order 3.

So $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A + I_3 = \begin{bmatrix} 2 & -5 & 0 \\ 8 & 5 & -3 \\ -4 & 1 & 9 \end{bmatrix}$$

Question 6: Find A + I_3

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A + I_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Subtraction of matrices

Subtraction of matrices is similar to addition of matrices, except that instead of adding the corresponding elements, we subtract the elements.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Find A−B

$$A - B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1-5 & 2-6 \\ 3-7 & 4-8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

Question 1: Find A – B

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} -5 & 6 \\ 7 & -8 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1--5 & 2-6 \\ 3-7 & 4--8 \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ -4 & 12 \end{bmatrix}$$

Question 2: Find A – B

$$A = \begin{bmatrix} 5 & 4 \\ 2 & 1 \\ 4 & 2 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 1 \\ 4 & 3 \\ 5 & 7 \end{bmatrix} \quad A - B = \begin{bmatrix} 6 & 3 \\ -2 & -2 \\ -1 & -5 \end{bmatrix}$$

Question 3: Find A – B

$$A = \begin{bmatrix} -1 & 0 & 5 \\ 2 & 1 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 9 & -1 & -7 \\ 7 & 8 & 9 \end{bmatrix} \quad A - B = \begin{bmatrix} -10 & 1 & 12 \\ -5 & -7 & -2 \end{bmatrix}$$

Question 4: Find A + B – C

$$A = \begin{bmatrix} 5 & 4 \\ 2 & 1 \\ 4 & 2 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 1 \\ 4 & 3 \\ 5 & 7 \end{bmatrix} \quad C = \begin{bmatrix} 6 & 3 \\ -2 & -2 \\ -1 & -5 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 4 & 5 \\ 6 & 4 \\ 9 & 9 \end{bmatrix}$$

$$A + B - C = \begin{bmatrix} 4-6 & 5-3 \\ 6--2 & 4--2 \\ 9--1 & 9--5 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 8 & 6 \\ 10 & 14 \end{bmatrix}$$

Question 5: Find A + B and A−B

$$A = \begin{bmatrix} 1 & 0 & -5 \\ 8 & -2 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 2 & 3 \\ 5 & 1 & 7 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 5 & 2 & -2 \\ 13 & -1 & 16 \end{bmatrix}$$

$$A - B = \begin{bmatrix} -3 & -2 & -8 \\ 3 & -3 & 2 \end{bmatrix}$$

Question 6: Find A – B and A – C

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 3 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -4 & 3 \\ 9 & -4 & -3 \end{bmatrix} \quad C = \begin{bmatrix} 0 & -4 \\ 9 & -4 \end{bmatrix}$$

$$A - B = \begin{bmatrix} -1 & 6 & -3 \\ -9 & 7 & 9 \end{bmatrix}$$

A – C can not be solved. Matrices A and C are not the same order, since A is 2 × 3 and C is 2 × 2. So this subtraction is not defined. (A and B are the same order, each being 2 × 3 matrices, so we can subtract.)

Question 7: Find the values of xand y given the following.

$$\begin{bmatrix} -3 & x \\ 2y & 0 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ -5 & 1 \end{bmatrix}$$

First add the matrices in the LHS

$$\begin{bmatrix} -3 & x \\ 2y & 0 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6+x \\ 2y-3 & 1 \end{bmatrix}$$

Now we can write

$$\begin{bmatrix} 1 & 6+x \\ 2y-3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ -5 & 1 \end{bmatrix}$$

Equating

6 + x = 7

x = 7–6 = 1

x= 1

2y – 3 = –2

2y = –2+3 = –1

y = –1.

Commutative and Associative Laws of matrices addition / subtraction

1. Matrix addition is commutative, that is, A + B = B +A

2. Matrix subtraction is commutative, that is, A –B =–B + A

3. Matrix addition is associative, that is, (A + B) + C = A + (B + C)

(it could also be stated as A + (B + C) = (A + B) + C).

Question 1: Given matrices A and B prove that matrix addition is commutative

$$A = \begin{bmatrix} 7 & 10 & 12 \\ 5 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & -8 & 0 \\ -4 & 6 & -11 \end{bmatrix}$$

We have to prove that A + B = B +A

$$A + B\begin{bmatrix}7 & 10 & 12\\5 & 1 & 3\end{bmatrix} + \begin{bmatrix}3 & -8 & 0\\-4 & 6 & -11\end{bmatrix} = \begin{bmatrix}10 & 2 & 12\\1 & 7 & -8\end{bmatrix}$$

$$B + A = \begin{bmatrix}3 & -8 & 0\\-4 & 6 & -11\end{bmatrix} + \begin{bmatrix}7 & 10 & 12\\5 & 1 & 3\end{bmatrix} = \begin{bmatrix}10 & 2 & 12\\1 & 7 & -8\end{bmatrix}$$

Question 2: Given matrices A and B prove that matrix subtraction is commutative

$$A = \begin{bmatrix}5 & 3\\4 & 9\\10 & 8\\6 & 12\end{bmatrix} \quad B = \begin{bmatrix}3 & 13\\7 & 9\\2 & 1\\8 & 6\end{bmatrix}$$

We have to prove that A –B = – B + A

$$A - B = \begin{bmatrix}5 & 3\\4 & 9\\10 & 8\\6 & 12\end{bmatrix} - \begin{bmatrix}3 & 13\\7 & 9\\2 & 1\\8 & 6\end{bmatrix} = \begin{bmatrix}2 & -10\\-3 & 0\\8 & 7\\-2 & 6\end{bmatrix}$$

$$-B + A = \begin{bmatrix}-3 & -13\\-7 & -9\\-2 & -1\\-8 & -6\end{bmatrix} + \begin{bmatrix}5 & 3\\4 & 9\\10 & 8\\6 & 12\end{bmatrix} = \begin{bmatrix}2 & -10\\-3 & 0\\8 & 7\\-2 & 6\end{bmatrix}$$

A –B = – B + A, so matrix subtraction is commutative.

Question 3: Given the matrices A, B and C, prove that Matrix addition is associative

$$A = \begin{bmatrix}3 & 2 & 4\\-1 & 0 & -5\end{bmatrix} \quad B = \begin{bmatrix}-2 & 3 & -1\\4 & 2 & 0\end{bmatrix} \quad C = \begin{bmatrix}8 & -1 & 5\\6 & 1 & 2\end{bmatrix}$$

We have to prove that (A + B) + C = A + (B + C)

First find (A + B) + C

$$A + B\begin{bmatrix}3 & 2 & 4\\-1 & 0 & -5\end{bmatrix} + \begin{bmatrix}-2 & 3 & -1\\4 & 2 & 0\end{bmatrix} = \begin{bmatrix}1 & 5 & 3\\3 & 2 & 5\end{bmatrix}$$

$$(A + B) + C = \begin{bmatrix}1 & 5 & 3\\3 & 2 & 5\end{bmatrix} + \begin{bmatrix}8 & -1 & 5\\6 & 1 & 2\end{bmatrix} = \begin{bmatrix}9 & 4 & 8\\9 & 3 & -3\end{bmatrix}$$

Now find A + (B + C)

$$B = \begin{bmatrix} -2 & 3 & -1 \\ 4 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 8 & -1 & 5 \\ 6 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 4 \\ 10 & 3 & 2 \end{bmatrix}$$

$$A + (B + C) = \begin{bmatrix} 3 & 2 & 4 \\ -1 & 0 & -5 \end{bmatrix} + \begin{bmatrix} 6 & 2 & 4 \\ 10 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 4 & 8 \\ 9 & 3 & -3 \end{bmatrix}$$

So (A + B) + C = A + (B + C) and it is proved that matrix addition is associative.

Scalar Multiplication

There are three types of multiplication for matrices: scalar multiplication, vector multiplication and matrix multiplication. Scalar multiplication is easy. You just take a regular number (called a "scalar") and multiply it on every entry in the matrix.

Let us make it clear with an example.

For the following matrix A, find 2A and –1A.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

To multiply a matrix by a scalar (that is, a single number), we simply multiply each element in thematrix by this number.
To do the first scalar multiplication to find 2A, just multiply a 2 on every entry in the matrix.

$$2A = 2 \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 \times 2 & 2 \times 2 \\ 3 \times 2 & 4 \times 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

To do the second scalar multiplication to find −1A, just multiply a −1 on every entry in the matrix.

$$-1A = -1 \times \begin{bmatrix} 1 \times -1 & 2 \times -1 \\ 3 \times -1 & 4 \times -1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}$$

So

$$2A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

$$-1A = \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}$$

<u>Question 1: </u> Find A−5B

$A = \begin{bmatrix} 3 & -2 \\ -9 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -4 & 1 \\ 0 & -5 \end{bmatrix}$

First find 5B

$5B = \begin{bmatrix} -20 & 5 \\ 0 & -25 \end{bmatrix}$

$A - 5B = \begin{bmatrix} 3 & -2 \\ -9 & 1 \end{bmatrix} - \begin{bmatrix} -20 & 5 \\ 0 & -25 \end{bmatrix} = \begin{bmatrix} 23 & -7 \\ -9 & 26 \end{bmatrix}$

Matrix multiplication

Matrixmultiplication is where a matrix is multiplied by another matrix. Multiplication of matrices can be easily understood if we follow a step by step approach, first seeing what is vector multiplication and then to matrix multiplication. In any case, if we have to multiply two matrices, they should be conformable for multiplication. The rule of compatibility is as follows.

If we have to multiply two matrices A and B, they are conformable for multiplication only if the number of columns in A is equal to the number of rows in B.

If A is of order m×n and B is of order m×n, then A and Bare conformable for addition only if n of A is equal to m of B. To go to a specific example if matrix A is of order 2×3 and matrix B is of order 3×1, we can say that A and B are conformable for multiplication since the number of columns in A and the number of rows in B are equal to 3. The order of the two matrices will give us another useful information as well. The new matrix that we get after multiplication will have the order of 2×1. This we get by taking the number of rows of A and the number of columns of B. We will illustrate this further as we move on.

(a) Vector Multiplication

We have already stated that a vector is a matrix with just one row or one column. Consider the following example.

Find AB given

$A = \begin{bmatrix} 3 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$

First let us make sure that A and B are conformable for multiplication. A is of order 1×2 and B is of order 2×1. We see that the number of columns in A and the number of rows in B are the same, both are 2. So A and B are conformable for multiplication. Now let us predict the order of the new matrix AB, which will be equal to the number of rows in A and the number of columns in B. That is the new matrix AB will be of the order 1×1, which means it will have just one row and one column.

Matrix multiplication is based on combining rows from the first matrixwith columns from the second matrix in a special way. If we have a row, 3 7, and a column, 29, wecombine them by finding the products of corresponding values and then adding the products as shown:

$$AB = [3 \times 2 + 7 \times 9] = [6 + 63] = [69]$$

Note that AB is a 1×1 matrix as we predicted.

(I strongly recommend to students to do this predication exercise before starting with a problem so that you have an idea as to what your answer will look like. In matrix multiplication it will help you a lot.

$$\underbrace{\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \end{bmatrix}}_{2 \times 3} \cdot \underbrace{\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix}}_{3 \times 2} = \underbrace{\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}}_{2 \times 2}$$

You may even try to write a dummy result matrix before starting so that you will not be confused as to where to stop putting + sign. I will illustrate this later when we discuss matrix multiplication.)

Question 1

Find AB

$$A = [4 \quad 5 \quad 6] \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

First let us make sure that A and B are conformable for multiplication. A is of order 1×3 and B is of order 3×1. We see that the number of columns in A and the number of rows in B are the same, both are 3. So A and B are conformable for multiplication. Now let us predict the order of the new matrix AB, which will be equal to the number of rows in A and the number of columns in B. That is the new matrix AB will be of the order 1×1, which means it will have just one row and one column.

$$AB = [4 \times 1 \quad 5 \times 2 \quad 6 \times 3] = [4 \quad 10 \quad 18]$$

Question 2: Find AB

$$A = [1 \quad 23 \quad 4] \quad B = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$$

$$AB = [1 \times 5 \quad 2 \times 63 \times 7 \quad 4 \times 8]$$

$AB = [5 \quad 1221 \quad 32]$

(b) Matrix Multiplication

Matrix multiplication is an extension of vector multiplication.

