

MATHEMATICAL ECONOMICS

III Semester

**COMPLEMENTARY COURSE
B Sc MATHEMATICS**

(2011 Admission)



UNIVERSITY OF CALICUT

SCHOOL OF DISTANCE EDUCATION

Calicut University P.O. Malappuram, Kerala, India 673 635



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STUDY MATERIAL

Complementary Course for

B Sc Mathematics

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MATHEMATICAL ECONOMICS

Prepared by *Sri. Venugopalan. P.K,*
Assistant Professor,
Department of Statistics,
Sree Kerala Varma College,
Thrissur.

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MODULE I

FIRST-ORDER DIFFERENTIAL EQUATIONS

Definitions and Concepts

A differential equation is an equation which expresses an explicit relationship between Function $y = f(t)$ and one or more of its derivatives or differential equations include.

$$dy/dt = 5t + 9y' = 12y \quad \text{and} \quad y'' - 2y' + 19 = 0$$

Equations involving a single independent variable, such as those above, are called ordinary differential equations. The solution or integral of a differential equation is any equation, without derivative or differential, that is defined over an interval and satisfies the differential equation for all the values of the independent variable (s) in the interval. See Example 1.

The order of a differential equation is the order of the highest derivative in the equation. The degree of a differential equation is the highest power to which the derivative of highest order is raised. See Example 2.

Example 1. To solve the differential equation $y''(t) = 7$ for all the functions $y(t)$ which satisfy the equation, simply integrate both sides of the equation to find the integrals.

$$Y'(t) = \int 7 dt = 7t + c_1$$

$$Y(t) = \int (7t + c_1) dt = 3.5t^2 + c_1t + c$$

This is called a *general solution* which indicates that when c is unspecified, a differential equation has an infinite number of possible solutions. If c can be specified, the differential equation has a *particular or definite solution* which alone of all possible solutions is relevant.

Example 2. The order and degree of differential equations are shown below.

1. $dy/dt = 2x + 6$ First order, First - degree
2. $(dy/dt)^4 - 5t^5 = 0$ First order, Fourth - degree
3. $d^2y/dt^2 + (dy/dt)^3 + x^2 = 0$ second - order, First degree
4. $(d^2y/dt^2)^7 + (d^3y/dt^3)^5 = 75y$ Third order, Fifth - degree

GENERAL FORMULA FOR FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS

For A first-order linear differential equation, dy/dt and must be first degree and no product $y(dy/dt)$ may occurs. For such an equation

$$dy/dt + vy = z$$

Where v and z may be constants or functions of time. The formula for a general solution is

$$y(t) = e^{-\int v dt} [A + \int z e^{\int v dt} dt]$$

Where A is an arbitrary constant. As solution is composed of two parts: $e^{-\int v dt} A$ is called the complementary function, and $e^{-\int v dt} \int z e^{\int v dt} dt$ is called the particular integral y_p equals the intertemporal equilibrium level of $y(t)$; the complementary function y_c represents the deviation from the equilibrium. For $y(t)$ to be dynamically stable, y_c must approach zero as t approaches infinity (that is, k in e^{kt} must be negative). The solution of a differential equation can always be checked by differentiation.

Example 3. The general solution for the differential equation $dy/dt + 4y = 12$ is calculated as follows. Since $v = 4$ and $z = 12$, substituting gives

$$y(t) = e^{-\int 4 dt} [A + \int 12 e^{\int 4 dt} dt]$$

$\int 4 dt + c$, c is always ignored and subsumed under A. Thus,

$$y(t) = e^{-4t} [A + \int 12 e^{4t} dt]$$

Integrating the remaining integral gives $\int 12 e^{4t} dt = 3e^{4t} + c$. Ignored and constant again and substituting

$$y(t) = e^{-4t} [A + 3e^{4t}] = Ae^{-4t} + 3$$

since $e^{-4t} e^{4t} \rightarrow e^0 = 1$ as $t \rightarrow \infty$ $y_c = Ae^{-4t} \rightarrow$

0 and $y(t)$ approaches $y_p = 3$ the intertemporal equilibrium level. $Y(t)$ is dynamically stable.

