STUDY MATERIALS

MATHEMATICAL TOOLS FOR ECONOMICS III

(The content of the study material is the same as that of Chapter I of Mathematics for Economic Analysis II of 2011 Admn.)

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MATHEMATICAL TOOLS FOR ECONOMICS IV

(The content of the study material is the same as that of Chapter II, IV & V of Mathematics for Economics Analysis II of 2011 Admn.)

(COMPLEMENTARY COURSE OF BA ECONOMICS III & IV SEMESTER)

(CUCBCSS -2014 admn. onwards)

(Chapters III and VI of Mathematics for Economic Analysis II to be deleted)

UNIVERSITY OF CALICUT

SCHOOL OF DISTANCE EDUCATION

Calicut University, P.O. Malappuram, Kerala, India-673 635

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MATHEMATICS FOR ECONOMIC ANALYSIS –II

III Semester

COMPLEMENTARY COURSE

BA ECONOMICS

(2011 Admission)

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MATHEMATICS FOR ECONOMIC ANALYSIS -II

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CHAPTER 1

THE DERIVATIVE AND THE RULES OF DIFFERENTIATION

I. 1: LIMITS

The concept of limit is useful in developing some mathematical techniques and also in analyzing various economic problems in economics.

For example, the rate of interest depends up on the availability of capital in the economy. But rate of interest can never fall to zero even if the quantity of capital available in the economy is very large. That shows, there is a minimum limit below which 'r' cannot fall. Let this minimum limit of the rate of interest 'r' be 5% then, if 'K' stands for capital and 'C' is a positive constant, we can write the function as

$$r = 5 + \frac{C}{K}$$

If K is small 'r' is large

If K is large 'r' is small

If K is very large (infinite) 'r' will never be less than

5%. Therefore the limit 'r' as 'K' increases is 5

It can be written as

$$\lim_{K \to \alpha} r = 5 \quad or \quad \mathrm{lt}_{K \to \alpha} r = 5$$

A function f(x) is said to approach the limit 'L' as 'x' approach to 'a' if the difference between f(x) and 'L' can be made as small as possible by taking 'x' sufficiently nearer to 'a'.

$$\lim_{x \to \alpha} f(x) = L$$

Example: Limit x^2 as 'x' approach to 4 is 16.

This can be expressed as

$$\lim_{x \to 4} x^2 = 16$$

i.e., the difference between x^2 and 16 can be made as small as possible by taking 'x' sufficiently nearer to 4.

Rules of Limits

1. If K is a constant

$$\lim_{x \to a} K = K$$

2. If 'n' is a positive integer

$$\lim_{x \to a} x^n = a^n$$

3. If 'K' is a constant

$$\lim_{x \to a} K\, f(x) = K \lim_{x \to a} f(x)$$

4. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

5. $\lim_{x \to a} [f(x)\,.\, g(x)] = \lim_{x \to a} f(x)\,.\, \lim_{x \to a} g(x)$

6. $\lim_{x \to a} [f(x) \div g(x)] = \lim_{x \to a} f(x) \div \lim_{x \to a} g(x)$

$$if\ [\lim_{x \to a} g(x) \neq 0\ ]$$

7. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$

Examples

1.	$\lim_{x \to 2} 10 = 10$	[Rule 1]
2.	$\lim_{x \to 5} x^2 = 5^2 = 25$	[Rule 2]
3.	$\lim_{x \to 2} 5\, x^4 = 5 \lim_{x \to 2} x^4 = 5\, X\, 2^4 = 5\, X\, 16 = 80$	[Rule 3]
4.	$\lim_{x \to 2} [x^3 + 3x] = \lim_{x \to 2} x^3 + \lim_{x \to 2} 3x$ $= 2^3 + 3 \times 2 = 8 + 6 = 14$	[Rule 4]
5.	$\lim_{x \to 4} [(x+8)(x-5)] = \lim_{x \to 4} (x+8)\,.\, \lim_{x \to 4} (x-5)$ $= (4 + 8)(4 - 5) = -12$	[Rule 5]

6.	$\lim\limits_{x \to 2} \dfrac{4x^2 - 5x + 10}{10x + 5} = \lim\limits_{x \to 2} 4x^2 - 5x + 10 \div \lim\limits_{x \to 2} 10x + 5$ $= \frac{4 X 2^2 - 5X2 + 10}{10X2 + 5} = \frac{16}{25}$	[Rule 6]
7.	If f(x) = (x + 2) $\lim\limits_{x \to 3} [x+2]^2 = [\lim\limits_{x \to 3} (x+2)]^2 = (3+2)^2 = 5^2 = 25$	[Rule 7]

INDETERMINATE FORM

If the value of the function becomes indeterminate like $\frac{0}{0}, \frac{\propto}{\propto}$ on substituting the value of the variables, mere substitution method may not be feasible. In such cases factorization of the function may remove the difficulties.

For example: $\lim\limits_{x \to 3} \frac{x^2-9}{x-3} = \frac{3^2-9}{3-3} = \frac{0}{0}$

$\therefore$ write $\frac{x^2-9}{x-3}$ *as* $\frac{x^2-3^2}{x-3}$

$= \frac{(x+3)(x-3)}{x-3} = x + 3 = 6$

[Note $(a^2 - b^2) = (a + b)(a - b)$]

Example 2: $\lim\limits_{x \to \alpha} \frac{x^2+1}{x^2-1} = \frac{\propto}{\propto}$

Here to find the limit value we have to divide the numerator and denominator by x^2

$$= \lim_{x \to \alpha} \frac{\frac{x^2}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}}$$

$$= \frac{1 + \frac{1}{x^2}}{1 - \frac{1}{x^2}}$$

Since, $\frac{1}{x^2} \to 0$ as $x \to \alpha$

We get $\frac{1}{1} = 1$

========

Example 3: $\lim_{x \to 1} \frac{x^2-4x+3}{x^2+2x-3}$

Factorizing the function we get

$$\frac{(x-3)(x-1)}{(x+3)(x-1)} = \frac{(x-3)}{(x+3)} = \frac{1-3}{1+3} = \frac{-2}{4} = \frac{-1}{2}$$

======

Problems:

1. Find $\lim_{x \to 2} \frac{x^2-3x+2}{x^2-5x+6}$ [Ans: -1]
2. Find $\lim_{x \to 1} \frac{x^2-1}{x-1}$ [Ans: 2]
3. Find $\lim_{r \to 2} \frac{r^2-5}{2r^3+6}$ [Ans: $\frac{-1}{22}$]
4. Find $\lim_{x \to 0} \frac{5x^2 + 8x}{x}$ [Ans: 8]
5. Find $\lim_{x \to \alpha} \frac{10x^2+5x}{5x^2}$ [Ans: 2]
6. Find $\lim_{x \to 6} \frac{x^2- x-30}{x^2-4x-12}$ [Ans: $\frac{11}{8}$]
7. Find $\lim_{x \to \alpha} \frac{3x^2-7x}{4x^2-21}$ [Ans: $\frac{3}{4}$]

1.2 CONTINUITY

A function is said to be continuous in a particular region if it has determinate value (or finite quantity) for every value of the variable in that region. A continuous function is one which has no breaks in its curve. It can be drawn without lifting the pencil from the paper.

Conditions for continuity

1. f(x) is defined ie exists at x = a
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

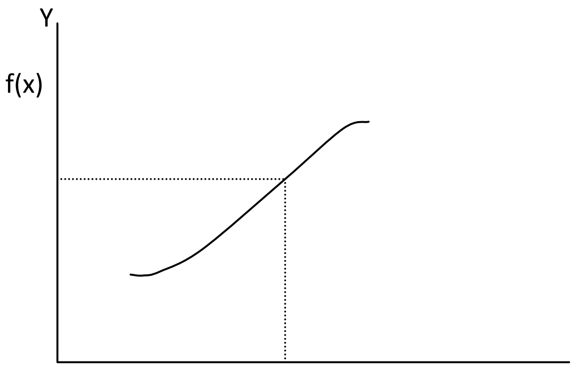
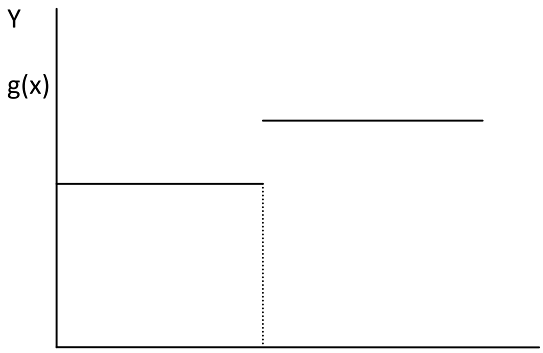


Fig 1.1 (a) x

Graph of continuous function



0 (b) x

Fig 1.2 Graph of discontinuous function

If the conditions for continuity are not satisfied for any value of x, f(x) is said to be discontinuous for that value of 'x'.

1. *Example:* If f(x) = $\frac{1}{x-2}$ and $\lim_{x \to a} f(x) = \frac{1}{a-2}$

   If $x \to 2$, f(x) does not have a finite value

   $\therefore$ f(x) is continuous for any value of 'a' except 2 and f(x) is not continuous at a = 2.

Problems:

Indicate whether the following function is continuous at the specific points.

1. f(x) = $5x^2 - 8x + 9$ at x = 3

   f(3) = $5(3)^2 - 8(3) + 9 = 45 - 24 + 9 = 30$

   $\lim_{x \to 3} 5(x)2 - 8x + 9 = 5(3)2 - 8(3) + 9 = 45 - 24 + 9 = 30$

$\lim_{x \to 3} f(x) = 30 = f(3)$

$\therefore$ f(x) is continuous at x = 3.

2. f(x) = $\frac{x-3}{x^2-9}$ at x = 3

   f(3) = $\frac{3-3}{3^2-9} = \frac{0}{0}$

   i.e., function is not defined at x = 3

   So cannot be continuous at x = 3 even though the limit value exist at x = 3

   $$\lim_{x \to 3} \frac{x-3}{x^2-9} = \lim_{x \to 3} \frac{x-3}{(x+3)(x-3)}$$

   $$= \lim_{x \to 3} \frac{1}{x+3} = \frac{1}{6}$$

   $\lim_{x \to 3} f(x) = \frac{1}{6} \neq$ f(3)

   $\therefore$ f(x) is discontinuous at x = 3.

3. Check for continuity of function at x = 1 when f(x) = $\frac{x^2+3x-4}{x-1}$

   [Ans: Not continuous]

4. Check whether the function $x^2 + 3x - 2$ is continuous for x = 2

   [Ans: Continuous]

1.3 THE SLOPE OF A CURVI LINEAR FUNCTION

The slope of a curvilinear function is differ from point to point; i.e., the slope of each point is different. The slope at a given point is measured by the slope of a line tangent to the function at that point. So in order to measure the slope of a curvilinear function at different points we requires separate tangent lines.

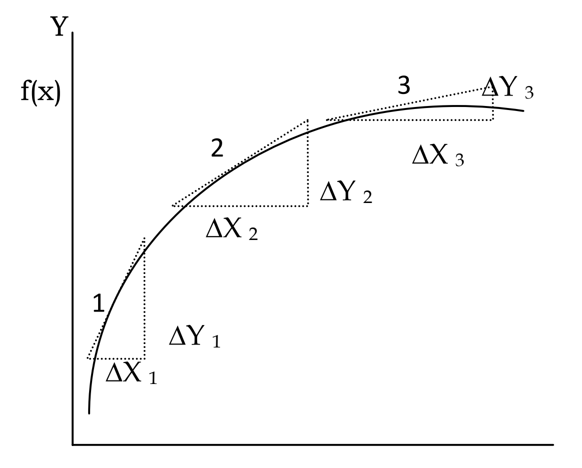


Fig 1.3

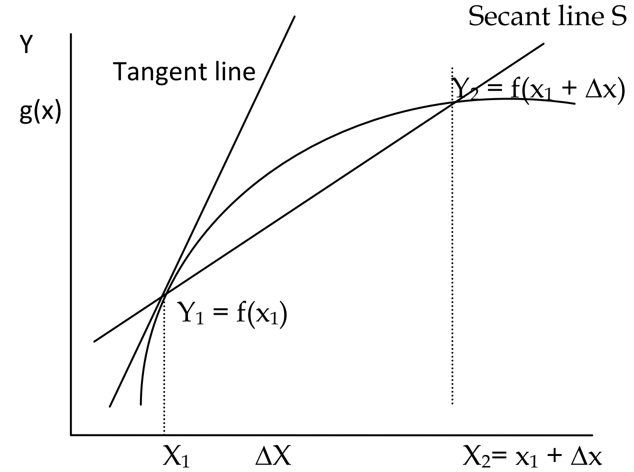


Fig 1.4

The slope of a tangent line is derived from the slopes of a family of secant lines (secant line 'S' is a straight line that intersects a curve at two points.)

Slope S = $\frac{y_2 - y_1}{x_2 - x_1}$

Here $x_2 = x_1 + \Delta x$, and $y_2 = f(x_1 + \Delta x)$ and $y_1 = f(x_1)$

We can express the slope 'S' as

$$\text{Slope S} = \frac{f(x_1 + \Delta x) - f(x_1)}{x_1 + \Delta x - x_1}$$

$$= \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

If the distance between x_2 and x_1 is made smaller and smaller ie. if $\Delta x \to 0$, the secant line pivots back to the left and draws progressively closer to the tangent line. If the slope of the secant line approaches a limit as $\Delta x \to 0$, the limit is the slope of the tangent line T, which is also the slope of the function at that point.

$$\text{Slop T} = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

Example: Find the slope of a curvilinear function, such as $f(x) = 4x^2$

Ans.

$$\text{Slop T} = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

Substituting the function we get

$$\text{Slope T} = \lim_{\Delta x \to 0} \frac{4(x+\Delta x)^2 - 4x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{4[x^2+2x\Delta x+(\Delta x)^2] - 4x^2}{\Delta x} \qquad [(x + \Delta x)^2 \text{ is in the } (a+b)^2 \text{ form}]$$

$$= \lim_{\Delta x \to 0} \frac{4x^2+8x\Delta x+4(\Delta x)^2 - 4x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{8x\Delta x+4\Delta x^2}{\Delta x}$$

Dividing through out by Δx we get

$$\text{Slope T} = \lim_{\Delta x \to 0} 8x + 4\Delta x$$

$\therefore$ Slope = 8x

=====

1.4 THE DERIVATIVE

The derivative of a function at a chosen input value is the best linear approximation of the function near that input value. For a real valued function of a single real variable, the derivative at a point equals the slope of the tangent line to the graph of the function at that point.

Given the function y = f(x), the derivative of f(x) with respect to x is the function f′(x) and is defined as

$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ Here f′(x) is the derivative of 'f' with respect to 'x.

Example:

Find the derivative of the following function using the definition of the derivative $f(x) = 4x^2 + 10x + 25$.

Solution: The definition of derivative is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Plugging the given function into the definitions of derivative we get

$\lim_{\Delta x \to 0} \frac{4(x+\Delta x)^2 + 10(x+\Delta x) + 25 - (4x^2 + 10x + 25)}{\Delta x}$
$\lim_{\Delta x \to 0} \frac{4(x^2 + 2x\Delta x + (\Delta x)^2)\ + 10x + 10\Delta x + 25 - 4x^2 + 10x + 25}{\Delta x}$
$\lim_{\Delta x \to 0} \frac{4x^2 + 8x\Delta x + 4(\Delta x)^2\ + 10x + 10\Delta x + 25 - 4x^2 + 10x + 25}{\Delta x}$
$\lim_{\Delta x \to 0} \frac{8x\Delta x + 4(\Delta x)^2\ + 10\Delta x}{\Delta x}$
dividing throughout by Δx we get
So the derivative *f′(x) = 8x +10*

1.5 DIFFERENTIABILITY AND CONTINUITY

A function is 'differentiable' at a point if the derivative exists at that point.

Conditions for differentiability at a point are :-

(a) A function must be continuous at that point.
(b) A function must have a unique tangent at that point.

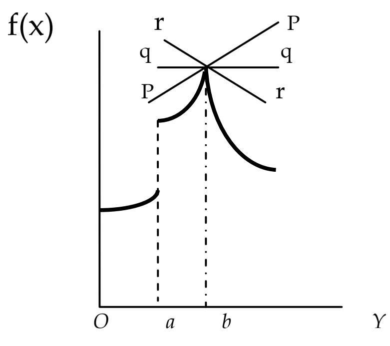


fig.1.5

In the *fig.1.5 f(x)* is not differentiable at 'a' because gap exist in the function at that point, and the derivative cannot be taken at any point where the function is discontinuous.

Continuity alone does not ensure differentiability. In the above figure *f(x)* is continuous at 'b', but it is not differentiable at 'b' because at a sharp point or kink (cusp) an infinite number of tangent lines line such as *pp, qq, rr* in the figure can be drawn. There is no one unique tangent line for that sharp point or kink.

1.6 DERIVATIVE NOTATION

The derivative of a function can be written in many ways.

If $y = f(x)$, its derivative can be expressed as $f'(x)$, y', $\frac{dy}{dx}$, $\frac{df}{dx}$, $\frac{d}{dx}f(x)$ or $D_x[f(x)]$

Example:- If $y = 10x^2 + 5x - 15$, the derivative can be written as

y', $\frac{d}{dx}$ $\frac{d}{dx}10x^2 + 5x - 15$ or $D_x[10x^2 + 5x - 15]$

1.7 RULES OF DIFFERENTIATION

Rule No. 1 : The constant Functions Rule

The derivative of a constant function is zero.

If y = K where 'K' is constant

Then $\frac{dy}{dx} = \frac{d}{dx}(K) = 0$

Ex. If $y = 10$ $\quad \frac{dy}{dx} = 0$

The derivative of the product of a constant and a function is equal to the product of the constant and the derivative of the function.

If $y = ax^n$ where 'a' is constant and $a \neq 0$

$$\frac{dy}{dx} = \frac{d}{dx}ax^n = a\frac{d}{dx}x^n$$

Rule No. 2: The Linear Function Rule

If $y = mx + c$ where 'm' and 'c' are constant

Then $\frac{dy}{dx} = \frac{d}{dx}y = \frac{d}{dx}(mx + c) = m$

Example

If $y = 10x + 100$ find $\frac{dy}{dx}$

$$\frac{dy}{dx} = 10$$

Rule No. 3: The Power Function Rule

The derivative of a power function $y = f(x) = x^n$ where 'n' is any real number is nx^{n-1}

ie If $y = x^n$ $\frac{dy}{dx} = nx^{n-1}$

Examples

(1) $y = x^5$; $\frac{dy}{dx} = 5x^{5-1} = 5x^4$

(2) $y = x^{-4}$ $\frac{dy}{dx} = -4 \times x^{-4-1} = -4x^{-5} = \frac{-4}{x^5}$

(3) $y = x^{5/3}$, $\frac{dy}{dx} = 5/3\, x^{5/3-1} = 5/3\, x^{2/3}$

(4) $y = x^{-5/2}$, $\frac{dy}{dx} = -5/2\, x^{5/2-1} = -5/2\, x^{-7/2}$

Rule No. 4: Addition Rule of Sum Rule

The derivative of a sum of two function is simply equal to the sum of the separate derivatives.