Let us explain with help of an example.

$$A = [1 \quad 2 \quad 3] \; B = \begin{bmatrix} 4 & 7 \\ 5 & 8 \\ 6 & 9 \end{bmatrix}$$

First let us make sure that A and B are conformable for multiplication. A is of order 1×3 and B is of order 3×2. We see that the number of columns in A and the number of rows in B are the same, both are 3. So A and B are conformable for multiplication. Now let us predict the order of the new matrix AB, which will be equal to the number of rows in A and the number of columns in B. That is the new matrix AB will be of the order 1×2, which means it will have just one row and 2 columns. Let us write a dummy for that. So our AB matrix will look like this dummy [ ]. So all we have to do is to fill this dummy.

We follow the same procedure we used for vector multiplication. The only difference is that here we repeat the same process once more, this time for the second column also.

$$AB = [1 \times 4 + 2 \times 5 + 3 \times 6 \quad 1 \times 7 + 2 \times 8 + 3 \times 9]$$

$$AB = [4 + 10 + 18 \quad 7 + 16 + 27]$$

$$AB = [32 \quad 50]$$

Question 1: Find AB given

$$A = [1 \quad 2 \quad 3] \; B = \begin{bmatrix} 4 & 7 & 10 \\ 5 & 8 & 0 \\ 6 & 9 & 11 \end{bmatrix}$$

First let us make sure that A and B are conformable for multiplication. A is of order 1×3 and B is of order 3×3. We see that the number of columns in A and the number of rows in B are the same, both are 3. So A and B are conformable for multiplication. Now let us predict the order of the new matrix AB, which will be equal to the number of rows in A and the number of columns in B. That is the new matrix AB will be of the order 1×3, which means it will have just one row and 3 columns. Let us write a dummy for that. So our AB matrix will look like this dummy [ ]. So all we have to do is to fill this dummy.

$$AB = [1 \times 4 + 2 \times 5 + 3 \times 6 \quad 1 \times 7 + 2 \times 8 + 3 \times 9 \quad 1 \times 10 + 2 \times 0 + 3 \times 11]$$

$$AB = [4 + 10 + 18 \quad 7 + 16 + 27 \quad 10 + 0 + 33]$$

$AB = \begin{bmatrix} 32 & 50 & 43 \end{bmatrix}$

Question 2: Find AB given

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 7 & 10 \\ 5 & 8 & 0 \\ 6 & 9 & 11 \end{bmatrix}$$

First let us make sure that A and B are conformable for multiplication. A is of order 2×3 and B is of order 3×3. We see that the number of columns in A and the number of rows in B are the same, both are 3. So A and B are conformable for multiplication. Now let us predict the order of the new matrix AB, which will be equal to the number of rows in A and the number of columns in B. That is the new matrix AB will be of the order 2×3, which means it will have 2 rows and 3 columns. Let us write a dummy for that. So our AB matrix will look like this dummy $\begin{bmatrix} & & \\ & & \end{bmatrix}$. So all we have to do is to fill this dummy.

$$AB = \begin{bmatrix} 1 \times 4 + 2 \times 5 + 3 \times 6 & 1 \times 7 + 2 \times 8 + 3 \times 9 & 1 \times 10 + 2 \times 0 + 3 \times 11 \\ 4 \times 4 + 5 \times 5 + 6 \times 6 & 4 \times 7 + 5 \times 8 + 6 \times 9 & 4 \times 10 + 5 \times 0 + 6 \times 11 \end{bmatrix}$$

$$AB = \begin{bmatrix} 4 + 10 + 18 & 7 + 16 + 27 & 10 + 0 + 33 \\ 16 + 25 + 36 & 28 + 40 + 54 & 40 + 0 + 66 \end{bmatrix}$$

$$AB = \begin{bmatrix} 32 & 50 & 43 \\ 77 & 122 & 106 \end{bmatrix}$$

Question 3: Find AB given

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 7 & 10 \\ 5 & 8 & 0 \\ 6 & 9 & 11 \end{bmatrix}$$

First let us make sure that A and B are conformable for multiplication. A is of order 3×3 and B is of order 3×3. We see that the number of columns in A and the number of rows in B are the same, both are 3. So A and B are conformable for multiplication. Now let us predict the order of the new matrix AB, which will be equal to the number of rows in A and the number of columns in B. That is the new matrix AB will be of the order 3×3, which means it will have 3 rows and 3 columns. Let us write a dummy for that. So our AB matrix will look like this dummy $\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$. So all we have to do is to fill this dummy.

$$AB = \begin{bmatrix} 1\times4+2\times5+3\times6 & 1\times7+2\times8+3\times9 & 1\times10+2\times0+3\times11 \\ 4\times4+5\times5+6\times6 & 4\times7+5\times8+6\times9 & 4\times10+5\times0+6\times11 \\ 7\times4+8\times5+9\times6 & 7\times7+8\times8+9\times9 & 7\times10+8\times0+9\times11 \end{bmatrix}$$

$$AB = \begin{bmatrix} 4+10+18 & 7+16+27 & 10+0+33 \\ 16+25+36 & 28+40+54 & 40+0+66 \\ 28+40+54 & 49+64+81 & 70+0+99 \end{bmatrix}$$

$$AB = \begin{bmatrix} 32 & 50 & 43 \\ 77 & 122 & 106 \\ 122 & 194 & 169 \end{bmatrix}$$

Question 4: Verify AI = IA = A, given $A = \begin{bmatrix} 1 & -4 & 7 \\ 3 & 2 & -5 \\ -1 & -6 & -8 \end{bmatrix}$

Here I is an identity matrix. The order of this identity matrix is not specified, so we presume it as 3.

Or

Show that for every square matrix A, there exists an identity matrix of the same order such that IA = AI =A

Solution

First find AI

$$AI = \begin{bmatrix} 1 & -4 & 7 \\ 3 & 2 & -5 \\ -1 & -6 & -8 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AI = \begin{bmatrix} 1\times1+-4\times0+7\times0 & 1\times0+-4\times1+7\times0 & 1\times0+-4\times0+7\times1 \\ 3\times1+2\times0+\ -5\times0 & 3\times0+2\times1+\ -5\times0 & 3\times0+2\times0+\ -5\times1 \\ -1\times1+-6\times0+\ -8\times0 & -1\times0+-6\times1+\ -8\times0 & -1\times0+-6\times0+\ -8\times1 \end{bmatrix}$$

$$AI = \begin{bmatrix} 1 & -4 & 7 \\ 3 & 2 & -5 \\ -1 & -6 & -8 \end{bmatrix} \textit{which is equal to } A$$

Now find IA.

$$IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -4 & 7 \\ 3 & 2 & -5 \\ -1 & -6 & -8 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1\times1+0\times3+0\times-1 & 1\times-4+0\times2+0\times-6 & 1\times7+0\times-5+0\times-8 \\ 0\times1+1\times3+0\times-1 & 0\times-4+1\times2+0\times-6 & 0\times7+1\times-5+0\times-8 \\ 0\times1+0\times3+1\times-1 & 0\times-4+0\times2+1\times-6 & 0\times7+0\times-5+1\times-8 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1 & -4 & 7 \\ 3 & 2 & -5 \\ -1 & -6 & -8 \end{bmatrix} \text{ which is equal to } A$$

Hence it is proved that AI = IA = A.

Properties of Matrix Multiplication

Let A, B and C be matrices of dimensions such that the following are defined. Then

1. Matrix multiplication is associative

(AB)C = A(BC)

2. Matrix multiplication is distributive

A(B + C) = AB + AC or (A + B)C = AC + BC

3. Matrix multiplication is not commutative

AB ≠ BA

Example 1: Given three matrices $A = \begin{bmatrix} 7 & 1 & 5 \end{bmatrix} B = \begin{bmatrix} 6 & 5 \\ 2 & 4 \\ 3 & 8 \end{bmatrix} C = \begin{bmatrix} 9 & 4 \\ 3 & 10 \end{bmatrix}$

Prove that matrix multiplication is associative.

Solution:

We have to prove that (AB)C = A(BC)

First find AB

$$AB = \begin{bmatrix} 7 & 1 & 5 \end{bmatrix} \begin{bmatrix} 6 & 5 \\ 2 & 4 \\ 3 & 8 \end{bmatrix}$$

First let us make sure that A and B are conformable for multiplication. A is of order 1×3 and B is of order 3×2. We see that the number of columns in A and the number of rows in B are the same, both are 3. So A and B are conformable for multiplication. Now let us predict the order of the new matrix AB, which will be equal to the number of rows in A and the number of columns in B. That is the new matrix AB will be of the order 1×2, which means it will have just one row and 2 columns. Let us write a dummy for that. So our AB matrix will look like this dummy[ ]. So all we have to do is to fill this dummy.

$AB = [7 \times 6 + 1 \times 2 + 5 \times 3 \quad 7 \times 5 + 1 \times 4 + 5 \times 8]$

$AB = [59 \quad 79]$

Now find (AB)C

$(AB)C = [59 \quad 79] \times \begin{bmatrix} 9 & 4 \\ 3 & 10 \end{bmatrix}$

First let us make sure that A and B are conformable for multiplication. A is of order 1×2 and B is of order 2×2. We see that the number of columns in A and the number of rows in B are the same, both are 2. So A and B are conformable for multiplication. Now let us predict the order of the new matrix AB, which will be equal to the number of rows in A and the number of columns in B. That is the new matrix AB will be of the order 1×2, which means it will have just one row and 2 columns. Let us write a dummy for that. So our AB matrix will look like this dummy $[\quad\quad]$. So allType equation here. we have to do is to fill this dummy.

$(AB)C = [59 \times 9 + 79 \times 3 \quad 59 \times 4 + 79 \times 10]$

$(AB)C = [531 + 237 \quad 236 + 790]$

$(AB)C = [768 \quad 1026]$

Now find BC

$BC = \begin{bmatrix} 6 & 5 \\ 2 & 4 \\ 3 & 8 \end{bmatrix} \times \begin{bmatrix} 9 & 4 \\ 3 & 10 \end{bmatrix}$

First let us make sure that A and B are conformable for multiplication. A is of order 3×2 and B is of order 2×2. We see that the number of columns in A and the number of rows in B are the same, both are 2. So A and B are conformable for multiplication. Now let us predict the order of the new matrix AB, which will be equal to the number of rows in A and the number of columns in B. That is the new matrix AB will be of the order 3×2, which means it will have 3 rows and 2 columns. Let us write a dummy for that. So our AB matrix will look like this dummy $\begin{bmatrix} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{bmatrix}$. So all we have to do is to fill this dummy.

$$BC = \begin{bmatrix} 6 \times 9 + 5 \times 3 & 6 \times 4 + 5 \times 10 \\ 2 \times 9 + 4 \times 3 & 2 \times 4 + 4 \times 10 \\ 3 \times 9 + 8 \times 3 & 3 \times 4 + 8 \times 10 \end{bmatrix}$$

$$BC = \begin{bmatrix} 54 + 15 & 24 + 50 \\ 18 + 12 & 8 + 40 \\ 27 + 24 & 12 + 80 \end{bmatrix}$$

$$BC = \begin{bmatrix} 69 & 74 \\ 30 & 48 \\ 51 & 92 \end{bmatrix}$$

Now find A(BC)

$$A(BC) = \begin{bmatrix} 7 & 1 & 5 \end{bmatrix} \begin{bmatrix} 69 & 74 \\ 30 & 48 \\ 51 & 92 \end{bmatrix}$$

First let us make sure that A and B are conformable for multiplication. A is of order 1×3 and B is of order 3×2. We see that the number of columns in A and the number of rows in B are the same, both are 3. So A and B are conformable for multiplication. Now let us predict the order of the new matrix AB, which will be equal to the number of rows in A and the number of columns in B. That is the new matrix AB will be of the order 1×2, which means it will have just one row and 2 columns. Let us write a dummy for that. So our AB matrix will look like this dummy [ ]. So all we have to do is to fill this dummy.

$$A(BC) = \begin{bmatrix} 7 \times 69 + 1 \times 30 + 5 \times 51 & 7 \times 74 + 1 \times 48 + 5 \times 92 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 483 + 30 + 255 & 518 + 48 + 460 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 768 & 1026 \end{bmatrix}$$

Hence we see that

$$(AB)C = \begin{bmatrix} 768 & 1026 \end{bmatrix}$$

And

$$A(BC) = \begin{bmatrix} 768 & 1026 \end{bmatrix}$$

Hence matrix multiplication is associative

<u>Example 2:</u> Given three matrices $A = \begin{bmatrix} 4 & 7 & 2 \end{bmatrix} B = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} C = \begin{bmatrix} 9 \\ 5 \\ 8 \end{bmatrix}$

Prove that matrix multiplication is distributive.