To check this answer, which is a general solution because A has not been specified start by taking the derivative.

From the original problem,

$$\frac{dy}{dt} + 4y = 12 \quad \frac{dy}{dt} = 12 - 4y$$

Substituting $y = Ae^{-4t} + 3$

$$\frac{dy}{dt} = 12 - 4(Ae^{-4t} + 3) = -4Ae^{-4t}$$

Example. Given $dy/dt + 3t^2 y = t^2$ Where $v = 3t^2$ and $z = t^2$. To find the general solution, first substitute

$$y(t) = e^{-\int 3t^2 dt} (A + \int t^2 e^{\int 3t^2 dt})$$

Integrating the remaining integral calls for the substitution method. Letting $u = t^3$, $du/dt = 3t^2$, and $dt = du/3t^2$,

$$\int t^2 e^{t^3} dt = \int t^2 e^u \frac{du}{3t^2} = \frac{1}{3} \int e^u du = \frac{1}{3} e^u = \frac{1}{3} e^{t^3}$$

Finally substituting

$$y(t) = e^{-t^3} \left(A + \frac{1}{3} e^{t^3} \right) = Ae^{-t^3} + \frac{1}{3}$$

As $t \rightarrow \infty$, $y_c = Ae^{-t^3} \rightarrow 0$ and $y(t)$ approaches $\frac{1}{3}$. The equilibrium is dynamically stable.

Differentiating check the general solution, $dy/dt = -3t^2 Ae^{t^3}$. From the original problem,

$$\frac{dy}{dt} + 3t^2 y = t^2 \quad \frac{dy}{dt} = t^2 - 3t^2 y$$

Substituting y

$$\frac{dy}{dt} = t^2 - 3t^2 (A e^{t^3} + 1/3) = -3t^2 A e^{t^3}$$

EXACT DIFFERENTIAL EQUATIONS AND PARTIAL INTEGRATION

Given a function of more than one independent variable, such as $F(y,t)$ where $M = aF_y$ and $N = aF_t$, the total Differential is written

$$dF(y,t) = M dy + N dt$$

Since F is a function of more than one independent variable, M and N are partial derivatives and equation is called a partial differential equation. If the differential is set equal to zero, so that $M dy + N dt = 0$, it is called an exact differential equation because the left side exactly equals the differential of the primitive function $F(y,t)$. For an exact differential equation, $\partial M/\partial t$ must equal $\partial N/\partial y$, that is, $\partial^2 F/(\partial t \partial y) = \partial^2 F/(\partial y \partial t)$. For proof of this proposition.

Solution of an exact differential equation calls for successive integration with respect to one independent variable at a time while holding constant the other independent variable(s). The procedure, called partial integration, reverses the process of partial differentiation.

Example. Solve the exact nonlinear differential equation

$$(6yt - 9y^2)dy + (3y^2 + 8t)dt = 0$$

1. Test to see if it is an exact differential equation. Here $M = 6yt - 9y^2$ and $N = 3y^2 + 8t$. Thus, $\partial M/\partial t = 6y$ and $\partial N/\partial y = 6y$. If $\partial M/\partial t \neq \partial N/\partial y$, it is not an exact differential equation.

2. Since $M = A$ Flay is a partial derivative, integrate M partially with respect to y by treating t as a constant, and add a new function $Z(t)$ for any additive terms of t which would have been eliminated by the original differentiation with respect to y . Note that ay replaces dy in partial integration.

$$F(y,t) = \int (6yt - 9y^2) \partial y + Z(t) = 3y^2t + 3y^3 + Z(t)$$

This gives the original function except for the unknown additive terms of t , $Z(t)$

3. Differentiate with respect to t , to find $\frac{\partial F}{\partial t}$ (earlier called N). Thus

$$\frac{\partial F}{\partial t} = 3y^2 + Z'(t)$$

Since $\partial F / \partial t = N$ and $N = 3y^2 + 8t$, substitute $\frac{\partial F}{\partial t} = 3y^2 + 8t$

$$3y^2 + 8t = 3y^2 + Z'(t) \quad Z'(t) = 8t$$

4. Next integrate $Z(t)$ with respect to t to find the missing t terms.

$$Z(t) = \int Z'(t) dt = \int 8t dt = 4t^2$$

5. Substitute and add a constant of integration

$$F(y,t) = 3y^2t + 3y^3 + 4t^2 + c$$

This is easily checked by differentiation.