If $u = f(x)$ and $v = g(x)$ are the two functions

then $\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$ (Derivative the first function + Derivative of second)

Ex. 1:

If $y = x^3 + 6x^2 - 5x + 10$

$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}6x^2 - \frac{d}{dx}5x + \frac{d}{dx}10$

$= 3x^2 + 6\frac{d}{dx}x^2 - s\frac{d}{dx}x + \frac{d}{dx}10$

$= 3x^2 + 12x - 5$

=========

Ex. 2

If $y = 10x^6 + 3x^4 + 4x^{-3} + 10x^2 + 5$ find $\frac{dy}{dx}$

$\frac{dy}{dx} = 60x^5 + 12x^3 - 12x^{-4} + 20x$

Rule No.5: Subtraction Rule

The derivative of a difference of two functions is equal to the difference of the separate derivates.

If $y = u - v$ then

$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$ (ie derivative of the first function – minus the derivative of the second function)

Ex. 1

If $y = x^5 - 4x^3 - 6x^2 + 10$

$\frac{dy}{dx} = \frac{d}{dx}x^5 - 4\frac{d}{dx}x^3 - 6\frac{d}{dx}x^2 + \frac{d}{dx}10$

$= 5x^4 - 12x^2 - 12x$

=========

Ex. 2

If Q = $20P^3 - 2P^4 - 4P - 50$ find $\frac{dQ}{dP}$

$\frac{dQ}{dP} = 20\frac{d}{dP}P^3 - 2\frac{d}{dP}P^4 - 4\frac{d}{dP}P - \frac{d}{dP}50$

$= 60P^2 - 8P^3 - 4$

=====

Rule No. 6: Product Rule or Multiplication Rule

The derivative of the product of two functions is equal to the first function multiplied by the derivative of the second function plus the second function multiplied by the derivative of the first function.

If $u = f(x)$ and $v = g(x)$ are the two given function

Then

$\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}$

Ex. 1

If $y = 3x^4(2x - 5)$ find $\frac{dy}{dx}$

Solution : Consider u $= 3x^4$, v = $(2x - 5)$

$\frac{du}{dx} = 12x^3$, $\frac{dv}{dx} = 2$

$\frac{dy}{dx}$= $(3x^4 \times 2)$ + (2x-5)× $12x^3$

= $6x^4$+ $24x^4$- $60x^3$

= $30x^4$- $60x^3$

=====

Ex.2

Find $\frac{dy}{dx}$ if $y = (x^4 + x^3)\ (x^2 + x)$

$\frac{dy}{dx}$= $u\frac{dv}{dx}$ + v.$\frac{du}{dx}$

= $[(x^4 + x^3) + (2x + 1) + (x^2+x)\ (4x^3+3x^2)]$

= $2x^5$+ x^4+$2x^4$+ x^3+$4x^5$+ $3x^4$+ $4x^4$+ $3x^3$

= $6x^5$- $10x^4$ + $4x^3$

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<u>Rule No. 7: Quotient Rule or Division Rule</u>

The derivative of the quotient of two functions is equal to the denominator multiplied by the derivative of the numerator minus the numerator multiplied by the derivative of the denominator, all divided by the square of the Denominator.

If $y = \frac{u}{v}$where $u = f(x)$ and $v = g(x)$ then

$$\frac{dy}{dx} = \frac{v.\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Ex. 1.

Given $f(x) = \frac{5x^3}{4x+3}$ find $\frac{dy}{dx}$

here $u = 5x^3$ $\qquad \frac{du}{dx} = 15x^2$

$v = 4x + 3$ $\qquad \frac{dv}{dx}= 4$

$$\frac{dy}{dx}= \frac{(4x+3)\times 15x^2 - 5x^3\times 4}{(4x+3)^2}$$

$$\frac{60x^3+45x^2 - 20x^3}{(4x+3)^2} = \frac{40x^3 + 45x^2}{(4x + 3)^2}$$

Ex.2.

If $y = \frac{x^6}{x^3}$ find $\frac{dy}{dx}$

$u = x^6 \qquad \frac{du}{dx} = 6x^5$

$v = x^3 \qquad \frac{dv}{dx} = 3x^2$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{x^3 \times 6x^5 - x^6 \times 3x^2}{(x^3)^2}$$

$$= \frac{6x^8 - 3x^8}{x^6} \qquad = \frac{3x^8}{x^6}$$

Rule No. 8: Chain Rule or Function of Function Rule

If 'y' is the function of 'u' where 'u' is the function of 'x',then the derivative of 'y' with respect to 'x' $\frac{dy}{dx}$ie is equal to the product of the derivative of 'y' with respect to 'u' and the derivative of 'u' with respect to 'x'.

ie. If $y = f(u)$, where $u = f(x)$

then $\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$

Ex. 1

If y = 7u + 3 *and* $u = 5x^2$ find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$$

$$= \frac{dy}{du} = 7, \ \frac{dy}{dx} 10x$$

$$\therefore \frac{dy}{dx} = 7 \times 10x = 70x$$

Ex.2

If $y = (5x^2 + 3)^4$ find $\frac{dy}{dx}$ using the chain rule

Let $u = 5x^2 + 3$

$\therefore y = u^4$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$\frac{dy}{du} = 4u^3$ $\qquad \frac{du}{dx} = 10x$

$\therefore \frac{dy}{dx} = 4u^3 \times 10x$

Substituting for $5x^2 + 3$ 'u' we get

$4(5x^2 + 3)^3 \times 10x$

$\frac{dy}{dx} = 40\text{x}(5x^2 + 3)^3$

Problem 1. If $y = z^3 + 10$ and z = x^7 find $\frac{dy}{dx}$

[Ans : $21x^{20}$]

Problem 2. Find $\frac{dy}{dx}$ If $y = (x^3 + 3x)^2$

[Ans: $6x^5 + 24x^3 + 18x$]

Rule No.9: Parametric Function Rule

If both 'x' and 'y' are the differential function of 't' then the derivative of 'y' with respect to 'x' is obtained by dividing the derivative of 'y' with respect to 't' by the derivative of 'x' with respect to 't'. ie. If $x = f(x)$ and $y = g(t)$ then

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

Examples

Find $\frac{dy}{dx}$ if $x = at^4$ and $y = 4at$

$\frac{dy}{dt} = 4a$ $\qquad \frac{dx}{dt} = 4at^3$

$\frac{dy}{dx} = \frac{4a}{4at^3}$ $\qquad = \frac{1}{t^3}$

Rule No. 10: Exponential Function Rule

The derivative of Exponential function is the exponential itself, if the base of the function is 'e', the natural base of an exponential.

If $y = e^x$, then $\frac{dy}{dx} = e^x$

If $y = e^{-x}$, then $\frac{dy}{dx} = -e^{-x}$

If $y = e^{ax}$, then $\frac{dy}{dx} = ae^{ax}$

Examples

1. If $y = e^{2x}$ find $\frac{dy}{dx}$

$$\frac{dy}{dx} = 2e^{2x}$$

If $y = e^u$, where $u = g(x)$, then

$\frac{dy}{dx} = \frac{d}{dx} e^u . \frac{du}{dx} = e^u \frac{du}{dx}$

Examples

Ex. 1

Find $\frac{dy}{dx}$ if $y = e^{x^2+3}$

Let $u = x^2 + 3$, then $\frac{dy}{du} = 2x$

$\therefore y = e^u$

$\frac{dy}{dx} = e^u \left(\frac{du}{dx}\right) = e^u \times 2x$

$= 2x \times e^{x^2+3}$ since $u = x^2 + 2$

=====

Ex. 2

Find $\frac{dy}{dx}$ if $y = e^{x^2}$

Let $u = x^2$ $\frac{du}{dx} = 2x$

$= e^{x^2} \times 2x$

=====

Ex. 3

Find $\frac{dy}{dx}$ if $y = e^{4x^3 + 6x^2 + 10}$

[Ans : $(e^{4x^3 + 6x^2 + 10})\ (12x^2 + 12x)$

Rule No.11. Logarithmic Function Rule

The derivative of a logarithmic function with natural base such as $y = log\ x$ is

$$\frac{dy}{dx} = \frac{d}{dx} (\log x) = \frac{1}{x}$$

If y = log u, when u = g(x), then

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$

$$= \frac{d(\log u)}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot \frac{du}{dx}$$

Ex. If y = log x^3 find $\frac{dy}{dx}$

Let u = x^3 then $\frac{du}{dx}$ = $3x^2$

Y = log u since x^3 = u

$\frac{dy}{dx} = \frac{1}{u} \cdot 3x^2$

$= \frac{1}{x^3} \cdot 3x^2$ since u = x^3

$= \frac{3x^2}{x^3} = \frac{3}{x}$

Rule No. 12: Differentiation of One function with respect to another Function

Let f(x) and g(x) be two functions of 'x' then derivative of f(x) with respect to g(x) is

$$\frac{d\,f(x)}{d\,g(x)} = \frac{df(x)}{dx} \div \frac{d\,g(x)}{dx}$$

(ie differentiate both function and divide)

Examples.

1. Differentiate (x^2 + 1) with respect to(2x-1)

$\frac{d}{dx}(x^2 + 1) = 2x$

$\frac{d}{dx}(x^2 - 1) = 2x$

$\therefore \frac{d\,f(x)}{d\,g(x)} = \frac{2x}{2x} = 1$

2. Differentiate ($3x^2$ + 2x) with respect to (x^2 + 5x)

$\frac{d}{dx}(3x^2 + 2x) = 6x + 2$

$\frac{d}{dx}(x^2 + 5x) = 2x + 5$

$\frac{d\,f(x)}{d\,g(x)} = \frac{df(x)}{dx} \div \frac{d\,g(x)}{dx}$

$= \frac{6x+2}{2x+5}$

3. Differentiate $\frac{4x+2}{x-2}$ with respect to $\frac{4x+2}{x-2}$

Let u $= \frac{4x+2}{x-2}$

$\therefore \frac{du}{dx} = \frac{[(x-2)(4+0)-(4x+2)(1-0)]}{(x-2)^2}$

$= \frac{4x-8-4x-2}{(x-2)^2}$

$= \frac{-10}{(x-2)^2}$

Let u $= \frac{3x+2}{x-2}$

$\therefore \frac{du}{dx} = \frac{[(x-2)\times 3-(3x+2)\times 1]}{(x-2)^2}$

$= \frac{3x-6-3x-2}{(x-2)^2}$

$= \frac{-8}{(x-2)^2}$

$\frac{du}{dx} = \frac{du}{dx} \div \frac{d\,v}{dx}$

$= \frac{-10}{(x-2)^2} \div \frac{-8}{(x-2)^2}$

$= \frac{-10}{-8} = \frac{10}{8}$

Rule No. 13. Inverse Function Rule

The derivative of the inverse function (ie $\frac{dx}{dy}$)is equal to the reciprocal of the derivative of the original function ie $\frac{1}{dy/dx}$

ie $\frac{dx}{dy} = \frac{1}{dy/dx}$

Examples:

1. Find the $\frac{dx}{dy}$ of the function $y = x^5$

$\frac{dy}{dx} = 5x^4$

$\therefore \frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{5x^4}$

2. Find the derivative of the inverse function

y = $(x^2 + 2)(x^2 + 4)$

$\frac{dy}{dx} = (x^2 + 2)(2x) + (x^2 + 4)(2x)$

$= 2x^3 + 4x + 2x^3 + 8x$

$= 4x^3 + 12x$

$\frac{dy}{dx} = \frac{1}{4x^3 + 12x}$

1.8 HIGHER ORDER DERIVATIVES (DERIVATIVES OF DERIVATIVES)

The first order derivatives of 'y' with respect to 'x' is $\frac{dy}{dx}$ or $f'(x)$, then second order derivative is obtained by differentiating the first order derivatives with respect to 'x' and is writer as

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ and is denoted as $f''(x)$

It measures the slope and the rate of change of the first derivative, just as the first derivative measures the slope and the rate of change of the original or primitive function. The third order derivative $f'''(x)$ measures the slope and the rate of change of the second order derivative etc.

Ex. If $f(x) = 5x^4 + 4x^3 + 3x^2 + 10x$

$f'(x)$	$= 20x^3 + 12x^2 + 6x + 10$
$f''(x)$	$= 60x^2 + 24x + 6$
$f'''(x)$	$= 120x + 24$
$f^4(x)$	120
$f^5(x)$	0

Problems

1. If $y = 5x^2 + 3x^2 - 8x + 2$ find $\frac{d^2y}{dx^2}$ [Ans: 30]
2. If $y = x^4 + x^2 - x$ find $f''(x)$ and $f'''(x)$ [Ans: $f''(x) = 12x^2 + 2$, $f'''(x) = 24x$]

1.9 IMPLICIT DIFFERENTIATION

An equation $y = 10x^2+5x+2$ directly expresses 'y' in terms of 'x'. Hence this function is called 'Explicit Function'.

As equation $x^3y + xy + 2x = 0$ does not directly express 'y' in terms of 'x'. This types of function is called 'Implicit Function'. Some implicit functions can be easily converted to explicit function by solving for the dependent variable in terms of the independent variable, others cannot. For those not readily convertible, the derivative may be found by implicit differentiation.

In order to find the derivative of implicit function every term in both sides of the function is to be differentiated with respect to 'x'.

Examples:

1. Find $\frac{dy}{dx}$ of the following 3x-2y = 4

$$= \frac{d}{dx} 3x - \frac{d}{dx} 2y = \frac{d}{dx} (4)$$

$$= 3 - 2\frac{dy}{dx} = 0$$

$$= -2\frac{dy}{dx} = -3$$

$$\therefore \frac{dy}{dx} = \frac{-3}{-2} = 3/2 = 1.5$$

2. Find the derivative of $4x^2 - y^3 = 97$

Take the derivative with respect to 'x' of both sides

$$\frac{d}{dx} 4x^2 - \frac{d}{dx} y^3 = \frac{d}{dx} 97$$

$$\frac{d}{dx} 4x^2 = 8x, \quad \frac{dy}{dx} 97 = 0$$

$$\frac{d}{dx} y^3 = 3.y^2 \frac{d}{dx} (y)$$

$$= 3y^2 \frac{dy}{dx}$$

$\therefore$ The derivative of the function is

$$8x - 3y^2\left(\frac{dy}{dx}\right) = 0$$

$$-3y^2\frac{dy}{dx} = -8x$$

$$\frac{dy}{dx} = \frac{-8x}{-3y^2} = \frac{8x}{3y^2}$$

3. Find the derivative of the function $3y^5 - 6y^4 + 5x^6 = 243$ [Ans: $\frac{-30x^5}{15y^4-24y^3}$]

USE OF THE DERIVATIVE IN MATHEMATICS AND ECONOMICS

1.10 INCREASING AND DECREASING FUNCTIONS

A function is and increasing function if the value of the function increases with an increase in the value of the independent variable and decrease with a decrease in the value of the independent variable.

For a function y f(x)

When $x_1 < x_2$ then $f(x_1) \leq f(x_2)$ the function is increasing

When $x_1 < x_2$ then $f(x_1) < f(x_2)$ the function is strictly increasing

A function is a decreasing function if the value of the function increases with a fall in the value of independent variable and decrease with an increase in the value of independent variable.

For a function y = f(x)

When $x_1 < x_2$ then $f(x_1) \geq f(x_2)$ the function is decreasing

When $x_1 < x_2$ then $f(x_1) > f(x_2)$ the function is strictly decreasing

The derivative of a function can be applied to check whether given function is an increasing function or decreasing function in a given interval.

If the sign of the first derivative is positive, it means that the value of the function f(x) increase as the value of independent variable increases and vice versa.

i.e. a function y = f(x) differentiable in the interval [a, b] is said to be an increasing function if and only if its derivatives an [a, b] is non negative.

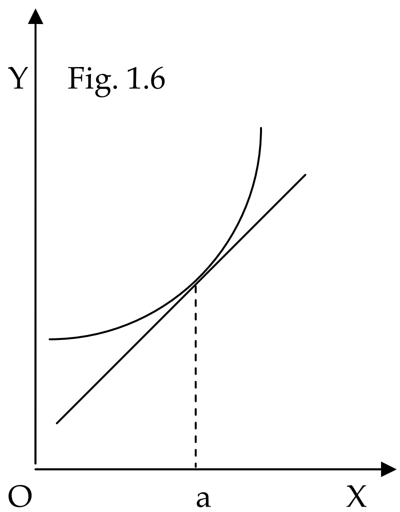
i.e. $\frac{dy}{dx} > 0$ in the interval [a, b]

A function y = f(x) differentiable in the interval [a, b] is said to be a decreasing function if and only if its derivative an [a, b] is non positive.

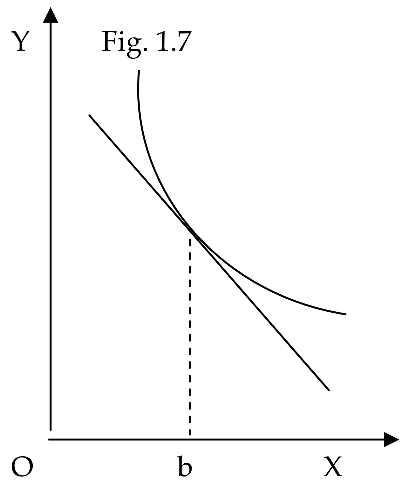
i.e. $\frac{dy}{dx} < 0$ in the interval [a, b]

The derivative of a curve at a point also measures the slope of the tangent to the curve at that point. If the derivative is positive, then it means that the tangent has a positive slope and the function (curve) increases as the value of the independent variable increase. Similar interpretation is given to the to the decreasing function and negative slope of the tangent.

A function that increases or decreases over its entire domain is called monotonic function.



Increasing function at x = a slope > 0



Decreasing function at x = b slope < 0

Example:1 Check whether the function $3x^3 + 3x^2 + x - 10$ is increasing or decreasing

Solution: $\frac{d}{dx}\ 3x^3 + 3x^2 + x - 10$

$= 9x^2 + 6x + 1 = (3x+1)^2$

The function is always positive because $(3x+1)^2$ is positive since square of a real number is always positive. Therefore the function is increasing.

Example:2 Test whether the following functions are increasing, decreasing or stationary state at x = 4

(a) $2x^2 - 5x + 10$ [Ans: Increasing}

(b) $X^3 - 10x^2 + 2x + 5$ [Ans: Decreasing]

(c) $Y = x^4 - 6x^3 + 4x^2 - 13$ [Ans: Stationary]

1.11 CONCAVITY AND CONVEXITY

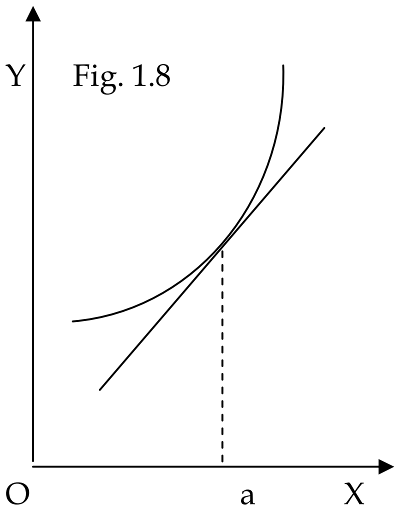
Convex Function

A function is convex at x = a if in an area very close to [a, f(a)] the graph of the function lies completely above its tangent line.