Solution

We have to prove that A(B + C) = AB + AC

First find B + C

$$B + C = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} + \begin{bmatrix} 9 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \\ 9 \end{bmatrix}$$

Now find A(B + C)

$$A(B + C) = [4 \quad 7 \quad 2] \times \begin{bmatrix} 15 \\ 10 \\ 9 \end{bmatrix}$$

First let us make sure that A and B are conformable for multiplication. A is of order 1×3 and B is of order 3×1. We see that the number of columns in A and the number of rows in B are the same, both are 3. So A and B are conformable for multiplication. Now let us predict the order of the new matrix AB, which will be equal to the number of rows in A and the number of columns in B. That is the new matrix AB will be of the order 1×1, which means it will have just one row and one column. Let us write a dummy for that. So our AB matrix will look like this dummy [ ]. So all we have to do is to fill this dummy.

$$A(B + C) = [4 \times 15 + 7 \times 10 + 2 \times 9]$$

$$A(B + C) = [60 + 70 + 18] = [148]$$

Now find AB

$$AB = [4 \quad 7 \quad 2] \times \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$$

First let us make sure that A and B are conformable for multiplication. A is of order 1×3 and B is of order 3×1. We see that the number of columns in A and the number of rows in B are the same, both are 3. So A and B are conformable for multiplication. Now let us predict the order of the new matrix AB, which will be equal to the number of rows in A and the number of columns in B. That is the new matrix AB will be of the order 1×1, which means it will have just one row and one column. Let us write a dummy for that. So our AB matrix will look like this dummy [ ]. So all we have to do is to fill this dummy.

$$AB = [4 \times 6 + 7 \times 5 + 2 \times 1] = [24 + 35 + 2] = [61]$$

Now find AC

$$AC = [4 \quad 7 \quad 2] \times \begin{bmatrix} 9 \\ 5 \\ 8 \end{bmatrix}$$

First let us make sure that A and B are conformable for multiplication. A is of order 1×3 and B is of order 3×1. We see that the number of columns in A and the number of rows in B are the same, both are 3. So A and B are conformable for multiplication. Now let us predict the order of the new matrix AB, which will be equal to the number of rows in A and the number of columns in B. That is the new matrix AB will be of the order 1×1, which means it will have just one row and one column. Let us write a dummy for that. So our AB matrix will look like this dummy [ ]. So all we have to do is to fill this dummy.

$$AC = [4 \times 9 + 7 \times 5 + 2 \times 8] = [36 + 35 + 16] = [87]$$

Now find AB + AC

$$AB + AC = [61] + [87] = [148]$$

Associative: A(BC) = (AB)C

Distributive: A(B + C) = AB + AC

Distributive: (A + B)C = AC + BC

C(AB) = (CA)B = A(CB), where c is a constant, please notice that A·B ≠ B·A

<u>Example 3</u>

Given the two matrices A and B, prove that matrix multiplication is not commutative.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \; B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Solution

We have to show that AB ≠ BA

$$AB = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

First let us make sure that A and B are conformable for multiplication. A is of order 2×2 and B is of order 2×2. We see that the number of columns in A and the number of rows in B are the same, both are 2. So A and B are conformable for multiplication. Now let us predict the order of the new matrix AB, which will be equal to the number of rows in A and the number of columns in B. That is the new matrix AB will be of the order 2×2, which means it will have 2 rows and 2 columns. Let us write a dummy for that. So our AB matrix will look like this dummy $\begin{bmatrix} & \\ & \end{bmatrix}$. So all we have to do is to fill this dummy.

$$AB = \begin{bmatrix} 1 \times 2 + 1 \times 1 & 1 \times 1 + 1 \times 1 \\ 2 \times 2 + 1 \times 1 & 2 \times 1 + 1 \times 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

Now find BA.

First let us make sure that B and A are conformable for multiplication. B is of order 2×2 and A is of order 2×2. We see that the number of columns in B and the number of rows in A are the same, both are 2. So B and A are conformable for multiplication. Now let us predict the order of the new matrix BA, which will be equal to the number of rows in B and the number of columns in A. That is the new matrix BA will be of the order 2×2, which means it will have just two rows and 2 columns. Let us write a dummy for that. So our BA matrix will look like this dummy $\begin{bmatrix} & \\ & \end{bmatrix}$. So all we have to do is to fill this dummy.

$$BA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 \times 1 + 1 \times 2 & 2 \times 1 + 1 \times 1 \\ 1 \times 1 + 1 \times 2 & 1 \times 1 + 1 \times 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

So we see that

$$AB = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

Hence AB ≠ BA, matrix multiplication is not commutative.

Exercises

<u>Question 1:</u> Verify the associative property of matrix multiplication for the following matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 5 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \\ -2 & -1 \end{bmatrix}$$

Solution:

Here we to show thatA(BC) = (AB)C usingassociativeproperty.

First find BC

$$BC = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 5 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 4 \\ -2 & -1 \end{bmatrix}$$

Let us make sure that B and C are conformable for multiplication. B is of order 2×3 and C is of order 3×2. We see that the number of columns in B and the number of rows in C are the same, both are 3. So B and C are conformable for multiplication. Now let us predict the order of the new matrix BC, which will be equal to the number of rows in B and the number of columns in C. That is the new matrix BC will be of the order 2×2, which means it will have 2 rows and 2 columns. Let us write a dummy for that. So our BA matrix will look like this dummy $\begin{bmatrix} & \\ & \end{bmatrix}$. So all we have to do is to fill this dummy.

$$BC = \begin{bmatrix} 2\times 0 + 0\times 1 + -1\times -2 & 2\times 2 + 0\times 4 + -1\times -1 \\ 3\times 0 + 5\times 1 + 1\times -2 & 3\times 2 + 5\times 4 + 1\times -1 \end{bmatrix}$$

$$BC = \begin{bmatrix} 2 & 5 \\ 3 & 25 \end{bmatrix}$$

Now find A(BC)

$$A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 3 & 25 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 8 & 55 \\ 18 & 115 \end{bmatrix}$$

Now find AB

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & -1 \\ 3 & 5 & 1 \end{bmatrix}$$

First let us make sure that A and B are conformable for multiplication. A is of order 2×2 and B is of order 2×3. We see that the number of columns in A and the number of rows in B are the same, both are 2. So A and B are conformable for multiplication. Now let us predict the order of the new matrix AB, which will be equal to the number of rows in A and the number of columns in B. That is the new matrix AB will be of the order 2×3, which means it will have 2 rows and 3 columns. Let us write a dummy for that. So our AB matrix will look like this dummy $\begin{bmatrix} & & \\ & & \end{bmatrix}$. So all we have to do is to fill this dummy.

$$AB = \begin{bmatrix} 1\times 2 + 2\times 3 & 1\times 0 + 2\times 5 & 1\times -1 + 2\times 1 \\ 3\times 2 + 4\times 3 & 3\times 0 + 4\times 5 & 3\times -1 + 4\times 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 8 & 10 & 1 \\ 18 & 20 & 1 \end{bmatrix}$$

Now find (AB)C

$$AB(C) = \begin{bmatrix} 8 & 10 & 1 \\ 18 & 20 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 2 \\ 1 & 4 \\ -2 & -1 \end{bmatrix}$$

First let us make sure that AB and C are conformable for multiplication. AB is of order 2×3 and Cis of order 3×2. We see that the number of columns in ABand the number of rows in C are the same, both are 3. So AB and C are conformable for multiplication. Now let us predict the order of the new matrix AB(C), which will be equal to the number of rows in AB and the number of columns in C. That is the new matrix AB(C) will be of the order 2×2, which means it will have 2 rows and 2columns. Let us write a dummy for that. So our AB(C) matrix will look like this dummy $\begin{bmatrix} & \end{bmatrix}$. So all we have to do is to fill this dummy.

$$AB(C) = \begin{bmatrix} 8\times 0 + 10\times 1 + 1\times -2 & 8\times 2 + 10\times 4 + 1\times -1 \\ 18\times 0 + 20\times 1 + 1\times -2 & 18\times 2 + 20\times 4 + 1\times -1 \end{bmatrix}$$

$$AB(C) = \begin{bmatrix} 8 & 55 \\ 18 & 115 \end{bmatrix}$$

So we see that

$$A(BC) = \begin{bmatrix} 8 & 55 \\ 18 & 115 \end{bmatrix}$$

$$AB(C) = \begin{bmatrix} 8 & 55 \\ 18 & 115 \end{bmatrix}$$

A(BC) = (AB)C (associative property).

Question 2

Swami restaurant sells 1000 ghee roast, 600 masala dosa and 1200 uthappam in a week. The price of one ghee roast is Rs. 45, one masladosa is Rs. 60 and one uthappam is Rs. 50. The coast of a ghee roast is Rs.38, a masala dosa is Rs. 42 and one uthappam is Rs. 32. Find the profit of Swami restaurant.

Solution

Profit = Total Revenue – Total Cost

TR = P×Q

TC = C×Q

Let us form matrices with the information.

Let matrix Q represent quantity sold, matrix P represent prices and matrix C represent cost.

$$Q = \begin{bmatrix} 1000 \\ 600 \\ 1200 \end{bmatrix} \quad P = \begin{bmatrix} 0.45 \\ 0.60 \\ 0.50 \end{bmatrix} \quad C = \begin{bmatrix} 0.38 \\ 0.42 \\ 0.32 \end{bmatrix}$$

$$TR = P \times Q = \begin{bmatrix} 0.45 \\ 0.60 \\ 0.50 \end{bmatrix} \times \begin{bmatrix} 1000 \\ 600 \\ 1200 \end{bmatrix}$$

First let us make sure that P and Q are conformable for multiplication. P is of order 3×1 and Q is of order 3×1. We see that the number of columns in P (1) and the number of rows in Q (3) are NOT the same. So P and Q are NOT conformable for multiplication.

In such situations, one way out is we can take the transpose of one of the matrices and do the multiplication. Let us take the transpose of P.

$$P^T = [0.45 \quad 0.60 \quad 0.50]$$

Now we find TR as TR = $P^T \times Q$

Let us make sure that P^T and Q are conformable for multiplication. P^T is of order 1×3 and Q is of order 3×1. We see that the number of columns in P^T and the number of rows in Q are the same. So P^T and Q are conformable for multiplication. Now let us predict the order of the new matrix $P^T(Q)$, which will be equal to the number of rows in P^T and the number of columns in Q. That is the new matrix P^TQ will be of the order 1×1, which means it will have just one row and one column. Let us write a dummy for that. So our P^TQ matrix will look like this dummy [   ]. So all we have to do is to fill this dummy.

$$\mathrm{P^TQ} = [0.45 \quad 0.60 \quad 0.50] \times \begin{bmatrix} 1000 \\ 600 \\ 1200 \end{bmatrix}$$

$$\mathrm{P^TQ} = [0.45 \times 1000 + \quad 0.60 \times 600 \quad 0.50 \times 1200]$$

$$\mathrm{P^TQ} = [450 + \quad 360 \quad +600] = [1410]$$

So TR is Rs. 1410. Now we have to find TC.

$$TC = C \times Q = \begin{bmatrix} 0.38 \\ 0.42 \\ 0.32 \end{bmatrix} \times \begin{bmatrix} 1000 \\ 600 \\ 1200 \end{bmatrix}$$

First let us make sure that C and Q are conformable for multiplication. C is of order 3×1 and Q is of order 3×1. We see that the number of columns in C (1) and the number of rows in Q (3) are NOT the same. So C and Q are NOT conformable for multiplication.