INTEGRATING FACTORS

Not all equations are exact. However, some can be made exact by means of an integrating factor. This is a multiplier which permits the equation to be integrated.

EXAMPLE. Testing the nonlinear differential equation $5ytdy + (5y^2 + 8t)dt = 0$ reveals that it is not exact, with $M = 5yt$ and $N = 5y^2 + 8t$, $\frac{\partial M}{\partial t} = 5y \neq \frac{\partial N}{\partial y} = 10y$. Multiplying by an integrating factor of t , however, makes it exact: $5yt^2dy + (5y^2t + 8t^2) dt = 0$. Now $\frac{\partial M}{\partial t} = 10yt = \frac{\partial N}{\partial y}$, and the equation can be solved by the procedure outlined above.

To check the answer to a problem in which an integrating factor was used, take the total differential of the answer and then divide by the integrating factor.

RULES FOR THE INTEGRATING FACTOR

Two rules will help to find the integrating factor for a nonlinear first-order differential equation, if such a factor exists. Assuming $\frac{\partial M}{\partial t} \pm \frac{\partial N}{\partial y} \frac{1}{N} \left(\frac{\partial M}{\partial t} - \frac{\partial N}{\partial y} \right) = f(y)$ alone, then $e^{\int f(y)dy}$ is an integrating factor.

Rule 2. If $1/M (aN/ay - aM/at) = g(t)$ alone, then $e^{\int g(t)dt}$ is an integrating factor.

EXAMPLE. To illustrate the rules above, find the integrating factor given in example 7, where

$$5ytdy = (5y^2+8t)dt = 0, M = 5yt, N = 5y^2 = 8t \frac{\partial M}{\partial t} = 5y \neq \frac{\partial N}{\partial y} = 10y$$

Applying rule 1, $\frac{1}{5y^2+8t} (5y - 10y) = -\frac{5y}{5y^2+8t}$

Which is not function of y alone and will not supply an integrating factor for the equation. Applying Rule 2,

$$\frac{1}{5yt} (10y - 5y) = \frac{5y}{5yt} = \frac{1}{t}$$

Which is a function of t alone. The integrating factor, therefore, is $e^{\int (1/t)dt} = e^{\ln t} = t$

SEPERATION OF VARIABES

Solution of nonlinear first-order first degree differential equations is complex, (A first-order first degree differential equations is one in which the highest derivative is the derivative dy/dt and that derivative is raised to a power of 1. It is nonlinear if it contains a product of y and dy/dt, or y raised to a power other than 1) If the equation is exact or can be rendered exact by an integrating factor; the procedure outlined in Example 6 can be used. If, however, the equation can be written in the form of separated variables such that $R(y) dy + s(t) = 0$, where R and s, respectively, are functions of y and t alone, the equation can be solved simply by ordinary integration.

EXAMPLE. The following calculations illustrate the separation of variables procedure to solve the nonlinear differential equation.

$$dy/dt = y^2 t$$

First ,separating the variable by rearranging terms,

$$\frac{dy}{y^2} = tdt$$

Where $R = \frac{1}{y^2}$ and $s = t$, Then integrating both side

$$\int y^{-2} dy = \int tdt$$

$$-2y^{-1} + c_1 = \frac{t^2}{2} + c_2$$

$$-1/y = \frac{t^2 + 2c_2 - 2c_1}{2}$$

$$y = \frac{-2}{t^2 + c}$$

Letting, $c = 2c_2 - 2c_1$

Since the constant of integration is arbitrary until it is evaluated to obtain a particular solution, it will be treated generally and not specifically in the initial steps of the solution e^c and $\ln c$ can also be used to express the constant