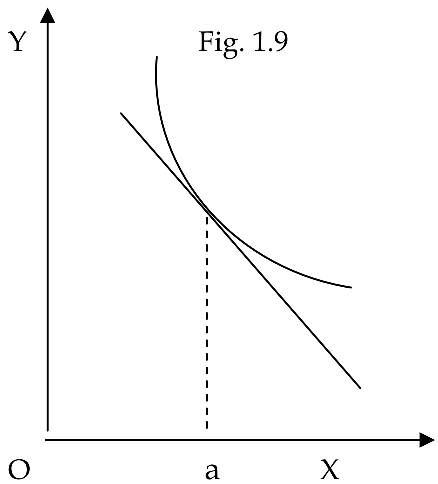
i.e. if second derivative of a function f(x) is positive i.e. $f''(x) > 0$, the function is said to be a convex function in the given interval.

$f''(a) > 0$ f(x) is convex at x = a.

Graphs of the function convex at x = a



$f'(a) > 0$, $f''(a) > 0$



$f'(a) < 0$, $f''(a) > 0$

Concave Function

A function f(x) is concave at x = a if in some small region close to the point [a, f(a)] the graph of the function lies completely below its tangent line.

i.e. if the second derivative of a function f(x) is negative i.e. $f''(x) < 0$, then the function is said t be a concave function in the given interval.

$f''(x) < 0$, f(x) is concave at x = a.

Graph of the function concave at x = a

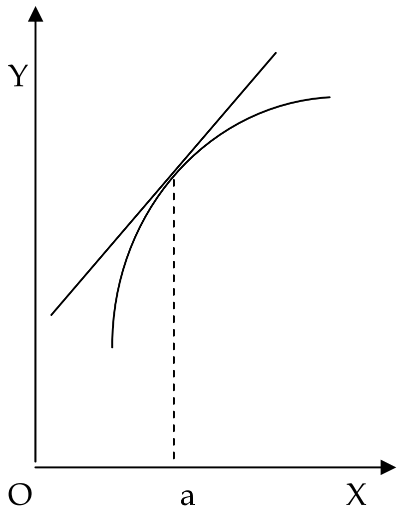


Fig. 1.10

$f'(a) > 0,\ f''(a) < 0$

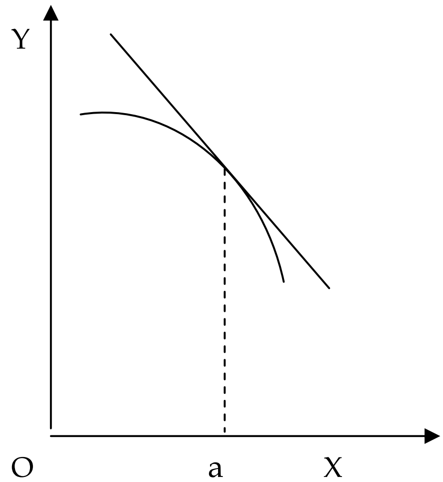


Fig. 1.11

$f'(a) < 0,\ f''(a) < 0$

Example: 1

Show that the curve of y = 2x – 3 +$^1/_x$ convex from below for all positive values of x.

Solution:

$$Y = 2x - 3 + {}^1/_x$$

$$\frac{dy}{dx} = 2 - {}^1/_{x^2}$$

$$\frac{d^2y}{dx^2} = {}^2/_{x^3}$$

Since $\frac{d^2y}{dx^2} = {}^2/_{x^3} > 0$ for $x > 0$ the curve is convex from below for all positive values of x.

Example: 2

Show that the demand curve $P = \frac{10}{(x+5)} - 8$ is downward sloping and convex from below

$$\frac{dp}{dx} = \frac{-10}{(x+5)^2}$$

Since $\frac{dp}{dx}$ (slope) is negative the demand curve is downward sloping.

$\frac{d^2p}{dx^2} = \frac{20}{(x+5)^3}$ is positive, the demand curve is convex from below.

Example:3

Test to see if the following functions are concave or convex at x = 3

(a) $Y = -2x^3 + 4x^2 + 9x - 15$ [Ans: Concave]

(b) $Y = (5x^2 - 8)^2$ [Ans: Convex]

1.12 RELATIVE EXTREMA

A relative extremum is a point at which a function is at a relative maximum or minimum. To be at a relative maximum or minimum at a point 'a', the function must be at a relative 'plateau', i.e. neither increasing nor decreasing at 'a'. If the function is neither increasing nor decreasing at 'a', the first derivative of the function at a must equal to zero or be undefined. A point in the domain of a function where the derivative equals zero is called critical point of critical value.

The second derivative is used to distinguish between relative maximum and minimum mathematically.

If we assume $f'(a) = 0$

(a) $f''(a) > 0$ shows function is convex, graphs of function lies completely above its tangent line at x = a and the function is at a relative minimum at x = a.

(b) $f''(a) < 0$ shows function is concave, graphs of function lies completely below its tangent line at x = a and the function is at a relative maximum at x = a.

(c) $f''(a) = 0$, the test is inconclusive.

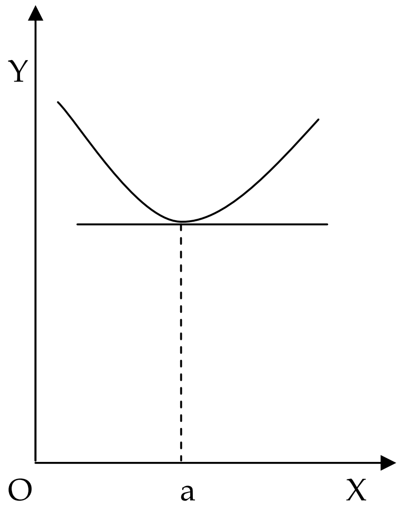


Fig. 1.12
Relative Minimum at x = a

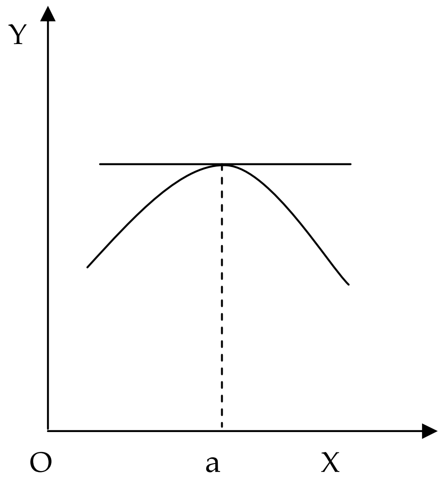


Fig. 1.13
Relative Maximum at x = a

Ex. 1

Find the relative extrema for the following functions by (a) finding the critical value (b) determining if at the critical values the function is at a relative maximum or minimum

(a)

$f(x) = -7x^2 + 126x - 23$

$f'(x) = -14x + 126$

$x = 9$ critical value

$f''(x) = -14$

$f''(x) = -14 < 0$ concave, relative minimum

(b)

$f(x) = 3x^3 - 36x^2 + 135x - 13$

[Ans: $x = 3$, $x = 5$ critical values]

$f''(x) = 18x - 72$

$f''(3) = -18 < 0$ [concave, relative maximum]

$f''(5) = 18 > 0$ [convex, relative minimum]

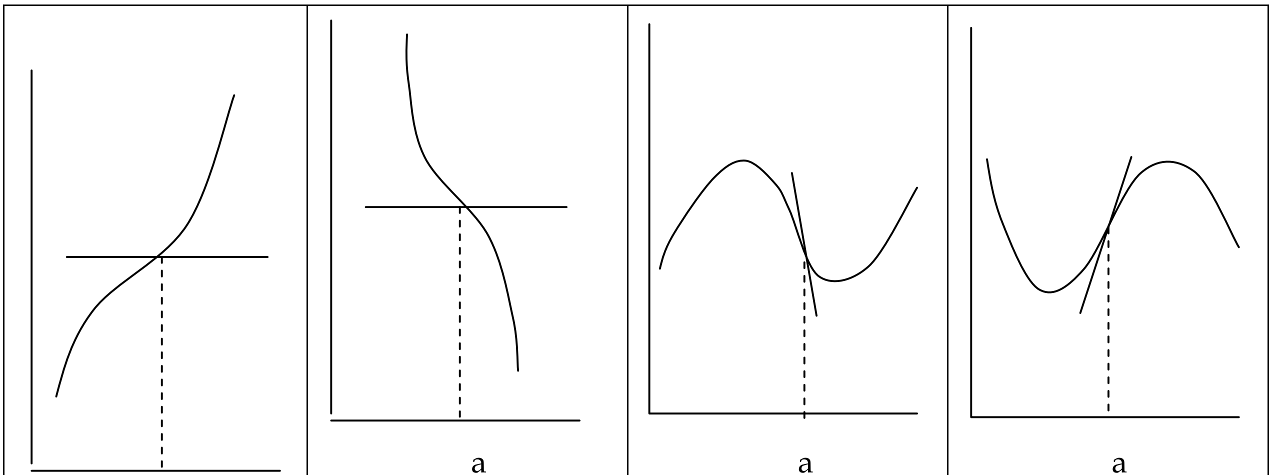
1.13 INFLECTION POINTS

An inflection point is a point on the graph where the function crosses its tangent line and changes from concave to convex or vice versa.

Inflective points occurs only where the second derivative equals zero or is undefined. The sign of the first derivative is immaterial.

Conditions

1. $f''(a) = 0$ or is undefined
2. Concavity changes at $x = a$
3. Graph crosses its tangent line at $x = a$



a	a	a	a
$f'(a) = 0$	$f'(a) = 0$	$f'(a) < 0$	$f'(a) > 0$
$f''(a) = 0$	$f''(a) = 0$	$f''(a) = 0$	$f''(a) = 0$

Example:

For the following function find the critical values and test to see it at the critical values the function is at a relative maximum, minimum or possible inflection point.

If $y = (5 = x)^3$

$y' = 3(5-x)^2 \times -1 = -3(5-x)^2 = 0$

$X = 5$ critical value.

y'' = 6 (5 – x)

$y''(5)$ = 6(5 – 5) = 0 test inconclusive

Since the second derivative is zero the function is at an inflection point at x = 5 not relative maximum or minimum.

1.14 CURVE SKETCHING

Whether we are interested in function as a purely mathematical object or in connection with some application to the real world; it is often useful to know what the graphs of the function looks like. We can obtain a good picture of the graph using certain crucial information provided by derivatives of the function and certain limits. Curve sketching is a practical application of differential calculus. We can make a fairly accurate sketch of any function using derivatives.

By using the first and second derivatives we can determine whether a function is maxima or minima at a particular value of 'x' the concavity of the function, the point of inflection etc.

The General guide lines for curve Sketching

The following steps are helpful when sketching curves. Since they are the general guidelines for all curve, each step may not always apply to all functions.

(a) Domain: Find the domain of the function. This will be useful when finding vertical asymptotes and determining critical numbers.

(b) Intercepts: Find the 'x' and 'y' intercepts of the function if possible. To find the 'x' intercept, we set y = 0 and solve the equation for x. Similarly for 'y'.

(c) Symmetry: Determine whether the function is an odd function, an even function or neither odd nor even. If f(-x) = f(x) for all 'x' in the domain, then function is even and symmetric about the 'y' axis. If f(x) = -f(x) for all 'x' in the domain, then the function is odd and symmetric about the origin.

(d) Asymptotes: For vertical asymptotes, check for rational functions zero denominators, or undefined by function points. For horizontal asymptotes, consider $\lim_{x \to \pm\propto} f(x)$

(e) Determine critical numbers: Checking where $f'(x) = 0$ or $f'(x)$ does not exist and finding intervals where 'f' is increasing or decreasing.

(f) Local Maximum/Minimum: Use the first derivative test to find the local maximums and minimums of the function.

(g) Concavity and points of Inflection: Determine when $f''(x)$ is positive and negative to find the intervals where the function is concave upward and concave downward and also determine inflection point checking where $f''(x) = 0$ or $f'''(x)$ does not exist.

(h) Plot intercepts, critical points, inflection points, asymptotes, and other points as needed.

(i) Connect plotted points with smooth curve.

1.15 OPTIMIZATION OF FUNCTION

Optimization is the process of finding the relative maximum or minimum of a function. Without the help of graph, this is done with the techniques of relative extrema and inflection points.

Steps

(1) Take the first derivative, set it equal to zero, and solve for the critical points. This step is known as necessary condition or first order conditions. It shows the function is neither increasing no decreasing; but at a plateau at that level of 'x'.

(2) Take the second derivative, evaluate it at the critical point and check the signs. If at a critical point 'a'.

$f''(a) < 0$, the function is concave at 'a' and hence at a relative maximum.

$f''(a) > 0$, the function is convex at 'a' and hence at a relative minimum.

$f''(a) = 0$, the test is inconclusive.

This test is known as second order condition or sufficient condition.

Relative Maximum	Relative Minimum
$f'(a) = 0$	$f'(a) = 0$
$f''(a) < 0$	$f''(a) > 0$

Example: Optimize $f(x) = 2x^3 - 30x^2 + 126x + 59$

$f'(x) = 6x^2 - 60x + 126$

Critical points is at $6x^2 - 60x + 126 = 0$

$= 6(x-3)(x-7) = 0$

$X = 3, x = 7$

$f''(x) = 12x - 60$

$f''(3) = 12 \times 3 - 60 = -24 < 0$ concave, relative maximum

$f''(7) = 12 \times 7 - 60 = 24 > 0$ convex, relative minimum.

i.e. the function is maximum at x = 3 and minimum at x = 7

Successive Derivative Test for Optimization

If $f'(a) = 0$ as in the case of point of inflection of different type, the second derivative test is inconclusive. In such cases without a graph, the successive derivative test is helpful for obtaining optimum value.

Conditions

(1) If the first non zero value of a higher ordered derivative, when evaluated at a critical point, is an odd numbered derivative (3^{rd} , 5^{th} etc.), the function is at an inflection point.

(2) If the first non zero value of a higher ordered derivative, when evaluated at a critical point 'a' is an even numbered derivative, the function is at a relative extremum at 'a'.

Marginal Concepts

Marginal cost in economics is defined as the change in total cost incurred from the production of an additional unit.

Marginal revenue is defined as the change in total revenue brought about by the sale of an extra good. Marginal Cost (MC), Marginal Revenue (MR) can each be expressed mathematically as derivatives of their respective total functions.

i.e. If TC = TC (Q); then MC = $\frac{dTC}{dQ}$

If TR = TR (Q); then MR = $\frac{dTR}{dQ}$

Examples: 1

If $TC = 25Q^2 + 7Q + 10$

Then MC = $\frac{dTC}{dQ} = \underline{\underline{50Q + 7}}$

If TR = 50Q - 2Q²

Then MR = $\frac{dTR}{dQ}$ = $\underline{\underline{50 - 4Q}}$

Ex. 2

Given the demand function P = 50 – 5Q, then the marginal revenue function can be found by first finding the total revenue function and then taking the derivative of the function with respect to Q.

Demand function P = 50 - 5Q

TR = P x Q = (50 – 5Q)Q

$= 50Q - 5Q^2$

MR = $\frac{dTR}{dQ}$ = $\underline{\underline{50 - 10Q}}$

Ex. 3

Find maximum profits 'π' for a firm, given total revenue $R = 4000Q - 33Q^2$ and total cost function $C = 2Q^3 - 3Q^2 + 400Q + 5000$, assuming Q > 0.

Profit function: π = R – C

$\Pi = (4000Q - 33Q^2) - (2Q^3 - 3Q^2 + 400Q + 5000)$

$= -2Q^3 - 30Q^2 + 3600Q - 5000$

Taking the first derivative, set it equal to zero, and solving this for Q we get critical points.

$\pi' = -6Q^2 - 60Q + 3600 = 0$

$= -6(Q^2 + 10Q + 600) = 0$

$= -6(Q + 30)(Q - 20) = 0$

Q = -30 Q = 20 critical points

[We can also use Quadratic formula]

Taking the second derivative we get

$\pi'' = -12Q - 60$

$\pi''(20) = -12(20) - 60 = -300 < 0$

[Taking Q = 20 ignoring negative value]

The function is concave and relative maximum. It means profit is maximum at Q = 20 When Q=20 the profit π is

$\pi(20) = 2(20)^3 - 30(20)^2 + 3600(20) - 5000 = \underline{39000}$

Ex. 4

Maximize the following total revenue TR function by finding the critical values, testing the second order conditions and calculating the maximum TR If TR = 32Q – Q²

$TR = 32Q - Q^2$

$TR' = 32 - 2Q$

Setting $TR' = 0$, $32 - 2Q = 0$

$32 = 2Q \qquad \therefore Q\ 32/2 = 16$ critical value

$TR'' = -2 < 0$ concave, relative maximum

TR when Q = 16 is $32(16) - 16^2 = \underline{\underline{256}}$

Ex. 5

Given the total revenue $R = 15Q - Q^2$ find the Average Revenue (AR), Marginal Revenue (MR) and the level of output Q that maximizes Total Revenue (TR).

$TR = R = 15Q - Q^2$

$AR = \frac{TR}{Q} = \frac{15Q - Q^2}{Q} = \underline{\underline{15 - Q}}$

$MR = TR' = \frac{dR}{dQ} = \underline{\underline{15 - 2Q}}$

Setting $TR' = MR = 0$ we get $15 - 2Q = 0$

$\therefore 15 = 2Q$ and $Q = \underline{\underline{7.5}}$ critical value.

$TR'' = -2 < 0$ concave, relative maximum.

TR when Q = 7.5 is

$15(7.5) - (7.5)^2$

$= 112.5 - 56.25 = 56.25$

The output that maximize TR=$\underline{7.5}$

AR when Q=7.5 is 15-7.5=$\underline{7.5}$

MR when Q=7.5 is 15-2(7.5)=15-15=0

Ex. 6.

From the total cost function $TC = Q^3 - 5Q^2 + 60Q$ find (1) the average cost AC function (2) the critical value at which AC is minimized and (3) the minimum average cost.

$$TC = Q^3 - 5Q^2 + 60Q$$

(1) $AC = \frac{TC}{Q} = \frac{Q^3 - 5Q^2 + 60Q}{Q} = \underline{\underline{Q^2 - 5Q + 60}}$

(2) AC is minimum at

$AC' = 2Q - 5 = 0, \therefore Q = \underline{\underline{2.5}}$

$AC'' = 2 > 0$ convex, relative minimum

(3) $AC(2.5) = (2.5)^2 - 5(2.5) + 60 = \underline{\underline{53.75}}$

Ex. 7:

A firm has the following total cost, C and demand function Q

$C = 1/3\, Q^3 - 7Q^2 + 111\, Q + 50$ and $P = 100 - Q$

(1) Write down the total revenue function R in terms of Q

(2) Formulate the total profit function

(1) Price function $P = 100 - Q$

Revenue = P x Q

The revenue function is $R = PQ = 100Q - Q^2$

(2) Profit function is TR – TC functions

$= (100Q - Q^2) - (1/3\, Q^3 - 7Q^2 + 111Q + 50)$

$\Pi = \underline{\underline{-1/3\, Q^3 + 6Q^2 - Q -}}\, 50$

Ex. 8.

A producer has the possibility of discriminating between the domestic and foreign markets for a product where the demands functions are

$Q_1 = 24 - 0.2P_1$ $\qquad$ $Q_2 = 10 - 0.05P_2$

Where TC = 35 + 40Q, what price will be firm charge (a) with discrimination and (b) without discrimination.