In such situations, one way out is we can take the transpose of one of the matrices and do the multiplication. Let us take the transpose of C.

$$C^T = [0.38 \quad 0.42 \quad 0.32]$$

Now we find TR as TR = $C^T \times Q$

Let us make sure that C^T and Q are conformable for multiplication. C^T is of order 1×3 and Q is of order 3×1. We see that the number of columns in P^T and the number of rows in Q are the same. So C^T and Q are conformable for multiplication. Now let us predict the order of the new matrix $C^T(Q)$, which will be equal to the number of rows in C^T and the number of columns in Q. That is the new matrix C^TQ will be of the order 1×1, which means it will have just one row and one column. Let us write a dummy for that. So our C^TQ matrix will look like this dummy [ ]. So all we have to do is to fill this dummy.

$$C^TQ = \begin{bmatrix} 0.38 & 0.42 & 0.32 \end{bmatrix} \begin{bmatrix} 1000 \\ 600 \\ 1200 \end{bmatrix}$$

$$C^TQ = \begin{bmatrix} 0.38 \times 1000 + & 0.42 \times 600 + & 0.32 \times 1200 \end{bmatrix}$$

$$C^TQ = [380 + 252 + 384] = [1016]$$

So we get TC = Rs. 1016

Now let us find profit using the formula $\pi = TR - TC = [1410] - [1016] = [394]$

So the profit is Rs. 394.

The same problem can be worked out in another manner.

We can first work out per unit profit by finding P – C, and then find total profit by multiplying P – C with Q. By this method also we should get profit as Rs. 394. Let us try that.

$$\pi = P - C = \begin{bmatrix} 0.45 \\ 0.60 \\ 0.50 \end{bmatrix} - \begin{bmatrix} 0.38 \\ 0.42 \\ 0.32 \end{bmatrix} = \begin{bmatrix} 0.07 \\ 0.18 \\ 0.18 \end{bmatrix}$$

Now find total profit

$$Total\ profit = \pi \times Q = \begin{bmatrix} 0.07 \\ 0.18 \\ 0.18 \end{bmatrix} \times \begin{bmatrix} 1000 \\ 600 \\ 1200 \end{bmatrix}$$

Here also we have to find transpose of Q.

$$Total\ profit = \pi \times Q^T = \begin{bmatrix} 0.07 \\ 0.18 \\ 0.18 \end{bmatrix} \times \begin{bmatrix} 1000 & 600 & 1200 \end{bmatrix}$$

$Total\ profit = \ \pi \times Q = [0.07 \times 1000 + 0.18 \times 600 + 0.18 \times 1200]$

$Total\ profit = \ [70 + 108 + 216] = [394]$

Total profit is Rs. 394.

Hence we see that the answers are the same by both the methods.

Question 3: Verify the distributive property of matrix multiplication for the following matrices.

$$A = \begin{bmatrix} 1 & 2 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$C = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Solution:

We have to show that the distributive property is satisfied: A(B + C) = AB + AC

$$A(B + C) = \ [1 \quad 2]\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = \ [1 \quad 2] \times \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$A(B + C) = [1 \times 4 + 2 \times 6] = \ [16]$$

Now find AB + AC

$$AB + AC = \ [1 \quad 2]\begin{bmatrix} 1 \\ 2 \end{bmatrix} + [1 \quad 2]\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \ [5] + [11] = [16]$$

So distributive property is proved A(B + C) = AB + AC

Question 4Prove the associative property given

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} -2 & 3 \\ 4 & 2 \end{bmatrix}, C = \begin{bmatrix} -1 & 5 \\ 1 & 2 \end{bmatrix}$$

Associative: A(BC) = (AB)C

Find A(BC)

$$A(BC) = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \left(\begin{bmatrix} -2 & 3 \\ 4 & 2 \end{bmatrix} \times \begin{bmatrix} -1 & 5 \\ 1 & 2 \end{bmatrix} \right)$$

$$A(BC) = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \left(\begin{bmatrix} 5 & -4 \\ -2 & 24 \end{bmatrix} \right)$$

$$A(BC) = \begin{bmatrix} 11 & 36 \\ -5 & 4 \end{bmatrix}$$

Now find (AB)C

$$(AB)C = \left(\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} -2 & 3 \\ 4 & 2 \end{bmatrix} \right) \left(\begin{bmatrix} -1 & 5 \\ 1 & 2 \end{bmatrix} \right)$$

$$(AB)C = \left(\begin{bmatrix} 2 & 13 \\ 2 & -3 \end{bmatrix} \times \begin{bmatrix} -1 & 5 \\ 1 & 2 \end{bmatrix} \right)$$

$$(AB)C = \begin{bmatrix} 11 & 36 \\ -5 & 4 \end{bmatrix}$$

Hence A(BC) = (AB)C

So matrix multiplication is associative.

Matrix Expression of a Set of Linear Equations

A system of linear equations can be expressed is matrix form. Let us make it clear with an example.

Consider the system of linear equations

$7x_1 + 3x_2 = 45$

$4x_1 + 5x_2 = 29$

This can be expressed in the matrix form AX = B

Here A is the coefficient matrix. So include the four coefficients of the equation 7, 3, 4 and 5 in the matrix.

So $A = \begin{bmatrix} 7 & 3 \\ 4 & 5 \end{bmatrix}$

X is the solution vector. So include the variables for which we have to find the solutions in this matrix, that is, x_1 and x_2. X will always be a matrix with one column, that is, a column vector.

So $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

B is the matrix containing the constants of the equation, that is, 45 and 29.

So $B = \begin{bmatrix} 45 \\ 29 \end{bmatrix}$

So we can re write the system of linear equations given in the question as

$AX = B$

$\begin{bmatrix} 7 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 45 \\ 29 \end{bmatrix}$

Now find AX

$AX = \begin{bmatrix} 7 & 3 \\ 4 & 5 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

A is of order 2×2 and X is of order 2×1. So A and X are conformable for multiplication and the resulting matrix AX will be of order 2 × 1.

$AX = \begin{bmatrix} 7x_1 + 3x_2 \\ 4x_1 + 5x_2 \end{bmatrix}$

Since by definition AX = B, we can write

$\begin{bmatrix} 7x_1 + 3x_2 \\ 4x_1 + 5x_2 \end{bmatrix} = \begin{bmatrix} 45 \\ 29 \end{bmatrix}$

To find the value of the unknowns, x_1 and x_2, we have to make use of a technique called inverse matrix which we will study in the next module. The formula for inverse of a matrix is $x = A^{-1}B$.

Additional Questions

Question 1 : Add the two matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

Question 2: Subtract

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1-5 & 2-6 \\ 3-7 & 4-8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

Question 3: Multiply by the scalar 5

$$5 \cdot \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 5 \cdot 1 & 5 \cdot 2 & 5 \cdot 3 \\ 5 \cdot (-1) & 5 \cdot (-2) & 5 \cdot (-3) \end{bmatrix} = \begin{bmatrix} 5 & 10 & 15 \\ -5 & -10 & -15 \end{bmatrix}$$

Question 4: Vector multiplication

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}_{1 \times 3} \cdot \underbrace{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}}_{3 \times 1} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = \underbrace{22}_{1 \times 1}$$

Question 5: Matrix multiplication

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}_{1 \times 3} \cdot \underbrace{\begin{bmatrix} 2 & 1 & 3 \\ 3 & 3 & 2 \\ 4 & 1 & 2 \end{bmatrix}}_{3 \times 3} = \begin{bmatrix} 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 \\ 1 \cdot 1 + 2 \cdot 3 + 3 \cdot 1 \\ 1 \cdot 3 + 2 \cdot 2 + 3 \cdot 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 20 \\ 10 \\ 13 \end{bmatrix}}_{1 \times 3}$$

Question 6: Find the product AB where A and B are matrices given by

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 6 \\ 1 & 4 \\ 5 & 2 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \cdot \begin{bmatrix} 3 & 6 \\ 1 & 4 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 31 & 28 \\ 42 & 40 \end{bmatrix}$$

Question 7:

Find the product AB where A and B are matrices given by:

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix}$$

Solution:

$$A \cdot B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 2\cdot(-2)+1\cdot 4 & 2\cdot 3+1\cdot(-1) \\ 3\cdot(-2)+5\cdot 4 & 3\cdot 3+5\cdot(-1) \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 14 & 4 \end{bmatrix}$$

Question 8:

Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$ $and\ B = \begin{bmatrix} 5 & 6 & 7 \\ 3 & 4 & 5 \end{bmatrix}$ $find\ A + B\ andB + A$

$$A + B = \begin{bmatrix} 1+5 & 2+6 & 3+7 \\ 7+3 & 8+4 & 9+5 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 10 \\ 10 & 12 & 14 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1-5 & 2-6 & 3-7 \\ 7-3 & 8-4 & 9-5 \end{bmatrix} = \begin{bmatrix} -4 & -4 & -4 \\ 4 & 4 & 4 \end{bmatrix}$$

Question 9: Given three matrices A, B and C, show that A−B = C

$$A = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}, B = \begin{bmatrix} 4 & 5 \\ 6 & 6 \end{bmatrix}\ and\ C = \begin{bmatrix} -1 & 0 \\ -2 & 0 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 5 \\ 6 & 6 \end{bmatrix} = \begin{bmatrix} 3-4 & 5-5 \\ 4-6 & 6-6 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -2 & 0 \end{bmatrix}$$

So A−B = C

Question 10: Given two matrices Aand B, find AB.

$$A = \begin{bmatrix} 1 & 6 & -2 \\ 0 & -3 & 10 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 0 & -1 & -1 \\ 2 & 4 & 0 & 6 \\ -1 & -2 & 4 & 1/2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 19 & 28-9 & 36 \\ -16 & -3240 & -13 \end{bmatrix}$$

Question 11: Find CD and DC, if they exist, given that C and D are the following matrices:

$$C = \begin{bmatrix} 2 & -1 \\ 0 & 3 \\ 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 1 & 4 & -1 \\ -2 & 0 & 0 & 2 \end{bmatrix}$$

C is a 3×2 matrix and *D* is a 2×4 matrix, so first I'll look at the dimension product for *CD*:

So the product CD is defined (that is, we can do the multiplication); also, we can tell that we are going to get a 3×4 matrix for my answer. Here is the multiplication:

$$CD = \begin{bmatrix} 2 & -1 \\ 0 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 & -1 \\ -2 & 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cdot 0+(-1)(-2) & 2\cdot 1+(-1)\cdot 0 & 2\cdot 4+(-1)\cdot 0 & 2\cdot(-1)+(-1)\cdot 2 \\ 0\cdot 0+3\cdot(-2) & 0\cdot 1+3\cdot 0 & 0\cdot 4+3\cdot 0 & 0\cdot(-1)+3\cdot 2 \\ 1\cdot 0+0\cdot(-2) & 1\cdot 1+0\cdot 0 & 1\cdot 4+0\cdot 0 & 1\cdot(-1)+0\cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0+2 & 2+0 & 8+0 & -2-2 \\ 0-6 & 0+0 & 0+0 & 0+6 \\ 0+0 & 1+0 & 4+0 & -1+0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 8 & -4 \\ -6 & 0 & 0 & 6 \\ 0 & 1 & 4 & -1 \end{bmatrix}$$

Matrix DC is not defined.

Question 12: A is a 3 × 2 matrix

B is a 2 × 3 matrix

C is a 2 × 2 matrix

D is a 3 × 3 matrix

Which of the following products does not exist?

(a) AB (b) AC (c) BD (d) CD

The correct answer is (d) CD.

The reason is

When we do multiplication of matrices, the number of columns of the 1st matrix must equal the number of rows of the 2nd matrix.

The number of columns of A (2) and the number of rows of B (2) are equal, so the product AB exists.

The number of columns of A (2) and the number of rows of C (2) are equal, so the product AC exists.

The number of columns of B (3) and the number of rows of D (3) are equal, so the product BD exists.