This solution can be checked as follows: Taking the derivatives of $y = -2(t^2+c)^{-1}$ by the generalized power function rule,

$$dy/dt = (-1)(-2)(t^2+c)^{-2}(2t) = \frac{4t}{(t^2+c)^2}$$

$dy/dt = y^2t$. substituting into

$$dy/dt = \frac{(-2)^2t}{(t^2+c)^2} = \frac{4t}{(t^2+c)^2}$$

EXAMPLE. Given the nonlinear differential equation

$$t^2 dy + y^3 dt = 0$$

Integrating the separated $f(t)$. But multiplying by $1/(t^2 y^3)$ to separate the variables gives

$$\frac{1}{y^3} dy + \frac{1}{t^2} dt = 0$$

Integrating the separated variables,

and $\int y^{-3} dy + \int t^{-2} dt = -1/2y^{-2} - t^{-1} + c$

$$\begin{aligned} F(y,t) &= -1/2y^{-2} - t^{-1} + c \\ &= 1/2y^2 - 1/t + c \end{aligned}$$

For complicated function, the answer is frequently left in this form. It can be checked by differentiating and comparing, which can be reduced through multiplication by $y^3 t^2$

ECONOMIC APPLICATIONS

Differential equations serve many functions in economics. They are used to determine the conditions for dynamic stability in microeconomic models of market equilibrium and to trace the time path of growth under various conditions in macroeconomic. Given the growth rate of a function, differential equations enable the economist to find the function whose estimate the demand growth is described; from point elasticity, they enable the economist to estimate the demand function. They were use to estimate capital functions from marginal cost and marginal revenue functions.

EXAMPLE Given the demand function $Q_d = c+bp$ and the supply function $Q_s = g+hp$, the equilibrium price is

$$\bar{p} = \frac{c-g}{h-p}$$

Assume that the rate of change of price in the market dp/dt is a positive linear function of excess demand $Q^d - Q^s$ such that

$$\frac{dp}{dt} = m(Qd-Qs) \quad m = a \text{ constant} > 0$$

The conditions for dynamic price stability in the market (i.e under what conditions $p(t)$ will converge to p as $t \rightarrow \alpha$) can be calculated as shown below. Substituting the given parameters for Qd and Qs

$$\frac{dp}{dt} = m[(c+bp)-(g+hp)] = m(c+bp-g-hp)$$

Rearranging to fit the general format, $dp/dt + m(h-b)p = m(c-g)$. Letting $v = m(h-b)$ and $z = m(c-g)$

$$P(t) = e^{-\int v dt} (A + \int Z e^{-\int v dt}) = e^{-vt} (A + \int Z e^{vt} dt)$$

$$e^{-vt} (A + \frac{Z e^{vt}}{v}) = A e^{-vt} + \frac{Z}{v}$$

At $t = 0$, $P(0) = A + z/v$ and $A = p(0) - z/v$

Substituting

$$P(t) = \frac{[p(0) - z]}{v} e^{-vt} + \frac{z}{v}$$

Finally, replacing $v = m(h-b)$ and $z = m(c-g)$,

$$P(t) = \frac{[p(0) - c - g]}{h-b} e^{-m(h-b)t} + m(h-b) \frac{c-g}{h-b}$$

The time path is

$$P(t) = [p(0) - p] e^{-m(h-b)t} + p$$

Since $p(0)$, p , $m > 0$, the term on the right-hand side converge towards zero as $t \rightarrow \alpha$ and thus $p(t)$ will converge toward p only if $h-b > 0$. For normal cases where demand is negatively sloped ($b < 0$) and supply is positively sloped ($h > 0$), the dynamic stability condition is assured. Markets which positively sloped demand functions or negatively sloped supply functions will also be dynamically stable as long as $h > b$.

PHASE DIAGRAMS FOR DIFFERENTIAL EQUATIONS

Many nonlinear differential equations cannot be solved explicitly as functions of time *phase diagrams*, however, offer qualitative information about the stability as functions that is helpful in determining whether the equations will converge to an intertemporal (steady-state) equilibrium or not. A phase diagram of a differential equation depicts the derivative which we

now express as y for simplicity of notation as a function of y . The steady-state solution is easily identified on a phase diagram as any point at which the graph crosses the horizontal axis, because at that point $y = 0$ and the function is not changing. For some equations there may be more than one intersection and hence more than one solution.