(a) With discrimination

$Q_1 = 24 - 0.2P_1$

(Multiply with 5 to get $1P_1$)

$5Q_1 = 120 - P_1$

$\therefore P_1 = 120 - 5Q_1$

$TR_1 = (120 - 5Q_1)Q_1 = 120Q_1 - 5Q_1^2$

$MR_1 = \underline{\underline{120 - 10Q_1}}$

The firm will maximize profit where $MC = MR_1 = MR_2$

$TC = 35 + 40Q$

$MC = 40$

When $MC = MR_1$, $40 = 120 - 10Q_1$, $\therefore Q_1 = 8$

When $Q_1 = 8$, $P_1 = 120 - 5(8) = 80$

In the second market with $Q_2 = 10 - 0.05P_2$

Multiplying with 20 on both sides

$20Q_2 = 200 - P_2$

$\therefore P_2 = 200 - 20Q_2$

$TR_2 = (200 - 20Q_2)Q_2 = 200Q_2 - 20Q_2^2$

$MR_2 = 200 - 40Q_2$

When $MC = MR_2$, $40 = 200 - 40Q_2$

$\therefore Q_2 = 4$

$P_2 = 200 - 20(4) = 120$

Hence $P_1 = 80$, $P_2 = 120$, $Q_1 = 8$, $Q_2 = 4$

(b) If the producer does not discriminate, $P_1 = P_2 = P$ and two demand function can be combines as follows.

$Q = Q_1 + Q_2 = 24 - 0.2P + 10 - 0.05P$

$= 34 - 0.25P$

$P = 136 - 4Q$

$TR = (136 - 4Q)Q = 136Q - 4Q^2$

MR = 136 – 82

At profit maximizing level MC = MR

$40 = 136 - 8Q$

$Q = 12$

At Q = 12   P = 136 – 4(12) = 88

CHAPTER 2

CALCULUS AND MULTIVARIABLE FUNCTIONS

2.1 FUNCTIONS OF SEVERAL VARIABLES

It is evident that all sciences are concerned with large number of inter-related variable quantities and that only by a process of severe simplification can functional relations between two variables be applied at all. For example, in economics, when one individual considers his purchases on a market, the demand for any good depends, not only on the price of the good, but also on his money income and on the prices of related goods.

To measure the effect of a change in a single independent variable on the dependent variable in a multivariable function, the partial derivative is needed. In a function Z = f(x, y), the two variables x and y can be varied in any way quite independently of each other. In particular, one of the variables can be allotted a fixed value and the other allowed to vary. Two functions can be obtained in this way, Z as a function of x only (y fixed) and z as a function of y only (x fixed). The derivative of each of these functions can be defined at any point and evaluated according to the familiar "partial derivatives" of the function Z = f(x, y), the term "partial" implying that they are defined only for very special variations of the independent variables. One partial derivative follows when x is varied and y kept constant, the other when y is varied and x kept constant.

The partial derivative of z with respect to x measures the instantaneous rate of change of z with respect to x while y is held constant. It is written as $\frac{\partial z}{\partial x}$, $\frac{\partial f}{\partial x}$, f_x (x, y), f_x or z_x. Expressed mathematically,

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x,\ y) - f(x,y)}{\Delta x}$$

The partial derivative of z with respect to y measures the instantaneous rate of change of z with respect to y while x is held constant. Expressed mathematically,

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x,\ y+\Delta y) - f(x,y)}{\Delta y}$$

Example 1:

Given Z = f(x, y) = $3x^2+xy+4y^2$. Find the partial derivatives.

When finding $\frac{\partial z}{\partial x}$ (or f_x), we must bear in mind that y is to be treated as a constant during differentiation. As such, y will drop out in the process if it is an additive constant (such as the term $4y^2$), but will be retained if it is a multiplicative constant (such as in term xy). Thus we have

$$\frac{\partial z}{\partial x} = f_x = 6x + y$$

Similarly, by treating x as a constant, we find that

$$\frac{\partial z}{\partial y} = f_x = x + 8y$$

Example 2:

Given z = f(u, v) = (u + 4) (3u + 2v), the partial derivatives can be found by use of the product rule. By holding v constant, we have

$$f_u = [(u+4)x3] + [1\ (3u+2v)]$$

$$= 2(3u+v+6)$$

Similarly, by holding u constant, we find that

$$f_v = [(u+4)2] + 0(3u+2v)$$

$$= \underline{\underline{2(u+4)}}$$

Example 3:

Given z = $(3x - 2y)/(x^2 + 3y)$, the partial derivatives can be found by use of the quotient rule.

$$\frac{\partial z}{\partial x} = f_x = \frac{[3(x^2 + 3y)] - [2x(3x - 2y)]}{(x^2+3y)^2}$$

$$= \underline{\underline{\frac{-3x^2 + 4xy + 9y}{(x^2+3y)^2}}}$$

$$\frac{\partial z}{\partial y} = f_y = \frac{[-2(x^2 + 3y)] - [3(3x - 2y)]}{(x^2+3y)^2}$$

$$= \underline{\underline{\frac{-x\ (2x+9)}{(x^2+3y)^2}}}$$

2.2 RULES OF PARTIAL DIFFERENTIATION

Partial derivatives follow the same basic patterns as the rules of differentiation in one-variable case. A few key rules are given below.

1. **Product Rule:**

   Given z = g(x, y) x h (x, y)

$$\frac{\partial z}{\partial x} = g(x, y) \times \frac{\partial h}{\partial x} + h(x, y) \times \frac{\partial g}{\partial x}$$

$$\frac{\partial z}{\partial y} = g(x, y) \times \frac{\partial h}{\partial y} + h(x, y) \times \frac{\partial g}{\partial y}$$

Example:

Given z = (3x + 5) (2x + 6y), by the product rule

$$\frac{\partial z}{\partial x} = (3x + 5)\,2 + (2x + 6y)3$$

$$= 6x + 10 + 6x + 18y$$

$$= \underline{\underline{12x + 18y + 10}}$$

$$\frac{\partial z}{\partial y} = (3x + 5)6 + (2x + 6y)0$$

$$= 18x + 30 + 0$$

$$= \underline{\underline{18x + 30}}$$

2. **Quotient Rule**

   Given z = $\frac{g(x,y)}{h(x,y)}$

$$\frac{\partial z}{\partial x} = \frac{h(x,y) \times \frac{\partial g}{\partial x} - g(x,y) \times \frac{\partial h}{\partial x}}{[h(x,y)]^2}$$

$$\frac{\partial z}{\partial y} = \frac{h(x,y) \times \frac{\partial g}{\partial y} - g(x,y) \times \frac{\partial h}{\partial y}}{[h(x,y)]^2}$$

Example:

Given z = (6x + 7y)/5x+3y), by the quotient rule

$$\frac{\partial z}{\partial x} = \frac{(5x+3y)6 - (6x+7y)5}{(5x+3y)^2}$$

$$= \frac{30x+18y-30x - 35y}{(5x+3y)^2}$$

$$= \frac{-17y}{(5x+3y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(5x+3y)7 - (6x+7y)3}{(5x+3y)^2}$$

$$= \frac{35x+21y-18x - 21y}{(5x+3y)^2}$$

$$= \frac{17x}{(5x+3y)^2}$$

3. **Generalized Power Function Rule:**

Given z = $[f(x, y)]^n$

$$\frac{\partial z}{\partial x} = n[f(x, y)]^{n-1} \times \frac{\partial f}{\partial x}$$

$$\frac{\partial z}{\partial y} = n[f(x, y)]^{n-1} \times \frac{\partial f}{\partial y}$$

Example:

Given z = $(x^4 + 9y^2)^5$

$$\frac{\partial z}{\partial x} = 5(x^4 + 9y^2)^4 \times 4x^3$$

$$= 20x^3(x^4 + 9y^2)^4$$

$$\frac{\partial z}{\partial y} = 5(x^4 + 9y^2)^4 \times 18y$$

$$= 90y\,(x^4 + 9y^2)^4$$

2.3 SECOND – ORDER PARTIAL DERIVATIVES

Given a function z = f(x, y), the second-order (direct) partial derivative signifies that the function has been differentiated partially with respect to one of the independent variables, twice while the other independent variable has been held constant.

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = fxx$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = fyy$$

In effect, *fxx* measures the rate of change of the first order partial derivative *fx* with respect to x while y is held constant. And *fyy* measures the rate of change of the first-order partial derivative *fy* with respect to y while x is held constant.

The cross (or mixed) partial derivatives *fxy* and *fyx* indicate that first the primitive function has been partially differentiated with respect to one independent variable and then that partial derivative in turn been partially differentiated with respect to the other independent variable.

$$fxy = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$fyx = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

By young's theorem, if both cross partial derivatives are continuous, they will be identical.

Example:

The first, second, and cross partial derivatives for $z = 2x^3 - 11x^2y + 3y^2$

First order partial derivatives

$$\frac{\partial z}{\partial x} = fx = 6x^2 - 22xy$$

$$\frac{\partial z}{\partial y} = fy = -11x^2 + 6y$$

Second – order direct partial derivatives

$$\frac{\partial^2 z}{\partial x^2} = fxx = 12x - 22y$$

$$\frac{\partial^2 z}{\partial y^2} = fyy = 6$$

Cross – partial derivatives

$$\frac{\partial^2 z}{\partial y \partial x} = f_{xy} = -22x$$

$$\frac{\partial^2 z}{\partial x \partial y} = f_{yx} = -22x$$

2.4 OPTIMIZATION OF MULTIVARIABLE FUNCTIONS

For a multivariable function such as z = *f*(x, y) to be at a relative minimum or maximum, three conditions must be met:

(1) The first – order partial derivatives must equal zero simultaneously. That is

$$\frac{\partial z}{\partial x} = fx = 0$$

$$\frac{\partial z}{\partial y} = fy = 0$$

This indicates that at the given point (a, b), called a critical point, the function is neither increasing nor decreasing with respect to the principal axes but is at a relative plateau.

(2) The second – order direct partial derivatives, when evaluated at the critical point (a,b) must both be negative for a relative maximum and positive for a relative minimum. That is,

For a relative maximum

$$\frac{\partial^2 z}{\partial x^2} = fxx < 0$$

$$\frac{\partial^2 z}{\partial y^2} = fyy < 0$$

When x = a and y = b, the values derived from the first condition

For a relative minimum

$$\frac{\partial^2 z}{\partial x^2} = fxx > 0$$

$$\frac{\partial^2 z}{\partial y^2} = fyy > 0$$

This ensures that from a relative plateau at (a, b) the function is concave and moving downward in relation to the principal axes in the case of a maximum and convex and moving upward in relation to the principal axes in the case of a minimum.

(3) The product of the second – order direct partial derivatives evaluated at the critical point must exceed the product of the cross partial derivatives also evaluated at the critical point. Since $f_{xy} = f_{yx}$ by Young's theorem, this condition can also be written as

$$f_{xx} \cdot f_{yy} - (f_{xy})^2 > 0$$

This added condition is needed to preclude an inflection point or saddle point.

If $f_{xx} \cdot f_{yy} - (f_{xy})^2 < 0$, the function is at an inflection point when f_{xx} and f_{yy} have the same signs and the function is at a saddle point when f_{xx} and f_{yy} have different signs.

If $f_{xx} \cdot f_{yy} - (f_{xy})^2 = 0$ the test is inconclusive.

Example:

Find the critical points and test whether the function is at a relative maximum or minimum, given

$$Z = 2y^3 - x^3 + 147x - 54y + 12$$

(a) Take the first – order partial derivatives, set them equal to zero, and solve for x and y

$$\frac{\partial z}{\partial x} = -3x^2 + 147 = 0$$

$$3x^2 = 147$$

$$x^2 = 49$$

$$x = \pm 7$$

$$\frac{\partial z}{\partial x} = 6y^2 - 54 = 0$$

$$6y^2 = 54$$

$$y^2 = 9$$

$$y = \pm 3$$

With $x = \pm 7$, $y = \pm 3$, there are four distinct sets of critical points:

(7, 3), (7, -3), (-7, 3), (-7, -3)

(b) Take the second – order direct partial derivatives, evaluate them at each of the critical points, and check the signs

$f_{xx} = -6x$

(i)	Point	(7, 3)	$f_{xx} =$	-42	< 0
(ii)	Point	(7, -3)	$f_{xx} =$	-42	< 0
(iii)	Point	(-7, 3)	$f_{xx} =$	42	> 0
(iv)	Point	(-7, -3)	$f_{xx} =$	42	> 0

$f_{yy} = 12y$

(i)	Point	(7, 3)	$f_{yy} =$	36	> 0
(ii)	Point	(7, -3)	$f_{yy} =$	-36	< 0
(iii)	Point	(-7, 3)	$f_{yy} =$	36	> 0
(iv)	Point	(-7, -3)	$f_{yy} =$	-36	<0

Since there are different signs for each of the second direct partials in (i) and (iv) , the function cannot be at a relative maximum or minimum at (7, 3) or (-7, -3). With both signs of the second direct partial derivatives negative in (ii) and positive in (iii), the function may be at a relative maximum at (7, -3) and at a relative minimum at (-7, 3), but the third condition must be tested first to ensure against the possibility of an inflection point.

(c) Take the cross partial derivatives and check to make sure that $f_{xx} \cdot f_{yy} - (f_{xy})^2 > 0$

$$f_{xy} = 0$$

$$\therefore f_{xx} \cdot f_{yy} - (f_{xy})^2 = -42 \times -36 - 0 = 1512 > 0$$

at (7, -3) and

$$f_{xx} \cdot f_{yy} - (f_{xy})^2 = 42 \times 36 - 0 = 1512 > 0$$

at (-7, 3)

The function is maximized at (7, -3) and minimized at (-7, 3).

2.5 CONSTRAINED OPTIMIZATION WITH LAGRANGE MULTIPLIERS

Differential calculus is also used to maximize or minimize a function subject to constant. Given a function $f(x, y)$ subject to a constraint g(x, y) = k, where k is a constant, a new function L can be formed by setting the constraint equal to zero (that is, k – g(x, y)

= 0), multiplying it by λ (that is, λ[k – g(x, y)]), and adding the product to the original function:

$$L = f(x, y) + \lambda\,[k - g(x, y)]$$

Here L is the Largrangian function, λ is the Lagrange multiplier, *f(x, y)* is the original or objective function, and g(x, y) is the constraint. Since the constraint is always set equal to zero, the product λ [k – g(x, y)] also equals zero, and the addition of term does not change the value of the objective function. Critical values x_0, y_0 and λ_0 at which the function is optimized, are found by taking the partial derivatives of L with respect to all three independent variables (x, y and λ), setting them equal to zero, and solving simultaneously.

$$\frac{\partial L}{\partial x} = L_x = 0$$

$$\frac{\partial L}{\partial y} = L_y = 0$$

$$\frac{\partial L}{\partial \lambda} = L_\lambda = 0$$

The second – order conditions can now be expressed in terms of a bordered Hessian $|\bar{H}|$ as

$$|\bar{H}| = \begin{vmatrix} fxx & fxy & gx \\ fyx & fyy & gy \\ gx & gy & 0 \end{vmatrix}$$

Where $fxx = \frac{\partial^2 f}{\partial x^2}$

$fyy = \frac{\partial^2 f}{\partial y^2}$

$fxy = \frac{\partial^2 f}{\partial x \partial y}$

If all the principal minors are negative, the bordered Hessian is positive definite, and a positive definite Hessian always satisfies the sufficient condition for a relative minimum.

If all principal minors alternate consistently in sign from positive to negative, the bordered Hessian is negative definite, and a negative definite Hessian always meets the sufficient condition for a relative maximum.

Example:

Optimize the function

$Z = 4x^2 + 3xy + 6y^2$

Subject to the constraint

$X + y = 56$

(1) Set the constraint equal to zero by subtracting the variables from the constant.

$56 - x - y = 0$

(2) Multiply this by λ

$\lambda (56 - x - y) = 0$

(3) Add the product to the objective function is order to form the Lagrangian function L

$L = 4x^2 + 3xy + 6y^2 + \lambda (56 - x - y)$

(4) Take the first order partial derivatives with respect to x, y and λ and set them equal to zero.

$$\frac{\partial L}{\partial x} = L_x = 8x + 3y - \lambda = 0$$

$$\frac{\partial L}{\partial y} = L_y = 3x + 12y - \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = L_\lambda = 56 - x - y = 0$$

(5) Solving these equations simultaneously, we get

$X = 36,\ y = 20$ and $\lambda = 348$

(6) Substituting the critical values (that is, x = 36, y = 20) in the objective function we get

$$\begin{aligned} Z &= 4(36)^2 + 3(36)(20) + 6(20)^2 \\ &= 4(1296) + 3(720) + 6(400) \\ &= 9744 \end{aligned}$$

(7) To see whether this is relative minimum or maximum, we consider second – order condition in terms of bordered Hessian.

$$|\bar{H}| = \begin{vmatrix} 8 & 3 & 1 \\ 3 & 12 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

Starting with the second principal minor $|\bar{H}|$

$|\overline{H_2}| = |\bar{H}| = 8(-1) - 3(-1) + 1(-9) = -14$

With $|\overline{H_2}| < 0$, $|\bar{H}|$ is positive definite, which means that Z is at a minimum.

2.6 SIGNIFICANCE OF THE LAGRANGE MULTIPLIER

The Lagrange multiplier $\boldsymbol{\lambda}$ approximates the marginal impact on the objective function caused by a small change in the constant of the constraint. With $\boldsymbol{\lambda} = 348$ in the example presented above, for instance, a one-unit increase (or decrease) in the constant of the constraint would cause z to increase (or decrease) by approximately 348 units. Lagrange multipliers are often referred to as shadow prices. In utility maximization subject to a budget constraint, for example, $\boldsymbol{\lambda}$ will estimate the marginal utility of an extra unit of income.

2.7 DIFFERENTIALS

The derivative $\frac{dy}{dx}$ may be treated as a ratio of differentials in which dy is the differential of y and dx the differential of x. Given a function of a single independent variable y = f(x), the differential of y, dy, measures the change in y resulting from a small change in x, written dx.

Given $y = 2x^2+5x+4$, the differential of y is found by first taking the derivative of y with respect to x, which measures the rate at which y changes for a small change in x.

$\frac{dy}{dx} = 4x + 5$ a derivative or rate of change and then multiplying that rate at which y changes for a small change in x by a specific change in x (that is, dx) to find the resulting change in y (that is, dy).

$dy = (4x + 5)dx$

Example:

If $y = 8x^3 + 10x^2 - 7$

$dy = (24x^2 + 20x)\ dx$

2.8 TOTAL AND PARTIAL DIFFERENTIALS

For a function of two or more independent variables, the total differential measures the change in the dependent variable brought about by a small change in each of the independent variables. If z = f(x, y), the total differential dz is expressed mathematically as

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

Where $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are the partial derivatives of z with respect to x and y respectively, and dx and dy are small changes in x and y. The total differential can thus be found by taking the partial derivatives of the function with respect to each independent variable and substituting these values in the formula above.