The number of columns of C (2) and the number of rows of D (3) are not equal, so the product CD does not exist

Question 13:

If $P = \begin{bmatrix} -2 & 5 \\ 1 & -3 \end{bmatrix}$ and $Q = \begin{bmatrix} 4 & -2 \\ 3 & 2 \end{bmatrix}$,then what is PQ ?

$$\begin{bmatrix} -2 & 5 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -2\times4+5\times3 & (-2)\times(-2)+5\times2 \\ 1\times4+(-3)\times3 & 1\times(-2)+(-3)\times2 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ -5 & -8 \end{bmatrix}$$

PQ = $\begin{bmatrix} 7 & 14 \\ -5 & -8 \end{bmatrix}$

Question 14:

If $P = \begin{bmatrix} -2 & 5 \\ 1 & -3 \end{bmatrix}$ and $Q = \begin{bmatrix} 4 & -2 \\ 3 & 2 \end{bmatrix}$,then what is QP ?

$$\begin{bmatrix} 4 & -2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -2 & 5 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 4\times-2+-2\times1 & 4\times5+(-2)\times(-3) \\ 3\times(-2)+2\times1 & 3\times5+2\times(-3) \end{bmatrix} = \begin{bmatrix} -10 & 26 \\ -4 & 9 \end{bmatrix}$$

QP = $\begin{bmatrix} -10 & 26 \\ -4 & 9 \end{bmatrix}$

Question 15

If $A = \begin{bmatrix} 1 & x \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $AB = BA$ then what is the value of x?

Answer :x = 2

$$AB = \begin{bmatrix} 1 & x \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1\times1+x\times1 & 1\times1+x\times2 \\ 2\times1+3\times1 & 2\times1+3\times2 \end{bmatrix} = \begin{bmatrix} 1+x & 1+2x \\ 5 & 8 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & x \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1\times1+1\times2 & 1\times x+1\times3 \\ 1\times1+2\times2 & 1\times x+2\times3 \end{bmatrix} = \begin{bmatrix} 3 & x+3 \\ 5 & x+6 \end{bmatrix}$$

If AB = BA, then the numbers in each cell of the two matrices must be equal

⇒ 1 + x = 3, so x=2

⇒ 1 + 2x = x + 3 ⇒ 2x = x + 2 ⇒ x = 2

⇒ 5 = 5 and

⇒ 8 = x + 6, so x=2

These equations are all satisfied by x

MODULE III

Matrix Inversion

Determinants

A determinant is a real number associated with every square matrix.Determinant is a single number obtained from a matrix that reveals a variety of the matrix's properties.The determinant tells us things about the matrix that are useful in systems of linear equations, helps us find the inverse of a matrix, is useful in calculus and more.

The determinant of a square matrix A is denoted by "det A" or $|A|$. Thus the symbol for determinant is two vertical lines either side. Note that it is exactly the same symbol as absolute value.

Example: **|A|** means the determinant of the matrix **A**

Calculating the Determinant

First of all the matrix must be square matrix to find the determinant. Let us start with a 2×2 matrix.

Finding determinant for a 2×2 matrix

Consider the following 2×2 matrix.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant in this case is the product of the elements on the main (principal) diagonal minus the product of the elements off the main diagonal. So here the elements on the main diagonal are a and d. So take their product, that is, a × d. Similarly find the products off the principal diagonal. Here the elements on the off the main diagonal are c and b. So take their product, that is, c × b. Now the deduct the product of the elements off the main diagonal from the product on the elements off the main diagonal.

Determinant A = |A| = ad – cb

Example: Find determinant of matrix A. $A = \begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$

$$|A| = (4 \times 8) - (3 \times 6) = 32 - 18 = 14$$

Question 1: Find the determinant of the matrix $\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix}$

$$|A| = (3 \times 6) - (4 \times 8) = 18 - 32 = -14$$

Question 2: Find the determinant of the matrix $A = \begin{bmatrix} 2 & 5 \\ 1 & -3 \end{bmatrix}$

$$|A| = (2 \times -3) - (1 \times 5) = -6 - 5 = -11$$

Question 3: Find the determinant of the matrix $A = \begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix}$

$|A| = (3 \times 1) - (2 \times -5) = 3 - -10 = 3 + 10 = 13$

Question 4: Find the determinant of the matrix $A = \begin{bmatrix} -2 & 4 \\ -6 & 2 \end{bmatrix}$

$|A| = (-2 \times 2) - (-6 \times 4) = -4 - -24 = -4 + 24 = 20$

Question 5: Find the determinant of the matrix $A = \begin{bmatrix} -3 & -1 \\ 4 & -5 \end{bmatrix}$

$|A| = (-3 \times -5) - (4 \times -1) = 15 - -4 = 15 + 4 = 19$

Question 6: Find the determinant of the matrix $A = \begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix}$

$|A| = (5 \times 2) - (-4 \times -3) = 10 - 12 = -2$

Question 7: Find the determinant of the matrix $A = \begin{bmatrix} 1 & -4 \\ 0 & 3 \end{bmatrix}$

$|A| = (1 \times 3) - (0 \times -4) = 3 - 0 = 3$

Question 8: Find the determinant of the matrix $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

$|A| = (2 \times 3) - (-1 \times 1) = 6 - -1 = 7$

Question 9: Find the determinant of the matrix $A = \begin{bmatrix} 0 & 4 \\ -1 & 1 \end{bmatrix}$

$|A| = (0 \times 1) - (-1 \times 4) = 0 - -4 = 4$

Question 10: Find the determinant of the matrix $A = \begin{bmatrix} -5 & -3 \\ -4 & -2 \end{bmatrix}$

$|A| = (-5 \times -2) - (-4 \times -3) = 10 - 12 = -2$

Question 11: Find the determinant of the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$|A| = (1 \times 1) - (0 \times 0) = 1 - 0 = 1$

Question 12: Find the determinant of the matrix $A = \begin{bmatrix} 10 & -3 \\ -4 & 2 \end{bmatrix}$

$|A| = (10 \times 2) - (-4 \times -3) = 20 - 12 = 8$

Question 13: Find the determinant of the matrix $A = \begin{bmatrix} 5 & 20 \\ -4 & 2 \end{bmatrix}$

$|A| = (5 \times 2) - (-4 \times 20) = 10 - -80 = 10 + 80 = 90$

Question 14: Find the determinant of the matrix $A = \begin{bmatrix} 1 & -3 \\ -4 & -2 \end{bmatrix}$

$|A| = (1 \times -2) - (-4 \times -3) = -2 - 12 = -14$

Question 15: Find the determinant of the matrix $A = \begin{bmatrix} 0 & -3 \\ -4 & 2 \end{bmatrix}$

$|A| = (0 \times 2) - (-4 \times -3) = 0 - 12 = -12$

Finding determinant of a 3×3 matrix

To find determinant of a 3×3 matrix, we use a slightly different method.

Let us illustrate with the aid of an example.

Consider the following matrix.

$$A = \begin{bmatrix} 3 & 6 & 5 \\ 2 & 1 & 8 \\ 7 & 9 & 1 \end{bmatrix}$$

There are different ways to find determinant of this matrix, that is, through any row or any column. Let us stick to finding determinant through first row.

First take the first element of first row, 3. Now discard the row and column to which 3 belongs, that is, discard the first row and first column. So what remains is $\begin{vmatrix} 1 & 8 \\ 9 & 1 \end{vmatrix}$. Find its determinant. So we get (1×1) − (9×8) = 1 – 72 = −71.

Now take the second element of first row, 6. Now discard the row and column to which 6 belongs, that is, discard the first row and second column. So what remains is $\begin{vmatrix} 2 & 8 \\ 7 & 1 \end{vmatrix}$. Find its determinant. So we get (2×1)−(7×8) = 2 – 56 = −54.

Now take the third element of first row, 5. Now discard the row and column to which 5 belongs, that is, discard the first row and third column. So what remains is $\begin{vmatrix} 2 & 1 \\ 7 & 9 \end{vmatrix}$. Find its determinant. So we get (2×9)−(7×1) = 18 – 7 = 11.

Now multiply the selected elements with its determinants. Another thing to do is give signs to the selected elements alternating +ve and –ve. So we give +ve to 3, -ve to 6 and again +ve to 5. So it will result in an arithmetic operation as shown below.

3 (−71) – 6(−54) + 5(11) = −213 + 324 + 55 = 166.

So the determinant of matrix A is 166. We can write $|A| = 166$

(the reason for giving +ve and –ve alternatively can be attributed to the position of the elements. For instance, take the first element 3. It can be represented as a_{11}. So 1+1 = 2, an even number, so give sign as +ve. Take the second element 6. It can be represented as a_{12}. So 1+2 = 3, an odd number, so give sign as −ve. Take the third element 5. It can be represented as a_{13}. So 1+3 = 4, an even number, so give sign as +ve. Without going into these complexities, you may remember to give + ve and – ve alternatively in the arithmetic operation).

Before we move to the next problem, let us write all that we did above in one step so that from next problem, you can save a lot of time.

So we re do the problem using the direct method, summarizing all that we did above in one step.

Given matrix was

$$A = \begin{bmatrix} 3 & 6 & 5 \\ 2 & 1 & 8 \\ 7 & 9 & 1 \end{bmatrix}$$

Now we find the determinant as follows in one step.

$$|A| = 3\begin{vmatrix} 1 & 8 \\ 9 & 1 \end{vmatrix} - 6\begin{vmatrix} 2 & 8 \\ 7 & 1 \end{vmatrix} + 5\begin{vmatrix} 2 & 1 \\ 7 & 9 \end{vmatrix}$$

$$= 3[1 - 72] - 6[2 - 56] + 5[18 - 7]$$

$$= 3[-71] - 6[-54] + 5[11]$$

$$= -213 + 324 + 55 = 166.$$

Question 1: Find the determinant of

$$A = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$$

Now we find the determinant as follows in one step.

$$|A| = 6\begin{vmatrix} -2 & 5 \\ 8 & 7 \end{vmatrix} - 1\begin{vmatrix} 4 & 5 \\ 2 & 7 \end{vmatrix} + 1\begin{vmatrix} 4 & -2 \\ 2 & 8 \end{vmatrix}$$

$$= 6[-14 - 40] - 1[28 - 10] + 1[32 - -4]$$

$$= 6[-54] - 1[18] + 1[36]$$

$$= -324 - 18 + 36 = -306.$$

Question 2: Find the determinant of

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix}$$

Now we find the determinant as follows in one step.

$|A| = 1\begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix} - 3\begin{vmatrix} 4 & 3 \\ 2 & 2 \end{vmatrix} + 2\begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix}$

$= 1[2 - 15] - 3[8 - 6] + 2[20 - 2]$

$= 1[-13] - 1[2] + 2[18]$

$= -13 - 2 + 36 = 21$

Question 3: Find the determinant of

$A = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 4 & 7 \\ 4 & 6 & 2 \end{bmatrix}$

Now we find the determinant as follows in one step.

$|A| = 1\begin{vmatrix} 4 & 7 \\ 6 & 2 \end{vmatrix} - 5\begin{vmatrix} 2 & 7 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 4 & 6 \end{vmatrix}$

$= 1[8 - 42] - 5[4 - 28] + 3[12 - 16]$

$= 1[-34] - 5[-24] + 3[-4]$

$= -34 - -120 + -12 = -34 + 120 - 12 = \ 74$

Properties of Determinants

1. Adding or subtracting any nonzero multiple of one row (or column) from another row (or column) will have no effect on the determinant.

2. Interchanging any two rows or columns of a matrix will change the sign, but not the absolute value, of the determinant.

3. Multiplying the elements of any row or column by a constant will cause the determinant to be multiplied by the constant.

4. The determinant of a triangular matrix, i.e., a matrix with zero elements everywhere above or below the principal diagonal, is equal to the product of the elements on the principal diagonal.

5. The determinant of a matrix equals the determinant of its transpose: $|A| = |A^T|$

6. If all the elements of any row or column are zero, the determinant is zero.

7. If two rows or columns are identical or proportional, i.e., linearly dependent, the determinant is zero.

Minors and Cofactors

We will first see minor and later see cofactor.