Diagrammatically, the stability of the steady-state solution(s) is indicated by the *arrows of motion*. The arrows of motion will point to the right (indicating y is increasing) any time graph of y is above the horizontal axis (including $y > 0$) and to the left (indicating y is decreasing) any time the graph of y is below the solution is stable; if the arrows of motion point away from a steady-state solution; the solution is unstable.

Mathematically; the slope of the phase diagram as it passes through a steady-state equilibrium point tells us if the equilibrium point is stable or not. When evaluated at a steady-state equilibrium point.

If $dy/dy < 0$, the equilibrium is stable

If $dy/dy > 0$, the point is unstable

EXAMPLE. Given the nonlinear differential equation

$$Y = 8y - 2y^2$$

A phase diagram can be constructed and employed in six easy steps.

1. The intertemporal or steady-state solution (s), where there is no pressure for change, is found by setting $Y = 0$ and solving algebraically.

$$Y = 8 - 4y = 0$$

$$2y(4 - y) = 0$$

$$Y = 0 \qquad y = 4 \text{ steady-state solutions}$$

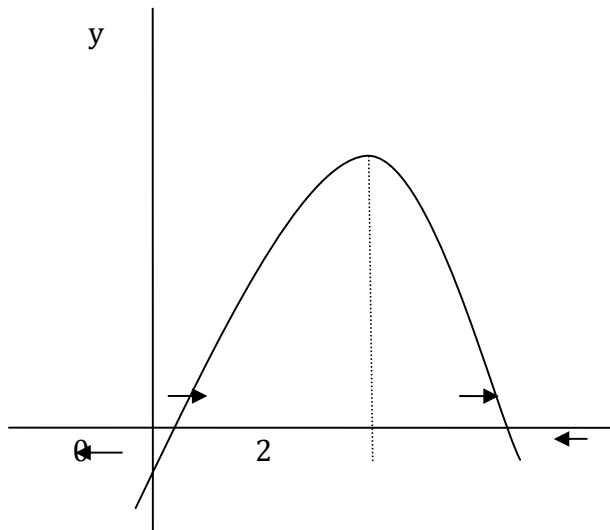
The phase diagram will pass through the horizontal axis at $y = 0, y = 4$

2. Since the function passes through the horizontal axis twice, it has one turning point. We next determine whether that point is a maximum or minimum.

$$dy/dy = 8 - 4y = 0 \qquad y = 2 \text{ is a critical value}$$

$$\frac{d^2y}{dx^2} = -4 < 0 \qquad \text{concave, relative maximum}$$

3. A rough, but accurate, sketch of the phase diagram can then easily be drawn see Fig.



4. The arrows of motion complete the graph. As explained above, where the graph lies above the horizontal axis, $y > 0$ and the arrows must point to the right; where the graph lies below the horizontal axis, $y < 0$ and the arrows must point to the left.
5. The stability of the steady-state equilibrium points can now be read from the graph. Since the arrows of motion point away from the first intertemporal equilibrium $y_1 = 0$, y_1 is an unstable equilibrium. With the arrows of motion pointing toward the second intertemporal equilibrium $y_1 = 4$, $y_2 = 4$ is a stable equilibrium.
6. The slope of the phase diagram at the steady-state solutions can also be used to test stability independently of the arrows of motion. Since the slope of the phase diagram is positive at $y_1 = 0$,

We can conclude y_1 is an unstable equilibrium. Since the slope of the phase diagram is negative at $y_2 = 4$, we know y_2 must be stable.