Example

Given $z = 2x^3 + 6xy + 5y^3$

$\frac{\partial z}{\partial x} = 6x^2 + 6y$; $\frac{\partial z}{\partial y} = 6x + 15y^2$

$dz = (6x^2 + 6y)dx + (6x + 15y^2)dy$

2.9 TOTAL DERIVATIVES

Given a case where z = f(x, y) and y = g(x), that is, when x and y are not independent, a change in x will directly affect z through the function f and indirectly through the function g. To measure the effect of a change in x on z when x and y are not independent, the total derivative must be found. The total derivative measures the direct effect of x on z, $\frac{\partial z}{\partial x}$, plus the indirect effect of x on z through $\frac{\partial z}{\partial y}\frac{dy}{dx}$. In brief, the total derivative is

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

Example:

Given $z = x^4 + 8y$ and $y = x^2 + 2x + 10$, the total derivative with respect to x is

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

$\frac{\partial z}{\partial x} = 4x^3$; $\frac{\partial z}{\partial y} = 8$; $\frac{dy}{dx} = 2x + 2$

Substituting, we get

$$\frac{dz}{dx} = 4x^3 + 8(2x + 2)$$

$$= 4x^3 + 16x + 16$$

2.10 IMPLICIT AND INVERSE FUNCTION RULES

Functions of the form y = f(x) express y explicitly in terms of x and are called explicit functions. Functions of the form f(x, y) = 0 do not express y explicitly in terms of x and are called implicit functions. For example, $x^2 + y^2 = 16$ is an implicit function between x and y.

If an implicit function f(x, y) = 0 exists and $\frac{\partial f}{\partial y} \neq 0$ at the point around which the implicit function is defined, the total differential is simply

$$\frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy = 0$$

Since a derivative is a ratio of differentials, we can then rearrange the terms to get the implicit function rule.

$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$$

Notice that the derivative $\frac{dy}{dx}$ is the negative of the reciprocal of the corresponding partial derivatives. That is,

$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y} = -\frac{1}{\frac{\partial f/\partial y}{\partial f/\partial x}}$$

Given a function y = f(x), an inverse function $x = f^{-1}(y)$ exists if each value of y yields one and only one value of x. Assuming the inverse function exists, the inverse function rule states that the derivative of the inverse function is the reciprocal of the derivative of the original function. Thus, if y = f(x) is the original function, the derivative of the original function is $\frac{dy}{dx}$, the derivative of the inverse function $[x = f^{-1}(y)]$ is $\frac{dx}{dy}$, and

$$\frac{dx}{dy} = \frac{1}{dy/dx} \text{ provided } \frac{dy}{dx} \neq 0$$

Example 1:

Given $6x^2 - 5y = 0$ the derivative $\frac{dy}{dx}$ is found as follows

$$\frac{dy}{dx} = \frac{-\partial f/_{\partial x}}{\partial f/_{\partial y}}$$

$$= \frac{-12x}{-5}$$

$$= \frac{12}{5}x$$

Example 2:

Given y = 40 – 4p, find the derivative for the inverse of the function.

$$\frac{dp}{dy} = \frac{1}{dy/_{dp}}$$

$$= \frac{1}{-4} \quad = -\frac{1}{4}$$

2.11 APPLICATION OF CALCULUS OF MULTIVARIABLE FUNCTIONS IN ECONOMICS

(1) Marginal Productivity

The marginal product of capital (MP_k) is defined as the change in output brought about by a small change in capital when all other factors of production are held constant. Given a production function such as

$$Q = 36\,KL - 2K^2 - 3L^2$$

the MP_K is measured by taking the partial derivative $\frac{\partial Q}{\partial K}$. That is

$$MP_K = \frac{\partial Q}{\partial K} = 36L - 4K$$

Similarly, the marginal product of labour (MP_L) is measured by taking the partial derivative $\frac{\partial Q}{\partial L}$.

That is,

$$MP_L = \frac{\partial Q}{\partial L} = 36K - 6L$$

Example:

Find the marginal productivity of inputs x and y, given the production function $Q = x^2+2xy+3y^2$

$$MP_x = \frac{\partial Q}{\partial x} = 2x + 2y$$

$$MP_y = \frac{\partial Q}{\partial y} = 2x + 6y$$

(2) Income and Cross Elasticities of Demand

Income elasticity of demand ε_y measures the percentage change in the demand for a good resulting from a small percentage change in income, when all other variables are held constant. Cross elasticity of demand ε_c measures the relative responsiveness of the demand for one product to changes in the price of another, when all other variables are held constant. Given the demand function

$$Q_1 = a - bP_1 + cP_2 + mY$$

Where y = income and P_2 = the price of a substitute good, the income elasticity of demand is

$$\varepsilon_y = \frac{\partial Q_1}{\partial Y}\left(\frac{Y}{Q_1}\right)$$

$$= m\,\frac{Y}{Q_1}$$

$$= \frac{mY}{a - bP1 + cP2 + mY}$$

And the cross elasticity of demand is

$$\varepsilon_c = \frac{\partial Q_1}{\partial P_2}\left(\frac{P_2}{Q_1}\right)$$

$$= c\,\frac{P_2}{Q_1}$$

$$= \frac{cP_2}{a - bP1 + cP2 + mY}$$

Example:

Given the demand for beef

$Q_b = 4850 - 5P_b + 1.5\ P_p + 0.1Y$ with Y = 10,000, $P_b = 200$, and the price of pork $P_p = 100$.

$$\varepsilon_y = \frac{\partial Q_b}{\partial Y}\left(\frac{Y}{Q_b}\right)$$

$$= 0.1 \text{ x } \frac{Y}{Q_b}$$

$$= \frac{0.1 \text{ x } 10000}{4850 - 5 \text{ x } 200 + 1.5 \text{ x } 100 + 0.1 \text{ x } 10000}$$

$$= \frac{1000}{4850 - 1000 + 150 + 1000}$$

$$= \frac{1000}{5000}$$

$$= \underline{\underline{0.2}}$$

$$\varepsilon_c = \frac{\partial Q_b}{\partial P_p}\left(\frac{P_p}{Q_b}\right)$$

$$= 1.5 \text{ x } \frac{100}{4850 - 5 \text{ x } 200 + 1.5 \text{ x } 100 + 0.1 \text{ x } 10000}$$

$$= \frac{1.5 \text{ x } 100}{4850 - 5 \text{ x } 200 + 1.5 \text{ x } 100 + 0.1 \text{ x } 10000}$$

$$= \frac{150}{5000}$$

$$= \underline{\underline{0.03}}$$

(3) Differentials and Incremental Charges

Frequently in economics we want to measure the effect of a change in an independent variable (labour, capital, items sold) on the dependent variable (costs, revenue, profit). If the change is a relatively small one, the differential will measure the effect. Thus, if z = f(x, y), the effect of a small change in x on z is given by the partial differential

$$dz = \frac{\partial z}{\partial x}\, dx$$

The effect of larger changes can be approximated by multiplying the partial derivative by the proposed change. Thus

$$\Delta z \approx \frac{\partial z}{\partial x} \Delta x$$

If the original function z = f(x. y) is linear.

$$\frac{dz}{dx} = \frac{\Delta z}{\Delta x}$$

and the effect of the change will be measured exactly

$$\Delta z = \frac{\partial z}{\partial x} \Delta x$$

Example:

A firm's costs are related to its output of two goods x and y. The functional relationship is

$$TC = x^2 - 0.5xy + y^2$$

The additional cost of a slight increment in output x will be given by the differential

$$dTC = MC_x\, dx$$

$$= (2x - 0.5y)dx$$

The costs of larger increments can be approximated by multiplying the partial derivatives with respect to x by the charge in x. Mathematically,

$$\Delta TC \approx \frac{\partial TC}{\partial x} \Delta x$$

$$\approx (2x - 0.5y)\,\Delta x$$

(4) Optimization of Multivariable Functions in Economics

In economics, we have to deal with multi-product firms or firms selling different varieties of the same product. Maximizing profits or minimizing costs under these conditions involve function of more than one variable. Thus, the basic rules for optimization of multivariate functions are required.

Example:

A firm producing two goods x and y has the profit function.

$$\Pi = 64x - 2x^2 + 4xy - 4y^2 + 32y - 14$$

To find the profit-maximizing level of output for each of the two goods.

(a) Take the first order partial derivatives, set them equal to zero, and solve for x and y simultaneously

$\Pi_x = 64 - 4x + 4y = 0$

$\Pi_y = 4x - 8y + 32 = 0$

When solved simultaneously, we get

X = 40 and y = 24

(b) Take the second-order direct partial derivatives and make sure both are negative, as is required for a relative maximum.

$\Pi_{xx} = -4 < 0$

$\Pi_{yy} = -8 < 0$

(c) Take the cross partial derivatives to make sure that $\Pi_{xx}\,\Pi_{yy} - (\Pi_{xy})^2 > 0$

$\Pi_{xx} = -4,\; \Pi_{yy} = -8,\; \Pi_{xy} = 4$

$\therefore\; \Pi_{xx}\,\Pi_{yy} - (\Pi_{xy})^2 = (-4) \times (-8) - 4^2 = 6 > 0$

Therefore, profit is indeed maximized at x = 40 and y = 24

At that point, π = 1650

(5) Constrained Optimization of Multivariable Functions in Economics

Solution to economic problems frequently have to be found under constraints (example, maximizing utility subject to a budget constraint or maximizing profit subject to resource constraint or minimizing costs subject to some such minimal requirement of output as a production quota). Use of Lagrangian function greatly facilitates this task.

Example:

Find the critical values for minimizing the costs of a firm producing two goods x and y when the total cost function is $C = 8x^2 - xy + 12y^2$ and the firm is bound by contract to produce a minimum combination of goods totaling 42, that is subject to the constraint $x + y = 42$.

(a) From the Lagrangian function

$L = 8x^2 - xy + 12y^2 + \lambda\,(42 - x - y)$

(b) Take the first order partial derivatives and set them equal to zero.

$L_x = 16x - y - \lambda = 0$

$L_y = -x + 24y - \lambda = 0$

$L_\lambda = 42 - x - y = 0$

(c) Solving simultaneously, we get

X = 25, y = 17 and λ = 383

(d) Consider the second-order condition in terms of bordered Hessian $|\overline{H}|$to ensure that the critical values x = 25, y = 17 and λ = 383 minimizes the function

$$|\overline{H}| = \begin{vmatrix} 16 & -1 & 1 \\ -1 & 24 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$|\overline{H_2}| = 1(-1 - 24) - 1(16 + 1)$

$= -25 - 17 = -42 < 0$

With $|\overline{H_2}| < 0$, $|\overline{H}|$ is positive definite and c is minimized.

The minimum value of c is attained by substituting the critical values in the objective function. That is,

$C = 8 \times (25)^2 - (25)(17) + 12 \times (17)^2$

$= 8 \times (625) - 425 + 12 \times 289$

$= 5000 - 425 + 3468$

$= \underline{\underline{4621}}$

(6) Marginal Utility

The marginal utility of a commodity X (MU_x) is defined as the change in total utility brought about when all other commodities (quantity consumed) are held constant. Given a utility function

$U = 3x^2 + 5xy + 6y^2$

the MU_x is measured by taking the partial derivative $\frac{\partial U}{\partial x}$. That is

$MU_x = \frac{\partial U}{\partial x} = 6x + 5y$

Similarly, the marginal utility of commodity y (MU_y) is measured by taking the partial derivative $\frac{\partial U}{\partial y}$. That is

$$MU_y = \frac{\partial U}{\partial y} = 5x + 12\,y$$

(7) Marginal Rate of Substitution or Rate of Commodity Substitution

Marginal Rate of Substitution of x for y (MRS_{xy} or RCS_{xy}) is the rate at which x is substituted for y (number of y foregone to attain one more unit of x), when utility held constant. Given a utility function

$$U = 3x^2y + 5xy^2 + 7$$

The MRS_{xy} is measured by taking the ratio of partial derivatives of the utility function with respect to x and y. That is

$$MRS_{xy} = -\frac{\partial U/\partial x}{\partial U/\partial y} = -\frac{6xy+5y^2}{3x^2+10xy}$$

The negative sign prefixed indicates that an increase in x should result in a decrease in y.

(8) Marginal Rate of Technical Substitution ($MRTS_{LK}$)

Marginal Rate of Technical Substitution of Labour for capital ($MRTS_{LK}$) is defined as the rate of substitution of labour for capital, keeping output constant. Given a production function

$$Q = AL^{\alpha} K^{\beta}$$

The $MRTS_{LK}$ is measured by taking the ratio of partial derivatives of the production function with respect to labour and capital. That is,

$$MRTS_{LK} = -\frac{\partial Q/\partial L}{\partial Q/\partial K} = -\frac{\alpha AL^{\alpha-1} K^{\beta}}{\beta AL^{\alpha} K^{\beta-1}}$$

$$= -\frac{\alpha}{\beta}\frac{K}{L}$$

(negative sign shows the inverse relation)

CHAPTER III

SPECIAL DETERMINANTS AND MATRICES IN ECONOMICS

3.1. THE JACOBIAN

The Jacobian determinant permits testing for functional dependence. A Jacobian determinant, denoted as |J|, is composed of all the first order partial derivatives of a system of equations arranged in ordered sequence.

Given, $Y_1 = f_1(x_1, x_2)$

$Y_2 = f_2(x_1, x_2)$

$$\text{Then, } |J| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix}$$

Note that the elements of each row we the partial derivatives of one function with respect to each of the independent variables x_1, x_2, …… If |J|=0, the equations we functionally dependent and if $|J| \neq 0$, the equations we functionally independent. Sometimes Jacobian is also expressed as $|J| = \left|\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}\right|$

Example 1:

Use Jacobian to test for functional dependence

$Y_1 = 2x_1 + 3x_2$

$Y_2 = 4x_1^2 + 12x_1x_2 + 9x_2^2$

$$|J| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix}$$

$\frac{\partial y_1}{\partial x_1} = 2$

$\frac{\partial y_1}{\partial x_2} = 3$

$\frac{\partial y_2}{\partial x_1}$ = 8x$_1$ + 12x$_2$

$\frac{\partial y_2}{\partial x_2}$ = 12x$_1$ + 18x$_2$

$$\therefore |J| = \begin{vmatrix} 2 & 3 \\ 8x_1 + 12x_2 & 12x_1 + 18x_2 \end{vmatrix}$$

$|J| = 2(12x_1 + 18x_2) – 3 (8x_1 + 12x_2)$

$|J| = 0$

Since $|J| = 0$, there is functional dependence between the equations.

Example 2:

Use Jacobian to test for functional dependence for the following system of equations

$Y_1 = 6x_1 + 4x_2$

$Y_2 = 7x_1 + 9x_2$

$$|J| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix}$$

$\frac{\partial y_1}{\partial x_1} = 6$

$\frac{\partial y_1}{\partial x_2} = 4$

$\frac{\partial y_2}{\partial x_1} = 7$

$\frac{\partial y_2}{\partial x_2} = 9$

$$\therefore |J| = \begin{vmatrix} 6 & 4 \\ 7 & 9 \end{vmatrix}$$

$|J|= 26$

========

Example 3:

Test for functional dependence by means of Jacobian

$Y_1 = x_1^2 - 3x_2 + 5$

$Y_2 = x_1^4 - 6x_1^2x_2 + 9x_2^2$

$$|J| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix}$$

$\frac{\partial y_1}{\partial x_1} = 2x_1$

$\frac{\partial y_1}{\partial x_2} = -3$

$\frac{\partial y_2}{\partial x_1} = 4x_1^3 - 12x_1x_2$

$\frac{\partial y_2}{\partial x_2} = -6x_1^2 + 18x_2$

$$\therefore |J| = \begin{vmatrix} 2x_1 & -3 \\ 4x_1^3 - 12x_1x_2 & -6x_1^2 + 18x_2 \end{vmatrix}$$

$|J| = 2x_1 (-6x_1^2 + 18x_2) - -3 (4x_1^3 - 12x_1x_2)$

$|J| = 0$

Since $|J| = 0$, the equations are functionally dependent .

Example 4:

$Y_1 = 1.5x_1^2 + 12x_1x_2 + 24x_2^2$

$Y_2 = 2x_1 + 8x_2$

Find out Jacobian?

$$|J| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix}$$

$\frac{\partial y_1}{\partial x_1} = 3x_1 + 12x_2$

$\frac{\partial y_1}{\partial x_2} = 12x_1 + 48x_2$

$\frac{\partial y_2}{\partial x_1} = 2$

$\frac{\partial y_2}{\partial x_2} = 8$

$$\therefore |J| = \begin{vmatrix} 3x_1 + 12x_2 & 12x_1 + 48x_2 \\ 2 & 8 \end{vmatrix}$$

$|J| = (3x_1 + 12x_2)8 - 2(12x_1 + 48x_2)$

$|J| = 0$

3.2 THE HESSIAN

A Hessian determinant or simply a Hessian denoted as $|H|$ is a determinant composed of all the second order partial derivatives. In the two variable case,

Z = f(x, y) then

$$|H| = \begin{vmatrix} Zxx & Zxy \\ Zyx & Zyy \end{vmatrix}$$

Thus, $|H|$ is composed of second under partial derivatives, with second order direct partials on the principal diagonal and second order cross partials off the principal diagonals in the Hessian determinant Zyx = Zxy following Young's Theorem.

If the first element on the principal diagonal or first principal minor, $|H_1|$ = Zxx is positive and the second principal minor $|H_2| = \begin{vmatrix} Zxx & Zxy \\ Zyx & Zyy \end{vmatrix}$ is also positive, then the Hessian is called positive definite. That is, if $|H_1|>0$ and $|H_2| > 0$, then $|H|$ is positive definite.

Positive definite Hessian fulfils the second order condition for minimum. Similarly, if the first principal minor $|H_1|$ =Zxx is negative and second principal minor $|H_2|$ = $\begin{vmatrix} Zxx & Zxy \\ Zyx & Zyy \end{vmatrix}$ is positive, then the Hessian is negative definite. That is, if $|H_1| < 0$ and $|H_2| > 0$, the Hessian is negative definite. A negative definite Hessian fulfills second order condition for maximum.

Example 1:

Given Z = $3x^2 - xy + 2y^2 - 4x - 7y + 12$ use Hessian to check second order condition.

$$|H| = \begin{vmatrix} Zxx & Zxy \\ Zyx & Zyy \end{vmatrix}$$

Zx = 6x – y – 4

Zxx = 6

Zxy = -1

Zy = -x + 4y – 7

Zyy = 4

Zyx = -1

$$|H| = \begin{vmatrix} 6 & -1 \\ -1 & 4 \end{vmatrix}$$

The first principal minor $|H_1|$ = Zxx = 6 > 0

Second principal minor $|H_2| = \begin{vmatrix} 6 & -1 \\ -1 & 4 \end{vmatrix} = 23 > 0$

Since $|H_1| > 0$ and $|H_2| > 0$, the Hessian is positive definite and Z is minimized.

Example 2:

If Q = $5u^2 + 3uv + 2v^2$ find out if Q is positive or negative definite

$$|H| = \begin{vmatrix} Quu & Quv \\ Qvu & Qvv \end{vmatrix}$$

Qu = 10u + 3v

Quu = 10

Quv = 3

Qv = 3u + 4v

Quv = 3

Qvv = 4

$\therefore |H| = \begin{vmatrix} 10 & 3 \\ 3 & 4 \end{vmatrix}$

$|H1| = 10 > 0$

$H2 = \begin{vmatrix} 10 & 3 \\ 3 & 4 \end{vmatrix} = 31 > 0$

∴ Q is positive definite.

Example 3:

Examine the following function for maximum or minimum

$U = 2x^2 + y^2 - 4x + 8y$

$|H| = \begin{vmatrix} Qxx & Qxy \\ Qyx & Qyy \end{vmatrix}$

Ux = 4x - 4

Uxx = 4

Uxy = 0

Uv = 2y + 8

Uyx = 0

Uyy = 2

$\therefore |H| = \begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix}$

$|H1| = 4 > 0$

$H2 = \begin{vmatrix} 4 & 0 \\ 0 & 24 \end{vmatrix} = 8 > 0$

∴ U is minimum.