In the procedure for finding the determinant of a 3×3 matrix, we followed a procedure by which we eliminated the row and column of the chosen element and considered only the rest. Actually what we were doing was we were finding the minor of that element.

A minor for any element is the determinant that results when the row and column that element are in are deleted.

The notation M_{ij} is used to stand for the minor of the element in row i and column j. So M_{21} for instance would mean the minor for the element in row 2, column 1.

Consider the following matrix $\begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix}$

Minors can be found for each element.

$$Minor\ of\ 1 = M_{11} = \begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix} = 2 - 15 = -13$$

$$Minor\ of\ 3 = M_{12} = \begin{vmatrix} 4 & 3 \\ 2 & 2 \end{vmatrix} = 8 - 6 = 2$$

$$Minor\ of\ 2 = M_{13} = \begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix} = 20 - 2 = 18$$

$$Minor\ of\ 4 = M_{21} = \begin{vmatrix} 3 & 2 \\ 5 & 2 \end{vmatrix} = 6 - 10 = -4$$

$$Minor\ of\ 1 = M_{22} = \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = 2 - 4 = -2$$

$$Minor\ of\ 3 = M_{23} = \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = 5 - 6 = -1$$

$$Minor\ of\ 2 = M_{31} = \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} = 9 - 2 = 7$$

$$Minor\ of\ 5 = M_{32} = \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = 3 - 8 = -5$$

$$Minor\ of\ 2 = M_{33} = \begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix} = 1 - 12 = -11$$

Matrix of Minors: The matrix of minors is the square matrix where each element is the minor for the number in that position.

So the matrix of minors for the above matrix will be like this.

$$\begin{bmatrix} -13 & 2 & 18 \\ -4 & -2 & -1 \\ 7 & -5 & -11 \end{bmatrix}$$

We can work out more examples of finding minor as we proceed.

Singular and Non Singular Matrices

If the determinant of a matrix is equal to zero, that matrix is called a singular matrix. A singular matrix is also defined as a matrix for which there exist linear dependence between at least two rows or columns.

If the determinant of a matrix is not equal to zero, it is called non-singular matrix. For a non-singular matrix all the rows and columns are linearly independent.

Rank of a matrix

The maximum number of linearly independent rows in a matrix A is called the row rank of A, and the maximum number of linearly independent columns in A is called the column rank of A. But we can see that the row rank of A = the column rank of A. Because of this fact, there is no reason to distinguish between row rank and column rank; the common value is simply called the rank of the matrix.

Therefore, the rank of a matrix can be defined as the maximum number of linearly independent rows or columns in the matrix.

Rank of a matrix can also be defined as is the order of the largest non-zero minor.

For example, find rank of the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 8 \end{bmatrix}$

Find determinant A

$|A| = 8 - 2 = 6$

So we see that $|A| \neq 0$, the matrix is non-singular, that is, there is no linear dependence between any of the rows or columns. So the rank of matrix A is equal order (dimension) of matrix A. Since A is a 2×2 square matrix, we can take the order of A as 2. So rank of matrix A is 2.

Rank can also be interpreted like this. Rank of a matrix is the order of the largest non-zero minor. So for this matrix find the largest minor and check whether it is not equal to zero. The largest minor is the matrix itself. So find its value, that is, find the determinant.

$|A| = 8 - 2 = 6$

We said the largest minor of A is A itself and it is non zero. So the order of the largest minor, 2, is the rank of the matrix.

Rank is symbolised by the Greek alphabet ρ *(Rho).*

So instead of writing in words that rank of matrix A is 2, symbolically we can write $\rho(A) = 2$.

Example 1

Find rank of the matrix $A = \begin{bmatrix} 6 & 4 \\ 9 & 6 \end{bmatrix}$

$|A| = 36 - 36 = 0$

Since |A| = 0, matrix A is singular. So rank of the matrix is 1.

Or

We said the largest minor of A is A itself and it is equal to zero. So take minors of lower order. 6 is a 1×1 minor of A, 4 is a 1×1 minor of A, 9 is a 1×1 minor of A and again 6 is a 1×1 minor of A. If any of these 1×1 minors is non zero, the rank of the matrix will be 1. We see that all the 1×1 minors of A are non zero. So the order of the largest non zerominor, 1, is the rank of the matrix. So the rank of matrix A is 1.

Question 1: Find the rank of the matrix $B = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$

Find determinant B. Here the largest minor is matrix B itself.

$|B| = 1\begin{vmatrix} -3 & 1 \\ 5 & 0 \end{vmatrix} - 2\begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix} + 1\begin{vmatrix} -2 & -3 \\ 3 & 5 \end{vmatrix}$

$= 1[0 - 5] - 2[0 - 3] + 1[-10 - -9]$

$= 1[-5] - 2[-3] + 1[-10 + 9]$

$= -5 + 6 + 1[-1]$

$= -5 + 6 - 1 = 0$

So 3×3 is not the largest non zero minor. Now take minors of lower order. The first minor of lowest order will be $\begin{vmatrix} 1 & 2 \\ -2 & -3 \end{vmatrix} = -3 - -2 = -3 + 2 = -1$. This is a non zero minor. So its order ,2, is the rank of the matrix. So rank of Matrix B is 2.

Rank can also be interpreted like this.

For matrix B, all three rows are linearly dependent (the first is equal to the sum of the second and third) so the rank must be less than 3.The first two rows are linearly independent, so the rank is 2.

Question 2: Find rank of the matrix $C = \begin{bmatrix} 5 & -9 & 3 \\ 2 & 12 & -4 \\ -3 & -18 & 6 \end{bmatrix}$

Take the largest non-zero minor, which is the matrix itself.

$|C| = 5\begin{vmatrix} 12 & -4 \\ -18 & 6 \end{vmatrix} - -9\begin{vmatrix} 2 & -4 \\ -3 & 6 \end{vmatrix} + 3\begin{vmatrix} 2 & 12 \\ -3 & -18 \end{vmatrix}$

$= 5[72 - 72] + 9[12 - 12] + 3[-36 - -36]$

$= 5[0] + 9[0] + 3[0] = 0$

So 3×3 is not the largest non zero minor. Now take minors of lower order. The first minor of lowest order will be $\begin{vmatrix} 5 & -9 \\ 2 & 12 \end{vmatrix} = 60 - -18 = 60 + 18 = 78 \neq 0$. This is a non zero minor. So its order, 2, is the rank of the matrix. So rank of Matrix C is 2.

(If this 2×2 minor is equal to zero, try other minors of order 2×2. If all of them are equal to zero, then try minors of lower order, that is 1×1. We will illustrate this in the next problem)

Question 3: Find rank of the matrix $D = \begin{bmatrix} -8 & 2 & -6 \\ 10 & -2.5 & 7.5 \\ 24 & -6 & 18 \end{bmatrix}$

Take the largest non-zero minor, which is the matrix itself.

$|C| = -8\begin{vmatrix} -2.5 & 7.5 \\ -6 & 18 \end{vmatrix} - 2\begin{vmatrix} 10 & 7.5 \\ 24 & 18 \end{vmatrix} + -6\begin{vmatrix} 10 & -2.5 \\ 24 & -6 \end{vmatrix}$

$= -8[-45 - -45] - 2[180 - 180] + 3[-60 - -60]$

$= -8[0] - 2[0] + 3[0] = 0$

So 3×3 is not the largest non zero minor. Now take minors of lower order.

The first minor of lowest order will be $\begin{vmatrix} -8 & 2 \\ 10 & -2.5 \end{vmatrix} = 20 - 20 = 0$

Since this 2×2 minor is equal to zero, try other minors of order 2×2.

$\begin{vmatrix} 2 & -6 \\ -2.5 & 7.5 \end{vmatrix} = 15 - 15 = 0$

$\begin{vmatrix} 10 & -2.5 \\ 24 & -6 \end{vmatrix} = -60 - -60 = 0$

$\begin{vmatrix} -2.5 & 7.5 \\ -6 & 18 \end{vmatrix} = -45 - -45 = 0$

Since all the 2×2 minors are equal to zero, the rank is not equal to 2. Now we have to try minors of lower order, that is 1×1.

The first 1×1 minor is the first element in the matrix. Check whether it is non zero. Here the first 1×1 minor, that is the first element of the matrix, is −8 which is non zero.

So the rank of matrix D is 1.

Cofactors

From minors we move on to cofactors.

A cofactor is a minor with a sign. For a cofactor the notation C_{ij} is used. C_{ij} stand for the cofactor of the element in row i and column j. So C_{21} for instance would mean the cofactor for the element in row 2, column 1.

This is how we give sign to a minor to convert it to a cofactor.

$|C_{ij}| = (-1)^{i+j}|M_{ij}|$

To convert M_{11} to cofactor we write

$|C_{11}| = (-1)^{1+1}|M_{11}|$

$|C_{11}| = (1)|M_{11}|$

$|C_{11}| = |M_{11}|$

To convert M_{12} to cofactor we write

$|C_{12}| = (-1)^{1+2}|M_{12}|$

$|C_{12}| = (-1)|M_{12}|$

$|C_{11}| = -|M_{11}|$

To convert M_{13} to cofactor we write

$|C_{13}| = (-1)^{1+3}|M_{13}|$

$|C_{13}| = (1)|M_{12}|$

$|C_{13}| = |M_{13}|$

To convert M_{22} to cofactor we write

$|C_{22}| = (-1)^{2+2}|M_{22}|$

$|C_{22}| = (1)|M_{22}|$

$|C_{22}| = |M_{22}|$

Thus we may say that if the sum of the subscripts i and j is an even number, $|C_{ij}| = |M_{ij}|$, since −1 raised to an even power gives +ve value.

If the sum of the subscripts i and j is an odd number, $|C_{ij}| = -|M_{ij}|$, since −1 raised to an odd power gives −ve value.

Consider the following matrix $\begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix}$

Cofactors can be found for each element.

$$Cofactor\ of\ 1 = C_{11} = (-1)^{1+1}M_{11} = \begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix} = 2 - 15 = -13$$

$$Cofactor\ of\ 3 = C_{12} = (-1)^{1+1}M_{12} = (-1)\begin{vmatrix} 4 & 3 \\ 2 & 2 \end{vmatrix} = -1(8-6) = -2$$

$$Cofactor\ of\ 2 = C_{13} = (-1)^{1+3}M_{13} = \begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix} = 20 - 2 = 18$$

$$Cofactor\ of\ 4 = C_{21} = (-1)^{2+1}M_{21} = (-1)\begin{vmatrix} 3 & 2 \\ 5 & 2 \end{vmatrix} = -1(6-10) = 4$$

$$Cofactor\ of\ 1 = C_{22} = (-1)^{2+2}M_{22} = \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = 2 - 4 = -2$$

$$Cofactor\ of\ 3 = C_{23} = (-1)^{2+3}M_{23} = (-1)\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = -1(5-6) = 1$$

$$Cofactor\ of\ 2 = C_{31} = (-1)^{3+1}M_{31} = \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} = (9-2) = 7$$

$$Cofactor\ of\ 5 = C_{32} = (-1)^{3+2}M_{32} = (-1)\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = -1\ (3-8) = 5$$

$$Cofactor\ of\ 2 = C_{33} = (-1)^{3+3}M_{33} = \begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix} = 1 - 12 = -11$$

Cofactor Matrix: The matrix of cofactors is the square matrix where each element is the cofactor for the number in that position.