EXAMPLE. Without even resorting to a phase diagram, we can use the simple first-derivative evaluated at the intertemporal level (s) to determine the stability of differential equations. Given $y = 8y - 2y^2$

$$\frac{dy}{dt} = 8 - 4y$$

Evaluated at the steady-state levels, $y^1 = 0$ and $y^2 = 4$,

$$\frac{dy(0)}{dy} = 8 - 4(0) = 8 > 0 \quad \frac{dy(4)}{dy} = 8 - 4(4) = -8 < 0$$

$$\bar{y}_{1=0} \text{ is unstable} \quad \bar{y}_2 = 4 \text{ is stable}$$

USE OF DIFERENTIAL EQUATIONS IN ECONOMICS

1. find the demand function $q=f(p)$ if point elasticity ϵ is -1 for all $p>0$

$$\epsilon = \frac{p}{Q} \frac{dQ}{dp} = -1 \quad \frac{dQ}{dp} = -\frac{Q}{p}$$

separating the variables, $dQ/Q + dp/p = 0$

Integrating, $\ln Q + \ln p = \ln c$

$$Qp = c \quad Q = c/p$$

2. Find the demand function $f(p)$ if $\epsilon = -k$ a constant.

$$\epsilon = \frac{p}{Q} \frac{dQ}{dp} = -k \quad \frac{dQ}{dp} = -\frac{kQ}{p}$$

Separating the variables,

$$\frac{dQ}{Q} + \frac{k}{p} dp = 0$$

$$\ln Q + k \ln p = c$$

$$Qp^k = c \quad Q = cp^{-k}$$

FIRST-ORDER DIFFERENCE EQUATIONS

DEFINITIONS AND CONCEPTS

A difference equation express a relationship between a dependent variable and a lagged independent variable (or variables) which changes at discrete intervals of time, for example, $I_t = f(y_{t-1})$, where 1 and y are measured at the end of each year. The order of a difference equation is determined by the greatest number of periods lagged. A first-order difference equation expresses a time lag of one period; a second - order, two period; etc. The change in y as t changes from t to + 1 is called the *first difference of y*. It is written.

$$\frac{\Delta y}{\Delta x} = \Delta y_1 = y_{t+1} - y_t$$

Where Δ is an operator replacing d/dt that is used to measure continuous change in differential equations. The solution of a difference equation defines y for every of t and does not contain a difference expression. See examples 1 and 2.

EXAMPLE 1. Each of the following is a difference equation of the order indicated.

$$I_t = a(y_{t-1} - y_{t-2}) \quad \text{order 2}$$

$$Q_s = a + bpt - 1 \quad \text{order 1}$$

$$Y_{t+3} - 9y_{t+2} + 2y_{t+1} + 6y_t = 8 \quad \text{order 3}$$

$$\Delta y_t = 5y_t \quad \text{order 1}$$

Substituting from for Δy_t , above,

$$Y_t + 1 - y_t \qquad y_t + 1 = 6y_t \quad \text{order 1}$$

EXAMPLE 2. Given that the initial value of y is y_0 in the difference equation

$$y_t + 1 = by_t$$

a solution is found as follows by successive substitutions of $t = 0,1,2,3$ etc.

$$y_1 = by_0 \qquad y_3 = by_2 = b(b_2 y_0) = b^3 y_0$$

$$y_2 = by_1 = b(by_0) = b^2 y_0 \qquad y_4 = by_3 = (b^3 y_0) = b^4 y_0$$

Thus, for any period t ,

$$Y_t = b^t y_0$$

This method is called the iterative method. Since y_0 is a constant, notice the crucial role b plays in determining values for y as t changes.

GENERAL FORMULA FOR FIRST-ORDER LINEAR DIFFERENCE EQUATIONS

Given a first-order difference equation which is linear (i.e, all the variable are raised to the first power and there are no cross products),

$$y_t = by_{t-1} + a$$

where b and a are constants, the general formula for a definite solution is

$$y_t = (y_0 - a/1-b)b^t + a/1-b \quad \text{when } b \neq 1$$

$$y_t = y_0 + at \qquad \text{when } b = 1$$

If no initial condition is given, an arbitrary constant A is used for $y_0 - a/(1-b)$. This is called a *general solution*