Example 4:

Given the profit function π = $15Q_1 + 18Q_2 - 2Q_1^2 - 2Q_1Q_2 - 3Q_2^2$, use the Hessian for second order condition

$$|H| = \begin{vmatrix} \pi 11 & \pi 12 \\ \pi 21 & \pi 22 \end{vmatrix}$$

$\Pi_1 = 15 - 4Q_1 - 2Q_2$

$\Pi_{11} = -4$

$\Pi_{12} = -2$

$\Pi_2 = 18 - 2Q_1 - 6Q_2$

$\Pi_{21} = -2$

$\Pi_{22} = -6$

$$\therefore |H| = \begin{vmatrix} -4 & -2 \\ -2 & -6 \end{vmatrix}$$

$$|H1| = -4 < 0$$

$$|H2| = \begin{vmatrix} -4 & -2 \\ -2 & -6 \end{vmatrix} = 20 > 0$$

∴ U is minimum.

Since $|H1| < 0$ and $|H2| > 0$, Hessian is negative definite, and the profit function (Π) is maximized.

3.3 THE DISCRIMINANT

The discriminant is used for positive or negative definiteness of any quadratic form. The determinant of a quadratic form is called the discriminate and is denoted as $|D|$.

Given the function of two variable, Z= f(x, y), the quadratic form is

$$Z = ax^2 + bxy + cy^2$$

The discriminant $|D|$ is formed by placing the coefficients of squared terms on the principal diagonal and dividing the coefficients of the nonsquared terms equally between the off diagonal positions. That is,

$$|D| = \begin{vmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{vmatrix}$$

If the first principal minor $|D_1|$ = a is positive and the second principal minor $|D_2|$ $= \begin{vmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{vmatrix}$ is also positive then the discriminant $|D|$ is called positive definite. That is, if $|D_1| > 0$ and $|D_2| > 0$, $|D|$ is positive definite and z is positive for all non-zero values of x and y. If $|D_1| < 0$ and $|D_2| > 0$, the discriminant $|D|$ is negative definite and z is negative for all non-zero values of x and y.

Example 1:

Given Z = $2x^2 + 5xy + 8y^2$, use discriminate to test for definiteness.

$$|D| = \begin{vmatrix} 2 & \frac{5}{2} \\ \frac{5}{2} & 8 \end{vmatrix}$$

$$|D| = \begin{vmatrix} 2 & 2.5 \\ 2.5 & 8 \end{vmatrix}$$

$$|D_1| = 2 > 0$$

$$|D_2| = \begin{vmatrix} 2 & 2.5 \\ 2.5 & 8 \end{vmatrix} = 9.75 > 0$$

Since $|D_1| > 0$ and $|D_2| > 0$, z is positive definite.

Example 2:

Find discriminant if Q = $22Y_1^2 + 46Y_1Y_2 + 42Y_2^2$, use discriminate to test for definiteness.

$$|D| = \begin{vmatrix} 22 & \frac{46}{2} \\ \frac{46}{2} & 42 \end{vmatrix}$$

$$|D| = \begin{vmatrix} 22 & 23 \\ 23 & 42 \end{vmatrix}$$

$$|D_1| = 22 > 0$$

$$|D_2| = \begin{vmatrix} 22 & 23 \\ 23 & 42 \end{vmatrix} = 395 > 0$$

Since $|D_1| > 0$ and $|D_2| > 0$, Q is positive definite.

Example 3:

Is Q= $5U^2 + 3UV + 2V^2$ either positive or negative definite.

$$|D| = \begin{vmatrix} 5 & 1.5 \\ 1.5 & 2 \end{vmatrix}$$

$$|D_1| = 5 > 0$$

$$|D_2| = \begin{vmatrix} 5 & 1.5 \\ 1.5 & 2 \end{vmatrix} = 7.75 > 0$$

Since $|D_1| > 0$ and $|D_2| > 0$, Q is positive definite.

Example 4:

Use discriminant to determine the sign of definiteness, given

$$Y = -2x_1^2 + 4x_1x_2 - 5x_2^2 + 2x_2x_3 - 3x_3^2 + 2x_1x_3$$

$$|D| = \begin{vmatrix} -2 & \frac{4}{2} & \frac{2}{2} \\ \frac{4}{2} & -5 & \frac{2}{2} \\ \frac{2}{2} & \frac{2}{2} & -3 \end{vmatrix}$$

$$|D| = \begin{vmatrix} -2 & 2 & 1 \\ 2 & -5 & 1 \\ 1 & 1 & -3 \end{vmatrix}$$

$$|D_1| = -2 < 0$$

$$|D_2| = \begin{vmatrix} -2 & 2 \\ 2 & -5 \end{vmatrix} = 6 > 0$$

$$|D_3| = \begin{vmatrix} -2 & 2 & 1 \\ 2 & -5 & 1 \\ 1 & 1 & -3 \end{vmatrix}$$

$$|D_3| = -2\begin{vmatrix} -5 & 1 \\ 1 & -3 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} + 1\begin{vmatrix} 2 & -5 \\ 1 & 1 \end{vmatrix}$$

$$|D_3| = -7 < 0$$

Since $|D_1| < 0$, $|D_2| > 0$ and $|D_3| < 0$, Y is negative definite.

3.4 THE HIGHER ORDER HESSIANS

Given Y = f(x_1, x_2, x_3) the third order Hessian denoted as $|H|$ consist of the elements of various second order partial derivatives of Y. That is,

$$|H| = \begin{vmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{vmatrix}$$

Conditions for a relative minimum or maximum depend on the signs of first, second and third principal minors respectively. The principal minors are

$$|H_1| = Y_{11}$$

$$|H_2| = \begin{vmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{vmatrix}$$

$$|H_3| = |H|$$

If $|H_1| > 0$, $|H_2| > 0$ and $|H_3| > 0$ then Hessian is positive definite and fulfills the second order conditions for a minimum. If $|H_1| < 0$, $|H_2| > 0$ and $|H_3| < 0$, then Hessian is negative definite and fulfills the second order conditions for a maximum.

That is, if all principal minors of Hessian is positive, $|H|$ is positive definite and second order conditions for a relative minimum we met. If all principal minors of alternative sign between negative and positive |H| is negative definite and second order conditions for a relative maximum are met.

Example 1:

Given the function Y = $-5{x_1}^2 + 10\,x_1 + x_1x_3 - 2x_2^2 + 4x_2 + 2x_2x_3 - 4x_3^2$. Use Hessian to check second order condition

$$|H| = \begin{vmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{vmatrix}$$

$Y_1 = -10x_1 + 10 + x_3$

$Y_{11} = -10 \; Y_{12} = 0$

$Y_{13} = 1$

$Y_2 = -4x_2 + 4 + 2x_3$

$Y_{21} = 0 \quad Y_{22} = -4 \qquad Y_{23} = 2$

$Y_3 = x_1 + 2x_2 - 8x_3$

$Y_{31} = 1 \qquad Y_{32} = 2 \qquad Y_{33} = -8$

$$\therefore H = \begin{vmatrix} -10 & 1 & 1 \\ 0 & -4 & 2 \\ 1 & 2 & -8 \end{vmatrix}$$

$|H_1| = -10 < 0$

$|H_2| = \begin{vmatrix} -10 & 0 \\ 0 & -4 \end{vmatrix} = 40 > 0$

$|H_3| = -10 \begin{vmatrix} -4 & 2 \\ 2 & -8 \end{vmatrix} - 0 \begin{vmatrix} 0 & 2 \\ 1 & -8 \end{vmatrix} + 1 \begin{vmatrix} 0 & -4 \\ 1 & 2 \end{vmatrix} = -276$

$\therefore |H_1| < 0$, $|H_2| > 0$, $|H_3| < 0$, $|H|$ is negative definite and function is maximised.

Example 2:

Find extreme value of $Z = 2x_1^2 + x_1x_2 + 4x_2^2 + x_1x_3 + x_3^2 + 2$

$$|H| = \begin{vmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{vmatrix}$$

$Z_1 = 4x_1 + x_2 + x_3$

$Z_{11} = 4 \qquad Z_{12} = 1$

$Z_{13} = 1$

$Z_2 = x_1 + 8x_2$

$Z_{21} = 1 \quad Z_{22} = 8 \qquad Z_{23} = 2$

$Z_3 = x_1 + 2x_3$

$Z_{31} = 1$ $Z_{32} = 0$ $Z_{33} = 2$

$$\therefore |H| = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

$|H_1| = 4 > 0$

$|H_2| = \begin{vmatrix} 4 & 1 \\ 1 & 8 \end{vmatrix} = 31 > 0$

$|H_3| = 4 \begin{vmatrix} 8 & 0 \\ 0 & 2 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 1 & 8 \\ 1 & 0 \end{vmatrix}$

$= 54 > 0$

$\therefore |H_1| > 0$, $|H_2| > 0$ and $|H_3| > 0$, $|H|$ is positive definite and Z is minimum

Example 3:

Use Hessian to check second order condition

$Y = 3x_1^2 - 5x_1 - x_1x_2 + 6x_2^2 - 4x_2 + 2x_2x_3 + 4x_3^2 + 2x_3 - 3x_1x_3$

$$|H| = \begin{vmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{vmatrix}$$

$Y_1 = 6x_1 - 5 - x_2 - 3x_3$

$Y_{11} = 6$ $Y_{12} = -1$

$Y_{13} = -3$

$Y_2 = -x_1 + 12x_2 - 4 + 2x_3$

$Y_{21} = -1$ $Y_{22} = 12$ $Y_{23} = 2$

$Y_3 = 2x_2 + 8x_3 + 2 - 3x_1$

$Y_{31} = -3$ $Y_{32} = 2$ $Y_{33} = 8$

$\therefore |H| = \begin{vmatrix} 6 & -1 & -3 \\ -1 & 12 & 2 \\ -3 & 2 & 8 \end{vmatrix}$

$|H_1| = 6 > 0$

$|H_2| = \begin{vmatrix} 6 & -1 \\ -1 & 12 \end{vmatrix} = 71 > 0$

$|H_3| = 6 \begin{vmatrix} 12 & 2 \\ 2 & 8 \end{vmatrix} - -1 \begin{vmatrix} -1 & 2 \\ -3 & 2 \end{vmatrix} + -3 \begin{vmatrix} -1 & 12 \\ -3 & 2 \end{vmatrix}$

$= 448 > 0$

$\therefore |H_1| > 0$, $|H_2| > 0$ and $|H_3| > 0$, $|H|$ is positive definite Y is minimized.

Example 4:

Examine the profit function

$\pi = 180Q_1 + 200Q_2 + 150Q_3 - 3Q_1Q_2 - 2Q_2Q_3 - 2Q_1Q_3 - 4Q_1^2 - 5Q_2^2 - 4Q_3^2$

for maximum

$|H| = \begin{vmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{vmatrix}$

$\pi_1 = 180 - 3Q_2 - 2Q_3 - 8Q_1$

$\pi_{11} = -8$ $\qquad \pi_{12} = -3$

$\pi_{13} = -2$

$\pi_2 = 200 - 3Q_1 - 2Q_3 - 10Q_2$

$\pi_{21} = -3 \quad \pi_{22} = -10 \qquad \pi_{23} = -2$

$\pi_3 = 150 - 2Q_2 - 2Q_1 - 8Q_3$

$\pi_{31} = -2 \qquad \pi_{32} = -2 \qquad \pi_{33} = -8$

$\therefore |H| = \begin{vmatrix} -8 & -3 & -2 \\ -3 & -10 & -2 \\ -2 & -2 & -8 \end{vmatrix}$

$|H_1| = 8 < 0$

$|H_2| = \begin{vmatrix} -8 & -3 \\ -3 & -10 \end{vmatrix} = 71 > 0$

$|H_3| = -8 \begin{vmatrix} -10 & -2 \\ -2 & -8 \end{vmatrix} - -3 \begin{vmatrix} -3 & -2 \\ -2 & -8 \end{vmatrix} + -2 \begin{vmatrix} -3 & -10 \\ -2 & -2 \end{vmatrix}$

$= -520$

$\therefore |H_1| < 0$, $|H_2| > 0$ and $|H_3| < 0$, $|H|$ is negative definite and profit is maximised.

3.5 BORDERED HESSIAN FOR CONSTRAINED OPTIMISATION

Optimisation means minimisation or maximization. Differential calculus can be used to minimize or maximize a function subject to a constraint. Given a function, $Z = f(x, y)$ subject to a constraint $g(x, y) = k$, where k is a constant, a new function F can be formed by

1) Setting the constraint equal to zero
2) Multiplying the constraint by language multiplier, λ
3) Adding the product to the original function.

That is, $F = g(x, y) + \lambda [k - g(x, y)]$

The first order conditions are found by taking partial derivatives of F with respect to all three independent variables and setting them equal to zero. That is, $F_x = 0$, $F_y = 0$, $F_\lambda = 0$. The second order conditions can be expressed in terms of bordered Hessain, denoted as

$|\bar{H}|$ can be written as

$|\bar{H}| = \begin{vmatrix} F_{xx} & F_{xy} & g_x \\ F_{yx} & F_{yy} & g_y \\ g_x & g_y & 0 \end{vmatrix}$

or

$$|\overline{H}| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & F_{xx} & F_{xy} \\ g_y & F_{yx} & F_{xy} \end{vmatrix}$$

If $|\overline{H}| > 0$ the function is maximized ($\overline{H}$ is negative definite) and if $|\overline{H}| < 0$ the function is minimized $\overline{H}$ is positive definite).

Example 1:

Optimise the function Z = $4x^2$ + 3xy + $6y^2$ subject to the constraint x + y = 56

Setting the constraint equal to zero,

56 – x – y = 0

Multiplying with λ and adding to the original function, we have

Z = $4x^2$ + 3 xy +6 y^2 + λ [56 – x - y]

Taking the first order conditions

Zx = 0 $\Longrightarrow$ 8x + 3y – λ = 0

Zy = 0 $\Longrightarrow$ 3x + 12y - λ = 0

Z λ = 0 $\Longrightarrow$ 56 - x - y = 0

∴ Zxx = 8 Zxy = 3

Zxy = 3 Zyy = 12

g_x = 1 g_y = 1

$$\therefore |\overline{H}| = \begin{vmatrix} 8 & 3 & 1 \\ 3 & 12 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$|\overline{H}| = 8\begin{vmatrix} 12 & 1 \\ 1 & 0 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 3 & 12 \\ 1 & 1 \end{vmatrix}$$

$|\overline{H}|$ = -14 < 0

∴ The function is minimized

Example 2:

Maximise the utility function U = 2xy subject to a budget constraint equal to 3x + 4y = 90. Find out the critical values of $\bar{x}$, $\bar{y}$ and $\bar{\lambda}$ and use bordered Hessian to test for second order condition

Applying , we have

$U = 2xy + \lambda\,[90 - 3x - 4y]$

$U_x = 2y - 3\lambda = 0$

$U_{xx} = 0 \qquad U_{xy} = 2 \qquad g_x = 3$

$U_y = 2x - 4\lambda = 0$

$U_{yx} = 2 \qquad U_{yy} = 0 \qquad g_y = 4$

$U_\lambda = 90 - 3x - 4y = 0$

To apply Cramer's rule, in matrix form, Ax = B

$$\begin{vmatrix} 0 & 2 & -3 \\ 2 & 0 & -4 \\ -3 & -4 & 0 \end{vmatrix} \begin{vmatrix} x \\ y \\ \lambda \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 90 \end{vmatrix}$$

$$|A| = 0\begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} - 2\begin{vmatrix} 2 & -4 \\ -3 & 0 \end{vmatrix} + -3\begin{vmatrix} 2 & 0 \\ -3 & -4 \end{vmatrix}$$

$= 48$

$$|A_1| = \begin{vmatrix} 0 & 2 & -3 \\ 0 & 0 & -4 \\ 90 & -4 & 0 \end{vmatrix}$$

$$|A_1| = 0\begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} - 2\begin{vmatrix} 0 & -4 \\ 90 & 0 \end{vmatrix} + -3\begin{vmatrix} 0 & 0 \\ 90 & -4 \end{vmatrix}$$

$= 720$

$$x = \frac{|A_1|}{|A|} = \frac{720}{48} = 15$$

$$A_2 = \begin{vmatrix} 0 & 0 & -3 \\ 2 & 0 & -4 \\ -3 & 90 & 0 \end{vmatrix}$$

$|A_2| = 0\begin{vmatrix} 0 & -4 \\ -3 & 0 \end{vmatrix} - 0\begin{vmatrix} 2 & -4 \\ -3 & 0 \end{vmatrix} + \text{-}3\begin{vmatrix} 2 & 0 \\ -3 & 90 \end{vmatrix}$

$|A_2| = 540$

$y = \frac{|A_2|}{|A|} = \frac{540}{48} = 11.25$

$A_3 = \begin{vmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ -3 & -4 & 90 \end{vmatrix}$

$|A_3| = 0\begin{vmatrix} 0 & 0 \\ -4 & 90 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ -3 & 90 \end{vmatrix} + 0\begin{vmatrix} 2 & 0 \\ -3 & -4 \end{vmatrix}$

$\lambda = \frac{|A_3|}{|A|} = \frac{360}{48} = 7.5$

Using Bordered Hessian to test for second order condition

$|\bar{H}| = \begin{vmatrix} U_{xx} & U_{xy} & g_x \\ U_{yx} & U_{yy} & g_y \\ g_x & g_y & 0 \end{vmatrix}$

$|\bar{H}| = \begin{vmatrix} 0 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 0 \end{vmatrix}$

$|\bar{H}| = 0\begin{vmatrix} 0 & 4 \\ 4 & 0 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 3 & 0 \end{vmatrix} \text{-}3\begin{vmatrix} 2 & 0 \\ 3 & 4 \end{vmatrix}$

$|\bar{H}| = 48 > 0$

U is maximized and $|\bar{H}|$ is negative definite

Example 3:

Find maximum of U = xy + x under the condition that 6x + 2y = 110

U = xy + x + λ [110 – 6x – 2y]

Ux = 0 $\Longrightarrow$ y + 1 - 6 λ = 0

Uxx = 0 Uxy = 1 g_x = 6

Uy = 0 $\Longrightarrow$ x - 2 λ = 0

Uyx = 1 Uyy = 0 g_y = 2

$U_\lambda = 0 \Longrightarrow$ 100 - 6 x – 2y = 0

In matrix form A X = B

$$\begin{vmatrix} 0 & 1 & -6 \\ 1 & 0 & -2 \\ -6 & -2 & 0 \end{vmatrix} \begin{vmatrix} x \\ y \\ \lambda \end{vmatrix} = \begin{vmatrix} -1 \\ 0 \\ -110 \end{vmatrix}$$

$$|A| = 0 \begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix} - 1 \begin{vmatrix} 1 & -2 \\ -6 & 0 \end{vmatrix} + -6 \begin{vmatrix} 1 & 0 \\ -6 & -2 \end{vmatrix}$$

= 24

$$A1 = \begin{vmatrix} -1 & 1 & -6 \\ 0 & 0 & -2 \\ -110 & -2 & 0 \end{vmatrix}$$

$$|A1| = -1 \begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix} - 1 \begin{vmatrix} 0 & -2 \\ -110 & 0 \end{vmatrix} + -6 \begin{vmatrix} 0 & 0 \\ -110 & -2 \end{vmatrix}$$

= 224

$$x = \frac{|A_1|}{|A|} = \frac{224}{24} = 9.33$$

$$A2 = \begin{vmatrix} 0 & -1 & -6 \\ 1 & 0 & -2 \\ -6 & -110 & 0 \end{vmatrix}$$

$$|A2| = 0 \begin{vmatrix} 0 & -2 \\ -110 & 0 \end{vmatrix} - -1 \begin{vmatrix} 1 & -2 \\ -6 & 0 \end{vmatrix} + -6 \begin{vmatrix} 1 & 0 \\ -6 & -110 \end{vmatrix}$$

|A2| = 672.

$$y = \frac{|A_1|}{|A|} = \frac{672}{24} = 28$$

$$A3 = \begin{vmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ -6 & -2 & -110 \end{vmatrix}$$

$$|A3| = 0 \begin{vmatrix} 0 & 0 \\ -2 & -110 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 \\ -6 & -110 \end{vmatrix} + -1 \begin{vmatrix} 1 & 0 \\ -6 & -2 \end{vmatrix}$$

= 112

$\lambda = \frac{|A_3|}{|A|} = \frac{112}{24} = 4.66$

Forming Boarded Hessain

$$|\bar{H}| = \begin{vmatrix} U_{xx} & U_{xy} & g_x \\ U_{yx} & U_{yy} & g_y \\ g_x & g_y & 0 \end{vmatrix}$$

$$|\bar{H}| = \begin{vmatrix} 0 & 1 & 6 \\ 1 & 0 & 2 \\ 6 & 2 & 0 \end{vmatrix}$$

$$|\bar{H}| = 0\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 6 & 0 \end{vmatrix} + 6\begin{vmatrix} 1 & 0 \\ 6 & 2 \end{vmatrix}$$

$|\bar{H}| = 24$

Since $|\bar{H}| > 0$ U is maximized and $|\bar{H}|$ is negative definite.