So the cofactor matrix for the above matrix will be like this.

$$Cofactor\ matrix\ of\ A = \begin{bmatrix} -13 & -2 & 18 \\ 4 & -2 & 1 \\ 7 & 5 & -11 \end{bmatrix}$$

Adjoint of a matrix

So we moved from matrix of minors to matrix of cofactors. The adjoint of a matrix is obtained by transposing the coefficient matrix. Thus, an adjoint matrix is the transpose of the cofactor matrix. Adjoint of matrix A is represented Adj A.

$$If\ C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$Adjoint = C^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Example 1: Find the adjoint of the following matrix.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 5 & 3 & 4 \end{bmatrix}$$

Find minors and cofactors.

$$Cofactor\ of\ 2 = C_{11} = (-1)^{1+1}M_{11} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

$Cofactor\ of\ 3\ = C_{12} = \ (-1)^{1+1}M_{12}\ = (-1)\ \begin{vmatrix} 4 & 2 \\ 5 & 4 \end{vmatrix} = -1(16 - 10) = -6$

$Cofactor\ of\ 1\ = C_{13} = \ (-1)^{1+3}M_{13}\ = \begin{vmatrix} 4 & 1 \\ 5 & 3 \end{vmatrix} = 12 - 5 = \ 7$

$Cofactor\ of\ 4\ = C_{21} = \ (-1)^{2+1}M_{21}\ = (-1)\begin{vmatrix} 3 & 1 \\ 3 & 4 \end{vmatrix} = -1(12 - 3) = \ -9$

$Cofactor\ of\ 1\ = C_{22} = \ (-1)^{2+2}M_{22}\ = \begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix} = 8 - 5 = \ 3$

$Cofactor\ of\ 2\ = C_{23} = \ (-1)^{2+3}M_{23}\ = (-1)\begin{vmatrix} 2 & 3 \\ 5 & 3 \end{vmatrix} = -1(6 - 15) = \ 9$

$Cofactor\ of\ 5\ = C_{31} = \ (-1)^{3+1}M_{31}\ = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = (6 - 1) = \ 5$

$Cofactor\ of\ 3\ = C_{32} = \ (-1)^{3+2}M_{32}\ \ = (-1)\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = -1\ (4 - 4) = \ 0$

$Cofactor\ of\ 4\ = C_{33} = \ (-1)^{3+3}M_{33} = \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = 2 - 12 = \ -10$

So the cofactor matrix is.

$$C = \begin{bmatrix} -2 & -6 & 7 \\ -9 & 3 & 9 \\ 5 & 0 & -10 \end{bmatrix}$$

To get adjoint, we take the transpose of cofactor matrix C.

$$Adj\ A = \ C^T = \begin{bmatrix} -2 & -9 & 5 \\ -6 & 3 & 0 \\ 7 & 9 & -10 \end{bmatrix}$$

Inverse of a matrix

Inverse of a matrix A is denoted as A^{-1}.

Inverse can be found only for a square matrix, only for a non singular matrix..
Inverse of a matrix satisfies a unique relation as given below, where I represents an identity matrix.
$AA^{-1} = I = \ A^{-1}A$
Multiplying a matrix by its inverse gives an identity matrix.
We can write the formula for finding the inverse of a matrix A as

$$A^{-1} = \frac{1}{|A|}\ AdjA$$

Example 1: Find inverse of the following matrix.

$$A = \begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix}$$

Solution

$A^{-1} = \frac{1}{|A|}\ AdjA$

Find determinant A

$|A| = 4(12 - -1) - 1(-8-3) + \ -5(2-9)$

$|A| = 4(12+1) - 1(-11) + \ -5(-7)$

$|A| = 4(13) - 1(-11) + \ -5(-7)$

$|A| = 52 + 11 + 35 = 98$

$Since\ |A| = 98 \neq 0, it\ is\ non\ singular.$

Now find minor → cofactor → adjoint → inverse.

Minors can be found for each element.

$Minor\ of\ 4\ =\ M_{11}\ =\ \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = 12 - -1 =\ 13$

$Minor\ of\ 1\ =\ M_{12}\ =\ \begin{vmatrix} -2 & 1 \\ 3 & 4 \end{vmatrix} = -8 - 3 =\ -11$

$Minor\ of - 5\ =\ M_{13}\ =\ \begin{vmatrix} -2 & 3 \\ 3 & -1 \end{vmatrix} = 2 - 9 =\ -7$

$Minor\ of - 2\ =\ M_{21}\ =\ \begin{vmatrix} 1 & -5 \\ -1 & 4 \end{vmatrix} = 4 - 5 =\ -1$

$Minor\ of\ 3 =\ M_{22}\ =\ \begin{vmatrix} 4 & -5 \\ 3 & 4 \end{vmatrix} = 16 - -15 =\ 31$

$Minor\ of\ 1 =\ M_{23}\ =\ \begin{vmatrix} 4 & 1 \\ 3 & -1 \end{vmatrix} = -4 - 3 =\ -7$

$Minor\ of\ 3 =\ M_{31}\ =\ \begin{vmatrix} 1 & -5 \\ 3 & 1 \end{vmatrix} = 1 - -15 =\ 16$

$Minor\ of - 1\ =\ M_{32}\ =\ \begin{vmatrix} 4 & -5 \\ -2 & 1 \end{vmatrix} = 4 - 10 =\ -6$

$Minor\ of\ 4\ =\ M_{33}\ =\ \begin{vmatrix} 4 & 1 \\ -2 & 3 \end{vmatrix} = 12 - -2 =\ 14$

Find cofactors.

$Cofactor\ of\ 4\ = C_{11} =\ (-1)^{1+1}M_{11}\ = (1)13 = 13$

$Cofactor\ of\ 1\ = C_{12} =\ (-1)^{1+1}M_{12}\ = (-1) - 11 = 11$

$Cofactor\ of - 5\ = C_{13} =\ (-1)^{1+3}M_{13}\ = (1) - 7 = -7$

$Cofactor\ of - 2\ = C_{21} =\ (-1)^{2+1}M_{21}\ = (-1) - 1 = 1$

$Cofactor\ of\ 3\ = C_{22} =\ (-1)^{2+2}M_{22}\ = (1)31 =\ 31$

$Cofactor\ of\ 1 = C_{23} = (-1)^{2+3}M_{23} = (-1)(-7) = 7$

$Cofactor\ of\ 3 = C_{31} = (-1)^{3+1}M_{31} = (1)16 = 16$

$Cofactor\ of - 1 = C_{32} = (-1)^{3+2}M_{32} = (-1)(-6) = 6$

$Cofactor\ of\ 4 = C_{33} = (-1)^{3+3}M_{33} = (1)(14) = 14$

So the cofactor matrix is.

$$C = \begin{bmatrix} 13 & 11 & -7 \\ 1 & 31 & 7 \\ 16 & 6 & 14 \end{bmatrix}$$

To get adjoint, we take the transpose of cofactor matrix C.

$$Adj\ A = C^T = \begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix}$$

Now find inverse. Multiply the adjoint by $\frac{1}{|A|}, that\ is, \frac{1}{98}\ to\ find\ A^{-1}$

$$A^{-1} = \frac{1}{98}\begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix} = \begin{bmatrix} 13/98 & 1/98 & 16/98 \\ 11/98 & 31/98 & 6/98 \\ -7/98 & 7/98 & 14/98 \end{bmatrix} = \begin{bmatrix} 0.13 & 0.01 & 0.16 \\ 0.11 & 0.31 & 0.06 \\ -0.07 & 0.07 & 0.14 \end{bmatrix}$$

Thus we have solved the problem by finding the inverse.

You may confirm the answer by using the relation

$AA^{-1} = I = A^{-1}A$

Multiply A with A^{-1}. You should get an identity matrix.

In the last module we saw how we can write linear equations in the form of a matrix. We can also use matrices to find the values of the unknown variables. For this we use the concept of inverse of a matrix.

Solving Linear Equations with the Inverse

To solve a linear equation with inverse we use the formula

$x = A^{-1}B$ Where

X is the solution vector. So include the variables for which we have to find the solutions in this matrix, for example, x_1, x_2 and x_3.

A is the coefficient matrix.So it includes the coefficients of the equation

B is the matrix containing the constants of the equation.

Example 1

Solve for the unknowns x_1, x_2 and x_3 using inverse matrix method.

$4x_1 + x_2 - 5x_3 = 8$

$-2x_1 + 3x_2 + x_3 = 12$

$3x_1 - x_2 + 4x_3 = 5$

Let us first convert this information in the form of a matrix, using the format AX = B.

AX = B

$$\begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 5 \end{bmatrix}$$

Now use the formula$x = A^{-1}B$

We have already found inverse of this matrix in the previous problem. So we take inverse from the previous problem without working out again.

$$x = A^{-1}B = \begin{bmatrix} 13/98 & 1/98 & 16/98 \\ 11/98 & 31/98 & 6/98 \\ -7/98 & 7/98 & 14/98 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} (13/98)8 + & (1/98)12 + & (16/98)5 \\ (11/98)8 + & (31/98)12 + & (6/98)5 \\ (-7/98)8 + & (7/98)12 + & (14/98)5 \end{bmatrix}$$

$$= \begin{bmatrix} 104/98 + & 12/98 + & 80/98 \\ 88/98 + & 372/98 + & 30/98 \\ -56/98 + & 84/98 + & 70/98 \end{bmatrix} = \begin{bmatrix} 196/98 \\ 490/98 \\ 98/98 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

So we get $x_1 = 2$, $x_2 = 5$ and $x_3 = 1$.

The solution of linear equations can also be done using another method Cramer's Rule.

Cramer's Rule for Matrix Solutions

Here we use the formula $x_i = \frac{|A_i|}{|A|}$

Where

x_i *is the unknown variable*

$|A|$ *is the determinant of the coefficient matrix*

$|A_i|$is the determinant of a special matrix A_i formed from the original matrix A by replacing the columns of A by the matrix B.

Let us make it clear with the following example.

Example 1: Solve for the unknowns x_1, x_2 and x_3 using Cramer's rule

$4x_1 + x_2 - 5x_3 = 8$

$-2x_1 + 3x_2 + x_3 = 12$

$3x_1 - x_2 + 4x_3 = 5$

Let us first convert this information in the form of a matrix, using the format AX = B.

AX = B

$$\begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 5 \end{bmatrix}$$

The formula is $x_i = \frac{|A_i|}{|A|}$

So let us first find determinant A

$|A| = 4(12 - -1) - 1(-8 - 3) + \ -5(2 - 9)$

$|A| = 4(12 + 1) - 1(-11) + \ -5(-7)$

$|A| = 4(13) - 1(-11) + \ -5(-7)$

$|A| = 52 + 11 + 35 = 98$

Now let us find x_1.

$$x_1 = \frac{|A_1|}{|A|}$$

A_1 is a matrix we get by replacing the first column of matrix A by the elements of matrix B.

So A_1 will be

$$A_1 = \begin{bmatrix} 8 & 1 & -5 \\ 12 & 3 & 1 \\ 5 & -1 & 4 \end{bmatrix}$$

Find determinant A_1

$$|A_1| = 8(12 - -1) - 1(48 - 5) + -5(-12 - 15)$$

$$|A_1| = 8(13) - 1(43) - 5(-27)$$

$$|A_1| = 104 - 43 + 135 = 196$$

Now find determinant A_2

A_2 is a matrix we get by replacing the second column of matrix A by the elements of matrix B.

So A_2 will be

$$A_2 = \begin{bmatrix} 4 & 8 & -5 \\ -2 & 12 & 1 \\ 3 & 5 & 4 \end{bmatrix}$$

Find determinant A_2

$$|A_2| = 4(48 - 5) - 8(-8 - 3) + -5(-10 - 36)$$

$$|A_2| = 4(43) - 8(-11) - 5(-46)$$

$$|A_2| = 172 + 88 + 230 = 490$$

Now find determinant A_3

A_3 is a matrix we get by replacing the first column of matrix A by the elements of matrix B.

So A_3 will be

$$A_3 = \begin{bmatrix} 4 & 1 & 8 \\ -2 & 3 & 12 \\ 3 & -1 & 5 \end{bmatrix}$$

Find determinant A_3

$$|A_3| = 4(15 - -12) - 1(-10 - 36) + 8(2 - 9)$$

$$|A_3| = 4(27) - 1(-46) + 8(-7)$$

$$|A_3| = 108 + 46 - 56 = 98$$

So We have got the following values necessary to apply in the formula

$$x_i = \frac{|A_1|}{|A|}$$

$$|A| = 98, |A_1| = 196, |A_2| = 490, |A_3| = 98$$

Now let us apply the formula.

$$x_1 = \frac{|A_1|}{|A|} = \frac{196}{98} = 2$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{490}{98} = 5$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{98}{98} = 1$$

So we get $x_1 = 2$, $x_2 = 5$ and $x_3 = 1$.

Exercises

Question 1: Solve for the unknowns using Inverse Matrix Method

$$-x + 3y + z = 1$$

$$2x + 5y \quad = 3$$

$$3x + y - 2z = -2$$

Let us first convert this information in the form of a matrix, using the format AX = B.

AX = B

$$\begin{bmatrix} -1 & 3 & 1 \\ 2 & 5 & 0 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

Now use the formula $x = A^{-1}B$

Find the inverse of the coefficient matrix A. In this case the inverse is

$$A^{-1} = \begin{bmatrix} -10/9 & 7/9 & -5/9 \\ 4/9 & -1/9 & 2/9 \\ -13/9 & 10/9 & -11/9 \end{bmatrix}$$

$$x = A^{-1}B = \begin{bmatrix} -10/9 & 7/9 & -5/9 \\ 4/9 & -1/9 & 2/9 \\ -13/9 & 10/9 & -11/9 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

On multiplying we get

$$x = A^{-1}B = \begin{bmatrix} 21 \\ -3 \\ 39 \end{bmatrix}$$

So x = 21, y = −3 and z = 39.

Question 2: Solve for the unknowns using Inverse Matrix Method

x + y + z = 6
2y + 5z = −4
2x + 5y − z = 27

Let us first convert this information in the form of a matrix, using the format AX = B.

AX = B

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix}$$

Now use the formula $x = A^{-1}B$

Find the inverse of the coefficient matrix A. In this case the inverse is

$$A^{-1} = \frac{1}{-21}\begin{bmatrix} -27 & 6 & 3 \\ 10 & -3 & -5 \\ -4 & -3 & 2 \end{bmatrix}$$

$$x = A^{-1}B$$

$$x = A^{-1}B = \frac{1}{-21}\begin{bmatrix} -27 & 6 & 3 \\ 10 & -3 & -5 \\ -4 & -3 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix} = \frac{1}{-21}\begin{bmatrix} -105 \\ -63 \\ 42 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

Question 3: The equilibrium conditions for two related markets wheat and rice is given as

4w + 3r = 28

2w + 5r = 42

Find the equilibrium price for each market.

All we have to do is to find is values of w and r. We can solve this problem using matrices.

Let us first convert this information in the form of a matrix, using the format AX = B.

AX = B

$$\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} w \\ r \end{bmatrix} = \begin{bmatrix} 28 \\ 42 \end{bmatrix}$$

Now use the formula$x = A^{-1}B$

Let us first find $|A|$

$$|A| = |4\times 5 - 2\times 3| = |20-6| = 14$$

Let us find Cofactor matrix

$$Cofactor\ of\ 4 = C_{11} = (-1)^{1+1}M_{11} = (1)5 = 5$$

$$Cofactor\ of\ 3 = C_{12} = (-1)^{1+1}M_{12} = (-1)2 = -2$$

$$Cofactor\ of\ 2 = C_{21} = (-1)^{2+1}M_{21} = (-1)3 = -3$$

$$Cofactor\ of\ 5 = C_{22} = (-1)^{2+2}M_{22} = (1)4 = 4$$

$$\text{Cofactor matrix} = \mathrm{C} = \begin{bmatrix} 5 & -2 \\ -3 & 4 \end{bmatrix}$$

$$Adj\ A = C^T = \begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\ AdjA$$

$$A^{-1} = \frac{1}{14}\begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix}$$

$$x = A^{-1}B$$

$$x = A^{-1}B = \frac{1}{14}\begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 28 \\ 42 \end{bmatrix}$$

$$x = A^{-1}B = \frac{1}{14}\begin{bmatrix} 5\times 28 + -3\times 42 \\ -2\times 28 + 4\times 42 \end{bmatrix}$$

$$x = A^{-1}B = \frac{1}{14}\begin{bmatrix} 140-126 \\ -56+168 \end{bmatrix}$$

$$x = A^{-1}B = \frac{1}{14}\begin{bmatrix} 14 \\ 112 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 8 \end{bmatrix} = \begin{bmatrix} w \\ r \end{bmatrix}$$

w = 1 and r = 8.

Question 4: The equilibrium condition for three related markets is given by

11x – y – z = 31

− x + 6y – 2z = 26

− x – 2y + 7 z = 24

Find the equilibrium price for each market.

Let us first convert this information in the form of a matrix, using the format AX = B.

AX = B

$$\begin{bmatrix} 11 & -1 & -1 \\ -1 & 6 & -2 \\ -1 & -2 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 31 \\ 26 \\ 24 \end{bmatrix}$$

Now use the formula $x = A^{-1}B$

Find $|A|$

$|A| = 11(38) + 1(-9) - 1(8) = 401$

Find cofactors.

$$Cofactor\ of\ 11 = C_{11} = (-1)^{1+1}M_{11} = (1)\begin{vmatrix} 6 & -2 \\ -2 & 7 \end{vmatrix} = 38$$

$$Cofactor\ of -1 = C_{12} = (-1)^{1+1}M_{12} = (-1)\begin{vmatrix} -1 & -2 \\ -1 & 7 \end{vmatrix} = 9$$

$$Cofactor\ of -1 = C_{13} = (-1)^{1+3}M_{13} = (1)\begin{vmatrix} -1 & 6 \\ -1 & -2 \end{vmatrix} = 8$$

$$Cofactor\ of -1 = C_{21} = (-1)^{2+1}M_{21} = (-1)\begin{vmatrix} -1 & -1 \\ -2 & 7 \end{vmatrix} = 9$$

$$Cofactor\ of\ 6 = C_{22} = (-1)^{2+2}M_{22} = (1)\begin{vmatrix} 11 & -1 \\ -1 & 7 \end{vmatrix} = 76$$

$$Cofactor\ of -2 = C_{23} = (-1)^{2+3}M_{23} = (-1)\begin{vmatrix} 11 & -1 \\ -1 & -2 \end{vmatrix} = 23$$

$$Cofactor\ of -1 = C_{31} = (-1)^{3+1}M_{31} = (1)\begin{vmatrix} -1 & -1 \\ 6 & -2 \end{vmatrix} = 8$$

$$Cofactor\ of -2 = C_{32} = (-1)^{3+2}M_{32} = (-1)\begin{vmatrix} 11 & -1 \\ -1 & -2 \end{vmatrix} = 23$$

$$Cofactor\ of\ 7 = C_{33} = (-1)^{3+3}M_{33} = (1)\begin{vmatrix} 11 & -1 \\ -1 & 6 \end{vmatrix} = 65$$

So the cofactor matrix is.

$$C = \begin{bmatrix} 38 & 9 & 8 \\ 9 & 76 & 23 \\ 8 & 23 & 65 \end{bmatrix}$$

To get adjoint, we take the transpose of cofactor matrix C.

$$Adj\ A = C^T = \begin{bmatrix} 38 & 9 & 8 \\ 9 & 76 & 23 \\ 8 & 23 & 65 \end{bmatrix}$$

Now find inverse. Multiply the adjoint by $\frac{1}{|A|}$, *that is*, $\frac{1}{401}$ *to find* A^{-1}

$$A^{-1} = \frac{1}{401}\begin{bmatrix} 38 & 9 & 8 \\ 9 & 76 & 23 \\ 8 & 23 & 65 \end{bmatrix}$$

Use the formula$x = A^{-1}B$

$$x = A^{-1}B = \frac{1}{401}\begin{bmatrix} 38 & 9 & 8 \\ 9 & 76 & 23 \\ 8 & 23 & 65 \end{bmatrix}\begin{bmatrix} 31 \\ 26 \\ 24 \end{bmatrix}$$

$$x = A^{-1}B = \frac{1}{401}\begin{bmatrix} 38 \times 31 + 9 \times 26 + 8 \times 24 \\ 9 \times 31 + 76 \times 26 + 23 \times 24 \\ 8 \times 31 + 23 \times 26 + 65 \times 24 \end{bmatrix}$$

$$= \frac{1}{401}\begin{bmatrix} 1178 + 234 + 192 \\ 279 + 1976 + 552 \\ 248 + 598 + 1560 \end{bmatrix} = \frac{1}{401}\begin{bmatrix} 1604 \\ 2807 \\ 2406 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

So, x = 4, y = 7 and z = 6.

<u>Question 5: </u>Solve for the unknowns using Cramer's rule

7x + 2y = 60

x + 8y = 78

Let us first convert this information in the form of a matrix, using the format AX = B.

AX = B

$$\begin{bmatrix} 7 & 2 \\ 1 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 60 \\ 78 \end{bmatrix}$$

The formula is

$$x_i = \frac{|A_i|}{|A|}$$

So let us first find determinant A

$|A| = 7(8) - 2(1) = 56 - 2 = 54$

Now form A_1 and find $|A_1|$

$$A_1 = \begin{vmatrix} 60 & 2 \\ 78 & 8 \end{vmatrix}$$

$$|A_1| = |60 \times 8 - 78 \times 2| = |480 - 156| = 324$$

Now form A_2 and find $|A_2|$

$$A_2 = \begin{vmatrix} 7 & 60 \\ 1 & 78 \end{vmatrix}$$

$$|A_2| = |7 \times 78 - 1 \times 60| = |546 - 60| = 486$$

So we have

$|A| = 54$

$|A_1| = 324$

$|A_2| = 486$

$$x = \frac{324}{54} = 6$$

$$y = \frac{486}{54} = 9$$

Question 6: Solve for the unknowns using Cramer's rule

$11x - y - z = 31$

$-x + 6y - 2z = 26$

$-x - 2y + 7z = 24$

(Note that we have solved this problem earlier – question no.4 – using inverse matrix method)

Let us first convert this information in the form of a matrix, using the format AX = B.

AX = B

$$\begin{bmatrix} 11 & -1 & -1 \\ -1 & 6 & -2 \\ -1 & -2 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 31 \\ 26 \\ 24 \end{bmatrix}$$

The formula is

$$x_i = \frac{|A_i|}{|A|}$$

So let us first find determinant A

$$|A| = 11(42 - 4) - -1(-7 - 2) + \; -1(2 - -6)$$

$$|A| = 11(38) + 1(-9) \; - 1(8)$$

$$|A| = 418 - 9 - 8 = 401$$

$$|A| = 401$$

Now let us find x_1.

$$x_1 = \frac{|A_1|}{|A|}$$

A_1 is a matrix we get by replacing the first column of matrix A by the elements of matrix B.

So A_1 will be

$$A_1 = \begin{bmatrix} 31 & -1 & -1 \\ 26 & 6 & -2 \\ 24 & -2 & 7 \end{bmatrix}$$

Find determinant A_1

$$|A_1| = 31(38) + 1(230) \; - 1(-196) = 1604$$

Now find determinant A_2

A_2 is a matrix we get by replacing the second column of matrix A by the elements of matrix B.

So A_2 will be

$$A_2 = \begin{bmatrix} 11 & 31 & -1 \\ -1 & 26 & -2 \\ -1 & 24 & 7 \end{bmatrix}$$

Find determinant A_2

$$|A_2| = 11(230) - 31(-9) - 1(2) = 2807$$

Now find determinant A_3

A_3 is a matrix we get by replacing the first column of matrix A by the elements of matrix B.

So A_3 will be

$$A_3 = \begin{bmatrix} 11 & -1 & 31 \\ -1 & 6 & 26 \\ -1 & -2 & 24 \end{bmatrix}$$

Find determinant A_3

$$|A_3| = 11(196) + 1(2) + 31(8) = 2406$$

So We have got the following values necessary to apply in the formula

$$x_i = \frac{|A_i|}{|A|}$$

$$|A| = 401, |A_1| = 1604, |A_2| = 2807, |A_3| = 2406$$

Now let us apply the formula.

$$x = \frac{|A_1|}{|A|} = \frac{1604}{401} = 4$$

$$y = \frac{|A_2|}{|A|} = \frac{1604}{401} = 7$$

$$z = \frac{|A_3|}{|A|} = \frac{2406}{401} = 6$$

So we get x = 4, y = 7 and z = 6.

****