Example 4:

Minimise the total cost C = $45x^2$ + 90 xy +90 y^2 when the firm has to meet a production quota equal to 2x + 3y = 60

C = $45x^2$ + 90 xy +90 y^2 + λ [60 – 2x - 3y]

$C_x = 0 \Rightarrow$ 90x + 90y - 2 λ = 0

$C_y = 0 \Rightarrow$ 90x + 180y - 3 λ = 0

$C_\lambda = 0 \Rightarrow$ 60 - 2x − 3y = 0

In matrix form AX = B

$$\begin{vmatrix} 90 & 90 & -2 \\ 90 & 180 & -3 \\ -2 & -3 & 0 \end{vmatrix} \begin{vmatrix} x \\ y \\ \lambda \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ -60 \end{vmatrix}$$

Solving by Cramer's rule we will have

x = 12

y = 12

λ = 1080

Now forming bordered Hessian $|\bar{H}| = \begin{vmatrix} C_{xx} & C_{xy} & g_x \\ C_{yx} & C_{yy} & g_y \\ g_x & g_y & 0 \end{vmatrix}$

$$|\bar{H}| = \begin{vmatrix} 90 & 90 & 2 \\ 90 & 180 & 3 \\ 2 & 3 & 0 \end{vmatrix}$$

$|\bar{H}| = -450$

Since $|\bar{H}| < 0$ C is minimized

<u>Reference</u>

1. Edward Dowling : "Introduction of Mathematical Economics" Shaum Outline series
2. Alpha C Chiang : "Fundamental Methods of Mathematical Economics"
3. Taro Yamane : "Mathematics for Economics : An Elementary Survey"
4. Barry Bressler : "A Unified Introduction to Mathematical Economics"
5. Manmohan Gupta : "Mathematics for Business and Economics"

CHAPTER IV

INTEGRAL CALCULUS: THE INDEFINITE INTEGRAL

Differentiation is the process of finding out the rates of changes. If *Y=f(x)* is a function of x, we can find the rate of change of Y with respect to x by differentiating the function and thereby obtaining $\frac{dy}{dx}$ or $f'(x)$. Thus if the hypothetical total utility function is U=q^2, where q is the quantity consumed, then $\frac{dU}{dq}$=2q is the rate of change of total utility as given by the differential coefficient of U with respect to q, and is the marginal utility function.

But here an important question is, if the marginal utility function is 2q, what is the total utility function? The problem is reversed here: because we are now given the marginal utility function and we are to find the total utility function, or in other words we are given the change in quantities and we have to calculate the total effects of the change. This reverse process involves summation of the differences. Process of such calculations is called integration.

Thus, reversing the process of differentiation and finding the original function from the derivative is called integration or antidifferentiation. The original function *F(x)* is called the integral, or antiderivative, of *F′(x)*.

Definition

If the differential coefficient of *F(x)* w.r.t. x is *f(x),* then the integral of *f(x)* with respect to x is *F(x)*.

In symbols : if $\frac{d[F(x)]}{dx}$ = *f(x)*, then

$$\int f(x)dx = F(x)$$

The sign '$\int$ ' is used to denote the process of integration. The differential symbol '*dx*' is written by the side of the function to be integrated to indicate the independent variable with respect to which the original differentiation was made and with respect to which we are now to integrate. In this way $\int f(x).dx$ means integration of *f(x)* w.r.t. x.

That is, suppose we want to integrate x^n.

We know that $\frac{d}{dx}x^{n+1} = (n+1)x^{n+1-1} = (n+1)x^n$

$$\therefore \quad \frac{1}{n+1}.\frac{d}{dx}x^{n+1} = x^n$$

Since x^n is differential coefficient of $(\frac{x^{n+1}}{n+1})$

$\therefore$ the integral of x^n is $\frac{x^{n+1}}{n+1}$

In symbols therefore, $\int x^n dx = \frac{x^{n+1}}{n+1}$

It may be noted that differentiating $\frac{x^{n+1}}{n+1}$, we obtain x^n.

<u>Indefinite Integration (Constant of Integration)</u>

Let us have a function F(x) = $\frac{X^{n+1}}{n+1}$ +C, then

$\frac{d^{F(x)}}{dx} = \frac{(n+1)x^n}{n+1}$; because differential coefficient of constant is zero

Hence by the definition of an integral $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$$\therefore \int f(x)dx = F(x) + C$$

where C is called the constant of integration. Since C can take any constant value, the integral is called indefinite integral. Here the left-hand side of the equation is read, "the indefinite integral of *f* with respect to x." The symbol $\int$ is the integral sign, *f(x)* is the integrand, and C is the constant of integration.

<u>Rules of Integration</u>

The following are the basic rules of integration.

<u>Rule 1. Power Rule</u>

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C \qquad (n \neq {}^-1)$$

Examples

1. $\int x^5 dx = \frac{x^6}{6} + C$

2. $\int x\,dx = \frac{1}{2}x^2 + C$

3. $\int 1dx = \int x^0\,dx = x + C$

4. $\int \sqrt{x^3}dx = \frac{x^{5/2}}{5/2}$+ C = $\frac{2}{5}\sqrt{x^5}$+C

5. $\int \frac{1}{x^4}dx = \frac{x^{-4+1}}{-4+1}$+ C = $\frac{1}{3x^3}$ + C

Rule 2. Exponential Rule

$\int e^x dx = e^x + C$, since $\frac{de^x}{dx} = e^x$ and

$\int a^x dx = \frac{a^x}{log_e a}$

ie. $\int a^{kx}\,dx = \frac{a^{kx}}{k \ln a}$+ C

Rule 3. Logarithmic Rule

$\int \frac{1}{x}dx = \log x + C, \qquad Since\ \frac{d(logx)}{dx} = \frac{1}{x}$

Rule 4. Integral of sum and difference

Integral of sum (or difference) of a number of functions is equal to the sum (or difference) of their separate integrals.

i.e. = $\int [f(x) + g(x)]\,dx = \int f(x)dx + \int g(x)dx$, and

$\int [f(x) - g(x)]\,dx = \int f(x)dx - \int g(x)dx$

Example

1. $\int (x^3 - x + 1)dx$ = $\int x^3dx - \int xdx + \int 1dx$

= $\frac{x^4}{4} + c_1 - \frac{x^2}{2} + c_2 + x + C_3$

= $\frac{x^4}{4} - \frac{x^2}{2} + x + C$

Note : In the final answer we have to add up all the constants into a single constant C

Rule 5. The integral of a Constant

The integral of a constant k is

$$\int k\,dx = kx + C$$

Rule 6. The Integral of the Negative

The integral of the negative of a function is the negative of the integral of that function.

ie For $\int -f(x)dx, Here\ k = {}^{-}1$, thus

$$\int -f(x)dx = -\int f(x)dx$$

Rule 7. Integral of a Multiple

If a function is multiplied by a constant number, this will remain a multiple of the integral of the function

$$\int kf(x)dx = k\int f(x)dx$$

Example

1. $\int 4x^3dx = 4\int x^3dx = \frac{4x^4}{4} + C = x^4 + C$

2. $\int (5e^x - x^{-2} + \frac{4}{x}) = 5\int e^x dx - \int x^{-2}dx + 4\int \frac{1}{x}dx$

$= 5e^x + \frac{1}{x} + 4log\ x + C$

Standard Results

In the following +C is implied in all the results.

1. $\frac{d\left[\frac{x^{n+1}}{n+1}\right]}{dx} = x^n \therefore \int x^n dx = \frac{x^{n+1}}{n+1}$

2. $\frac{d(\log x)}{dx} = \frac{1}{x} \therefore \int \frac{1}{x}dx = \log x$

3. $\frac{d(e^x)}{dx} = e^x \therefore \int e^x dx = e^x$

4. $\frac{d}{dx} \cdot \frac{a^x}{\log a} = a^x \therefore \int a^x dx = \frac{a^x}{\log a}$

Initial conditions and Boundary Conditions

The initial condition mean y assumes a specific value when x equals zero (ie $y = y_0$ *when* $x = 0$) and the boundary condition means y assumes a specific value when x assumes a specific value (ie $y = y_0$ *when* $x = x_n$) In many problems an initial condition or a boundary condition uniquely determines the constant (+C) of integration. By permitting a unique determination of constant C, the initial or boundary condition clearly defines the original function.

Examples

1. Given the boundary condition $y = 11$ when $x = 3$, the integral $y = \int 2dx$ is evaluated as follows.

$$y = \int 2dx = 2x + C$$

Substituting y = 11 when x = 3, we have

$$11=2(3) + C$$

$$\text{ie. } C = 5$$

Therefore, $y = 2x + 5$. Note that even though C is specified, $\int 2dx$ remains an indefinite integral because x is unspecified. Thus, the integral $2x + 5$ can assume an infinite number of possible values.

2. Find the integral for $y = \int (x^{1/2} + 3x^{-1/2})dx$ given the initial condition $y = 0$ when $x = 0$

Here $y = \int (x^{1/2} + 3x^{-1/2})dx$

$$= \frac{2}{3}x^{3/2} + 6x^{1/2} + C$$

Substituting the initial condition $y = 0$ when $x = 0$ we get $c = 0$,

Hence $y = \frac{2}{3}x^{3/2} + 6x^{1/2}$

Integration By Substitution

Integration by substitution is a method of integration. Integration often is not easy. Many functions may not be integrated at all. In order to integrate any given function, we try to transform the function to a standard form by suitable substitution and then use the standard result. For example, Integration of a product or quotient of two differentiable functions, such as

$$\int 12x^2(x^3+2)dx$$

cannot be done directly by using the simple rules. However if the integrand can be expressed as a constant multiple of another function 'u' and its derivative $\frac{du}{dx}$, integration by substitution is possible. By expressing the integrand *f(x)* as a function of 'u' and its derivative $\frac{du}{dx}$, and integrating with respect to x.

$$\int f(x)dx = \int (u\frac{du}{dx})\,dx$$

$$\int f(x)dx = \int (u\ \ du) = F(u) + \ C$$

The substitution method reverses the operation of the chain rule and the generalized power function rule in differential calculus.

Examples

1. Find the integral of $\int 12x^2(x^3+2)dx$

   Here the integrand can be converted to a product of another function 'u'' and its derivative $\frac{du}{dx}$ times a constant multiple.

   Let $u =\ x^3$+ 2, then

   $\frac{du}{dx} =\ 3x^2$(take the derivate of *u)*

   Then solving algebraically for dx, we get

   $$dx = \frac{du}{3x^2}$$

   Then substitute 'u' for x^3 + 2, and $\frac{du}{3x^2}$ for dx in the original integrand

   $$\int 12x^2(\ x^3+2)dx \quad = \int 12x^2.u.\frac{du}{3x^2}$$

   $$= \int 4\,u\,du$$

   $$= 4\int u\,du,$$ (where 4 is a constant multiple of u)

   Then integrate w.r.t. 'u' using the power rule of integrate we get

   $$4\int u\,du = 4\,(\tfrac{1}{2}u^2) = 2u^2 + \ C$$

convert back to the terms of the original problem by substituting $x^3 + 2$ for u, we have

$$\int 12x^2(x^3+2)dx \quad = 2u^2 + C$$

$$= 2\,(x^3+2)^2 + C$$

when checking the answer by differentiating with the generalized power function rule or chain rule, we have

$\frac{d}{dx}[2(x^3 + 2)^2 + C] = 4\,(x^3+2)(3x^2) = 12x^2\,(x^3+2)$ which is the original function.

2. Integrate $(5x + 7)^8$
   Here we shall try to reduce it to the standard form $\int x^n dx$: for this the obvious way is to substitute $u = (5x + 7)$. Then the given integral becomes

$$I = \int u^8 dx$$

But dx too will have to be changed to *'du'* otherwise integration will not be possible.

Since u = 5x + 7

$$\frac{du}{dx} = 5$$

or,

$$du = 5.dx$$

$$dx = \frac{du}{5}$$

Therefore $I = \int u^8 dx = \int u^8 \frac{du}{5}$

$$= \frac{1}{5}\int u^8 du$$

$$= \frac{1}{5}\,\frac{u^{8+1}}{8+1} + C$$

$$= \frac{u^9}{45} + C$$

Putting back the value of $u = 5x + 7$, we get $I = \frac{(5x+7)^9}{45} + C$, which will be original function.

Integration By Parts

If an integral is a product or quotient of differentiable functions of x and cannot be expressed as a constant multiple of $u.\frac{du}{dx}$, integration by parts is frequently useful. This method is derived by reversing the process of differentiating a product.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Taking the integral of the derivative, we get

$$f(x)g(x) = \int f(x)g'(x)dx \; + \; \int g(x)f'(x)dx$$

Then solving algebraically for the first integral on the right hand side.

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)\,f'(x)dx$$

For more complicated functions integration tables are generally used. Integration tables provide formulas for the integrals of as many as 500 different functions.

Example

1. Integrate $\int 4x(x+1)^3$
   Here integration by parts is used to determine $\int 4x(x+1)^3$
   First separate the two parts amenable to the following formula.

$$\int [f(x)g'(x)]dx = f(x)g(x) = \int [g(x)f'(x)]dx$$

As a general rule, consider first the simpler function for f(x) and the more complicated function for $g'(x)$. By letting $f(x) = 4x$ and $g'(x) = (x+1)^3$ Then

$f'(x) = 4$, and

$g(x) = \int (x+1)^3 dx$, → This can be integrated by using the simple power rule

then g(x) = $\int (x+1)^3 dx = \frac{1}{4}(x+1)^4 + C_1$

Then substitute the values for $f(x)$, $f'(x)$ and $g(x)$ in the formula, we get

$$\int 4x(x+1)^3 dx \quad = f(x)g(x) - \int [g(x).\,f'(x)]dx$$

$= 4x\left[\frac{1}{4}(x+1)^4 + C_1\right] - \int\left[\frac{1}{4}(x+1)^4 + C_1\right](4)dx$

$= x\,(x+1)^4 + 4C_1x - \int[(x+1)^4 + 4C_1]\,dx$

Then use the power rule $\int x^n dx = \frac{x^{n+1}}{n+1}$ + C, we get

$\int 4x(x+1)^3dx \quad = x(x+1)^4 + 4C_1x - \frac{1}{5}(x+1)^5 - 4C_1x + C$

$= x(x+1)^4 - \frac{1}{5}(x+1)^5 + C$

Then check the answer by letting $y(x) = x(x+1)^4 - \frac{1}{5}(x+1)^5 + C$ and using the product and generalized power function rules,

$y'(x) = [x.4(x+1)^3 + (x+1)^4.1] - (x+1)^4$

$= 4x(x+1)^3$

=====

2. Find the integral $\int 2x\,e^x dx$

Here the integral $\int 2x\,e^x$ is determined as follows.

Let $f(x) = 2x$, and

$g'(x) = e^x$

then $f'(x) = 2$

$g(x) = \int e^x dx = e^x$

Substitute the values in the formula of integration by parts

$$\int [f(x)g'(x)]\,dx = f(x)g(x) - \int[g(x)f'(x)]dx, \text{ we have}$$

$$\int 2xe^x dx = f(x).g(x) - \int g(x).f'(x).dx$$

$$= 2x.e^x - \int e^x 2dx$$

$$= 2x.e^x - 2\int e^x dx$$

By applying the constant rule of integration, we have the original function as

$= 2xe^x - 2e^x + C$

Then check the answer by finding $y'(x)$ where

$y = 2xe^x - 2e^x + C$, as

$y'(x) = 2x.e^x + e^x + 2 - 2e^x = 2xe^x$

Economic Applications

Integrals are used in economic analysis in various ways.

Investment and Capital Formation

Capital formation is the process of adding to a given stock of capital, as the net investment is defined as the rate of change in capital stock formation K over time, t. If the process of capital formation is continuous over time, I(t) = $\frac{dK(t)}{dt}$. From the rate of investment, the level of capital stock can be estimated. Capital stock is the integral with respect to time of net investment

$$K_t = \int I(t)dt = \int \frac{dk}{dt}dt = \int dk = K(t) + C = K(t) + Ko$$

Here $\frac{dK}{dt}$ = $I(t)$ is an identity, it shows the synonymity between net investment and the increment of capital. Since I(t) is the derivative of K(t), it stands to reason that K(t) is the integral or antiderivative of I(t), as shown in the last equation.

$$K(t) = \int I(t)dt = \int \frac{dK}{dt}dt$$

$$or\ K(t) = \int I(t)dt = K(t) + C = K(t) + K_0$$

where C = the initial capital stock K_0

Example

1. The rate of net investment is given by I(t) = 140 $t^{3/4}$ and the initial stock of capital at t = 0 is 150. What is the time path of capital K?

By integrating I(t) with respect to t, we obtain

$$K_1 = \int I(t)dt$$

$$= \int 140t^{3/4}dt$$

$$= 140 \int t^{3/4}dt$$

By the power rule

$$K = 140\left(\frac{4}{7}t^{7/4}\right) + C$$

$$= 80\, t^{7/4} + C$$

But C = K_0 =150, Therefore

$$K = 80t^{7/4} + 150$$

Estimate Total Function From a Marginal Function

Given a total function (e.g, a total cost function) the process of differentiation can yield the marginal function (eg., marginal cost function). Because the process of integration is the opposite of differentiation, it should enable us, conversely, to infer the total function from a given marginal function.

For example, since marginal cost is the change in total cost from an incremental change in output MC = $\frac{dTC}{dQ}$, and only variable costs change with the level of output, then

$$TC = \int MC \; dQ = VC + C = VC + FC$$

Since C = the fixed or initial cost FC

Problems

I. Determine the following Intergrals

a) $\int 3.5 \, dx$

b) $\int \frac{-1}{2} dx$

c) $\int dx$

d) $\int x^5 dx$

e) $\int 4x^3 dx$

f) $\int x^{-5/2} dx$

g) $\int \frac{1}{3} dx$

h) $\int (5x^3 + 2x^2 + 3x) dx$

i) Find the integral for $y = \int \left(x^{1/2} + 3x^{-1/2}\right) dx$ given the initial condition $y = 0$ when $x = 0$

j) Find the integral for y = $\int (10x^4 - 3) dx$ given the boundary condition y = 21 when x = 1

II. Determine the following integral using the substitution method

a) $\int 10x(x^2 + 3)^4 dx$

b) $\int (x - 9)^{7/4} dx$

c) $\int \frac{x^2}{(4x^3+7)^2} dx$

d) $\int x^3 e^{x^4} dx$

III. Use Integration by parts to evaluate the following integral

a) $\int 15x(x+4)^{3/2} dx$

b) $\int \frac{2x}{(x-8)^3}\, dx$

c) $\int 16xe^{-(x+9)}\, dx$

IV. Economic Application

a) The rate of net investment is I =40$t^{3/5}$ and capital stock at t = 0 is 75. Find the capital function K

b) The rate of net investment is I =60$t^{1/3}$,and capital stock at t = 1 is 85. Find K

c) Marginal cost is given by MC = $\frac{dTC}{dQ}$=25+ 30Q -9Q^2, fixed cost is 55. Find i) Total cost ii)Average cost and iii) variable cost function

d) Marginal revenue function is given by MR = 60 – 2Q -2Q^2. Find i)TR function and ii) the demand function *P =f(Q)*

Answers

a) 3.5x + C

b) $\frac{-1}{2}$x +C

c) $\int x + c$

d) $\frac{1}{6}x^6$+C

e) $x^4 + C$

f) $\frac{-2}{3\sqrt{x^3}}$+ C

g) $\frac{1}{3}logx$ + C

h) $\frac{5}{4}x^4$+ $\frac{2}{3}x^3$+$\frac{3}{2}x^2$+C

i) $\frac{2}{3}x^{3/2}$ +6$x^{1/2}$

j) $y = 2x^5 - 3x + 22$

II)

a) $(x^2 + 3)^5 + C$
b) $\frac{4}{11}(x - 9)^{11/4} + C$
c) $-\frac{1}{12(4x^3+7)} + C$
d) $\frac{1}{4}e^{x^4} + C$

III)

a) $6x(x + 4)^{5/2} - \frac{12}{7}(x + 4)^{7/2} + C$
b) $\frac{-x}{(x-8)^2} - \frac{1}{x-8} + C$
c) $-16xe^{-(x+9)} - 16e^{-(x+9)} + C$

IV)

a) K = $25t^{8/5} + 75$

b) K = $45t^{4/3} + 40$

c) i)TC =25Q + $15Q^2 - 3Q^3$ + 55

ii) AC = 25 + 15Q - $3Q^2 + \frac{55}{Q}$

iii) VC = 25Q +$15Q^2$ - $3Q^3$

d) i) 60Q -- $Q^2 - \frac{2}{3}Q^3$ + C

ii) 60 – Q - $\frac{2}{3}Q^2$

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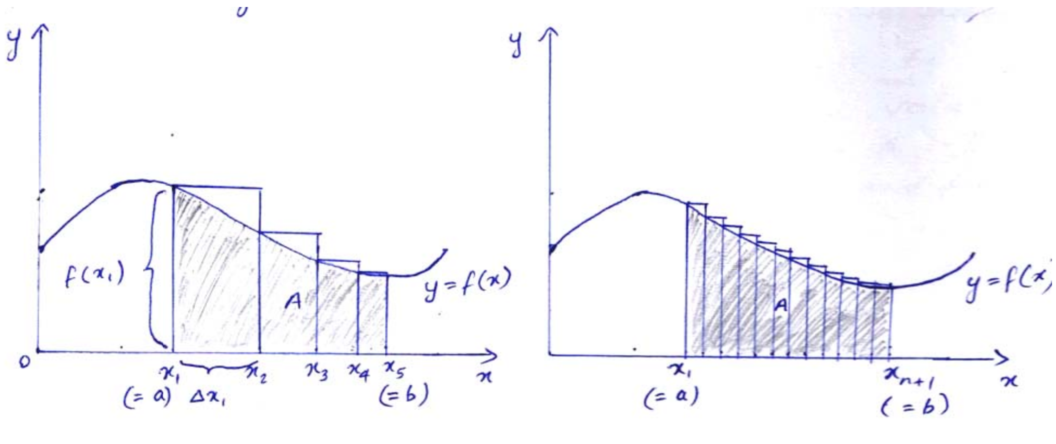
CHAPTER V

INTEGRAL CALCULUS : THE DEFINITE INTEGRAL

5.1. AREA UNDER A CURVE

The graph of a continuous function y = f(x) is drawn in Fig 5.1.



(a) Figure 5.1 (b)

If we seek to measure the (shaded) area A enclosed by the curve and the x axis between the two points a and b in the domain, we may proceed in the following manner. First, we divide the interval (a, b) into n subintervals (not necessarily equal in length). Four of these are drawn in Figure 5.1(a) – that is, n = 4 – the first being (x_1, x_2) and the last, (x_4, x_5). Since each of these represents a change in x, we may refer to them as Δx_1, Δx_2, Δx_3, and Δx_4 respectively. Now, on the subintervals let us construct four rectangular blocks such that the height of each block is equal to the highest value of the function attained in that block (which happens to occur at the left-side boundary of each rectangle here. The first block thus has the height $f(x_1)$ and the width Δx_1, and, in general, the i^{th} block has the height f(xi) and the width Δx_i. The total area A* of this set of blocks is the sum

A* = (n = 4 in Fig. 5.1 (a))

This, though, is obviously not the area under the curve we seek, but only a very rough approximation thereof.

What makes A* deviate from the true value of A is the unshaded portion of the rectangular block; these make A* an over estimate of A. If the unshaded portion can be shrunk in size and be made to approach zero, however, the approximation value A* will correspondingly approach the true value A. This result will be materialize when we try a finer and finer segmentation of the interval (a, b), so that n is increased and Δx_i is shortened indefinitely. Then the blocks will become more slender (if more numerous), and the protrusion beyond the curve will diminish, as we can be seen in Figure 5.1 (b). Carried to the limit, this "slandering" operation yield

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(xi)\Delta x_i = \lim_{n \to \infty} A^* = \text{area } A$$

Provided this limit exists. This equation, indeed, constitutes the formal definition of an area under a curve.

5.2 DEFINITE INTEGRAL

For a given indefinite integral of a continuous function f(x),

$$\int f(x)dx = F(x) + C$$

If we choose two values of x in the domain, say a and b (a < b), substitute them successively into the right side of the equation, and found the difference

$$[F(b) + c] - [F(a) + c] = F(b) - F(a)$$

We get a specific numerical value, free of the variable x as well as the arbitrary constant C. This value is called the definite integral of f(x) from a to b. We refer to a as the lower limit of integration and to b as the upper limit of integration.

In order to indicate the limits of integration, we now modify the integral sign to the form $\int_a^b f(x)dx$. The evaluation of the definite integral is then symbolized in the following steps:

$$\int_a^b f(x)dx = F(x)\Big]_a^b = F(b) - F(a)$$

Where the symbol $\Big]_a^b$ (also written as $\Big|_a^b$ or $\Big[\ldots\ldots\ldots\Big]_a^b$) is an instruction to substitute b and a, successively, for x in the result of integration to get F(b) and F(a), and then take their difference. As the first step, however, we must find the indefinite integral, although we may omit the constant c, since the latter will drop out in the process of difference – taking any way.

Example 1:

Evaluate $\int_1^5 3x^2\, dx$

Since the indefinite integral has the value $x^3 + c$, the definite integral is

$\int_1^5 3x^2\, dx = x3\Big]_b^a = 5^3 - 1^3 = 125 - 1 = 124$

Example 2:

Evaluate $\int_0^4 \left(\frac{1}{1+x} + 2x\right) dx$

$\int_0^4 \left(\frac{1}{1+x} + 2x\right) dx = [l_n 5 + 16] - [l_n 1 + 0]$

$= l_n 5 + 16$

A Definite Integral as an Area under a Curve

The area under a group of continuous function such as that in Figure 5.1 from 'a' to 'b' (a < b) can be expressed more succinctly as the definite integral of f(x) over the interval 'a' to 'b'. Put mathematically,

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

Here 'a' is the lower limit and 'b' is the upper limit of integration.

Properties of Definite Integrals

(i) Reversing the order of the limits changes the sign of the definite integral

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

(ii) If the upper limit of the integration equals the lower limit of integration, the value of the definite integral is zero.

$$\int_a^a f(x)dx = F(a) - F(a) = 0$$

(iii) The definite integral can be expressed as the sum of component subintegrals.

(iv) The sum or difference of two integrals with identical limits of integration is equal to the definite integral of the sum or difference of the two functions.

(v) The definite integral of a constant times a function is equal to the constant times the definite integral of the function.

5.3 AREA BETWEEN CURVES

The area of a region between two or more curves can be evaluated by applying the properties of definite integrals outlined above. The procedure can be demonstrated by using the example given below:

Example:

Using the properties of integrals, the area of the region between two functions such as $y_1 = 3x^2 - 6x + 8$ and $y_2 = -2x^2 + 4x + 1$ from $x = 0$ to $x = 2$ is found in the following way.

(a) Draw a rough sketch of the graph of the function and shade in the desired area

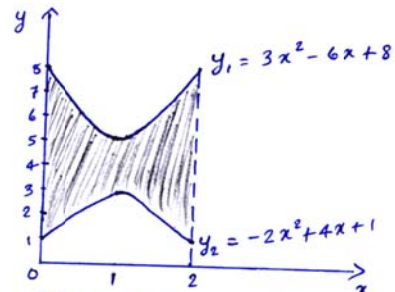


Figure 5. 2

(b) Note the relationship between the curves. Since y_1 lies above y_2, the desired region is simply the area under y_1 minus the area under y_2 between $x = 0$ and $x = 2$. Hence, A = $\int_0^2 (3x^2 - 6x + 8)dx - \int_0^2 (-2x^2 + 4x + 1)dx$

A = $\int_0^2 \left[(3x^2 - 6x + 8) - (-2x^2 + 4x + 1)\right] dx$

(Using property (iv))

A = $\int_0^2 (5x^2 - 10x + 7)dx$

$= \left[\frac{5}{3}\, x^3 - 5x^2 + 7x\right]_2^0$

$= \left[\frac{40}{3} - 20 + 14\right] - 0$

$= \frac{22}{3}$

<u>CONSUMERS' SURPLUS</u>

A demand function p = f(Q) represents the different prices consumers are willing to pay for different quantities of a good. If the equilibrium price is P_E and equilibrium quantity is Q_E, then consumers who would be willing to pay more than P_E benefit. Total benefit to consumers is called consumers' surplus. Mathematically,

Consumers' surplus = $\int_0^{Q_E} f(Q)dQ$ - $Q_E P_E$

<u>Example</u>

Given the demand function P = 42 – 5Q – Q^2. Assuming that the equilibrium price is 6, the consumer's surplus is evaluated as follows

at P = 6 $\quad 42 - 5Q - Q^2 = 6$

$Q^2 + 5Q - 36 = 0$

$(Q + 9)(Q - 4) = 0$

Therefore, Q = -9 or Q = 4

Since negative quantity is not feasible, we consider Q = 4

Consumers' Surplus = $\int_0^4 (42 - 5Q - Q^2)\, dQ - 4x6$

$= \left[42Q - \frac{5}{2}Q^2 - \frac{1}{3}\, Q^3\right]_0^4$ - 24

$= \left[168 - 40 - \frac{64}{3}\right]$ - 24

$= \frac{320}{3}$ - 24

$= \frac{248}{3}$

PRODUCERS' SURPLUS

A supply function Ps = fs (Qs) represents the prices at which different quantities of a good will be supplied. If the market equilibrium occurs at (Q_E, P_E), the producers who would supply at a lower price than P_E benefit. Total gain to producers is called producers' surplus. Mathematically,

Producers' Surplus = Q_E P_E - $\int_0^{Q_E} fs\ (Qs) dQs$

Example

Given the supply function P = $(Q + 3)^2$, find the producers' surplus at P_E = 81 and Q_E=6

Producers' Surplus = 6 x 81 - $\int_0^{6} (Q + 3)^2\ dQ$

$= 486 - \left[\frac{1}{3}\ (Q + 3)^3\right]_0^6$

$= 486 - [243 - 9]$

= 486 – 234 = 252

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CHAPTER 6

INTRODUCTION TO DIFFERENTIAL EQUATIONS AND DIFFERENCE EQUATIONS

Definition and concept of Differential Equations

The theory of differential equations is one of the most fascinating fields of mathematics and also one of great practical importance. Differential equations play a fundamental role in Physics, because they can express many of the laws of nature. Indeed, this is why systematic studies of differential equations were begun by Newton and Leibniz in the seventeenth century. Differential equations are also used in economics.

Consider the marginal revenue function MR = $\frac{dR}{dx}$. Here $\frac{dR}{dx} = f(x)$. This equation involves an independent variable x (quantity sold) and derivative of a dependent variable R w.r.t the independent variable x. This equation is termed as a Differential equation.

Thus, as the name suggests, differential equation in an equation and unlike the ordinary algebraic equations, in differential equations.

1. The unknown is a function (often of time), not a number.
2. The equation includes one or more of the derivatives of the function

Examples of differential equation include

$\frac{dy}{dt} = 5t + 9$ $y' = 12y$ and $y'' - 2y' + 19 = 0$

Definition

A differential equation is an equation which express an explicit or implicit relationship between a function y = f (t) and one or more of its derivatives or differentials $y', y'', \ldots\ldots\ldots y^{n.}$

Symbolically a differential equation may be written as $F(t, y', y'', \ldots\ldots, y^n) = 0$

By integrating both sides of the differential equation, we are able to find the relation between the dependent variable and the independent variable.

Example

To solve the differential equation y″ (t) = 7 for all the functions y (t) which satisfy the equation, simply integrate both sides of the equation and find integrals, i.e.,

$$y'(t) = \int 7\, dt = 7t + c_1$$

$$y(t) = \int (7t + c_1)dt = 3.5\, t^2 + c_1 t + c$$

This is called a general solution which indicates that when c is unspecified, a differential equation has an infinite number of possible solutions. If c can be specified, the differential equation has a particular or definite solution which alone of all possible solution is relevant.

KINDS OF DIFFERENTIAL EQUATION

Ordinary Differential Equation:

An ordinary differential equation is one involving a single independent variable, for which unknown is a function of only one variable.

For example $\frac{dx}{dt}$ = f (x, t) where x = x (t) is the unknown function.

Partial Differential Equation:

For a partial differential equation, the unknown or the dependent variable is a function of two or more variables, and one or more of the partial derivatives are included.

For example $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$ = kz is a partial differential equation in the unknown function z = z (x, y).

Order of Differential Equation

The order of a differential equation is the order of the highest derivatives in the equation.

For example

1. $\frac{dy}{dt} = 2x^2 - 3x + 4$ is the 1st order differential equation
2. $\frac{d^2y}{dt^2} = 3x - 4$ is the 2nd order differential equation

3. $\frac{d^3y}{dt^3}$ = 4x + 3 is the 3rd order differential equation
4. $\left(\frac{d^2y}{dt^2}\right)^7 + \left(\frac{d^3y}{dt^3}\right)^5 = 75y$ is the 3rd order differential equation

Degree of Differential Equation

The degree of a differential equation is the highest power to which the derivative of highest order is raised.

For example

1. $\frac{dy}{dt}$ = 2x + 6 is the first degree differential equation
2. $\left(\frac{dy}{dt}\right)^4 - 5t^2$ = 0 is the fourth degree differential equation
3. $\left(\frac{d^2y}{dt^2}\right)^7 + \left(\frac{d^3y}{dt^3}\right)^5 = 75y$ is the fifth degree differential equation
4. $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 12x$ is the first degree differential equation
5. $\left(\frac{d^3y}{dx^3}\right)^4 + \left(\frac{d^2y}{dx^2}\right)^6 = 4 - y$ is the fourth degree difference equation

Definition and concepts of Difference Equations

Economists often study the development through time of economic variables like national income, the interest rate, the money supply, the production of oil and the price of wheat etc. The focus is not on relating continuous changes of variables, but on discrete changes. In planning models, for example, the comparison is between the initial base year and the terminal year and the change in investment over the period is related to change in time over a period. Both the changes are said to be discrete. The behavior of these discrete equations that relate such quantities at different times are analysed in difference equation.

Definition

A difference equation expresses a relationship between a dependent variable and a lagged independent variable or variables whish changes at discrete intervals of time.

For exempt $I_t = f(y_t - 1)$, where I & Y are measured at the end of each year and thus difference equation relates the independent variable, the dependent variable and its successive difference.

Therefore F (t, y_t , Δy_t , $\Delta^2 y_t$, ...) = 0 is a difference equation

The order of Difference Equation

The order of a difference equation is determined by the greatest number of periods tagged. The first order difference equation expresses a time lag of one period; a second order difference equation expresses a time lag of two periods and so on. The change in y as t changes from t to t + 1 is called the first difference of y. It is written as

$\frac{\Delta y}{\Delta t} = \Delta y_t = \Delta y_{t+1} - y_t$

Where Δ is an operator replacing $\frac{d}{dt}$ which is used to measure continuous change in differential equation.

Example

1. $I_t = a\,(y_{t-1} - y_{t-2}) \longrightarrow$ is a difference equation of order 2
2. $Q_s = a + b\,P_{t-1} \longrightarrow$ is a difference equation of order 1
3. $y_{t+3} - 9\,y_{t+2} + 2\,y_{t+1} + 6\,y_t = 8 \longrightarrow$ is a difference equation of order 3
4. $\Delta y_t = 5\,y_t$, that is

   $y_{t+1} - y_t = 5\,y_t$

   i.e. $y_{t+1} = 6\,y_t \longrightarrow$ is a difference equation of order 1

5. $\Delta^2 y_t + \Delta y_t = y_t$ is equivalent to

   $y_{t+2} - 2\,y_{t+1} + y_t + y_{t+1} - y_t = y_t$

   or $y_{t+2} - y_{t-1} - y_t = 0 \longrightarrow$ is a difference equation of order 2

Problems

1. Find the order and degree of following differential equations

   a) $\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^3 = 18x$

   b) $\frac{dy}{dx} = 3\,x^2$

   c) $\frac{d^3y}{dx^3} + x^2 y \frac{d^2y}{dx^2} - 4y^4 = 0$

   d) $\left(\frac{dy}{dt}\right)^4 - 5t^2 = 0$

e) $\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^3 + x^2 = 0$

2. Find the order of the following difference equations

a) $\Delta y_t = 8$

b) $\Delta y_t = a\, y_t$

c) $\Delta^2 y_t$

d) $y_{t+2} + a_1 y_{t+1} + a_2 y_t = c$

Answers

I a) Third order, first degree

b) First order, first degree

c) Third order, first degree

d) First order, fourth degree

e) Second order, first degree

II a) First order

b) First order

c) Second order

d) Second order

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