EXAMPLE 3. Consider the difference equation $y_t = 7y_{t-1} + 16$ and $y_0 = 5$. In the equation, $b = 7$ and $a = 16$ since $b \neq 1$,

$$y_t = (5-16/1+7)(-7)^t + 16/1+7 = 3(-7)^t + 2$$

To check the answer, substitute $t = 0$ and $t = 1$

$$Y_0 = 3(-7)^0 + 2 = 5 \quad \text{since } (-7)^0 = 1$$

$$Y_1 = 3(-7)^1 + 2 = -19$$

Substituting $Y_1 = -19$ for y_t and $y_0 = 5$ for y_{t-1} in the original equation,

$$-19 = -7(5) + 16 = -35 + 16$$

STABILITY CONDITIONS

In the general form

$$Y_t = Ab^t + c$$

Where $A = y_0 - a/(1-b)$ and $c = a/(1-b)$, here Ab^t is called the *complementary function* and c is the *particular solution*. The particular solution expresses the intemporal equilibrium level of y ; the complementary function represents the deviation from that equilibrium. Equation will be dynamically stable, therefore, only if the complementary function $Ab^t \rightarrow 0$, as $t \rightarrow \infty$. All depends on the base b assuming $A = 1$ and $c = 0$ for the moment, the exponential expression b^t will generate seven different time paths depending on the value of b , as illustrated in Example 4. As seen there, if $|b| > 1$ the time path will explode and move towards equilibrium. If $b > 0$, the time path will oscillate between positive and negative values; $b < 0$, the time path will be non oscillating. If $A \neq 1$, the value of the multiplicative constant will scale up or down the magnitude of b^t , but will not change the basic pattern of movement. If $A = -1$, a mirror image of the time path of b^t with respect to the horizontal axis will be produced. If $c \neq 0$, the vertical intercept of the graph is affected and the graph shifts up or down accordingly.

THE HARROD MODEL

The Harrod Model attempts to explain the dynamics of growth the economy. It assumes

$$S_t = sy_t$$

Where s is a constant equal to both MPS and APS.

It also assumes the acceleration principle. The investment is proportional to the rate of change of national income overtime.

$$I_t = a(y_t - y_{t-1})$$

Where a is a constant equal to both the marginal and average capital output ratios.

In equilibrium, $I_t = S_t$

$$\therefore a(y_t - y_{t-1}) = sy_t = (a - s)y_t = ay_{t-1}$$

Dividing through by $a-s$ to conform to (17.3), $Y_t = [a/(a-s)] Y_{t-1}$, since $a/(a-s) \neq 1$,

$$Y_t = (Y_0 - 0) (a/a-s)^t + 0 = (a/a-s)^t Y_0$$

The stability of the time path thus depends on $a/(a-s)$. Since $a =$ the capital-output ratio, which is normally larger than 1, and since $s =$ MPS which is larger than 0 and less than 1, the base $a/(a-s)$ will be larger than 0 and usually larger than 1. Therefore, Y_t is explosive but non oscillating. Income will expand indefinitely, which means it has no bounds.

EXAMPLE 8. The warranted rate of growth (i.e., the path the economy must follow to have equilibrium between saving and investment each year) can be found as follows in the Harrod Model.

Y_t increases indefinitely. Income in one period is $a/(a-s)$ times of the income of the previous period.

$$Y_1 = (a/a-s)Y_0$$

The rate of growth G between the periods is defined as

$$G = \frac{Y_1 - Y_0}{Y_0}$$

Substituting

$$\begin{aligned} G &= \frac{[a(a-s)]Y_0 - Y_0}{Y_0} = \frac{[a(a-s) - 1] Y_0}{Y_0} \\ &= \frac{a}{a-s} - 1 = \frac{a}{a-s} - \frac{a-s}{a-s} = \frac{s}{a-s} \end{aligned}$$

The warranted rate of growth, therefore, is

$$G_w = \frac{s}{a-s}$$

Ref: Introduction to Mathematical Economics 3rd edition.

By EDWARD. T. DOWLING

MODULE II

PRODUCTION FUNCTION

The theory of production plays two important roles in the theory of relative prices. It provides a base for the analysis of relations between costs and volumes of input costs influence supplies which along with demand determine prices. The theory of production serves a base for the theory of demand for factors of production. Production means a transformation of inputs into outputs. The inputs are the things brought by the firm and the outputs are the things which the firm sells.

Production takes place by the combined forces of various factors of production such as land, labour capital and enterprise. In production there are two processes, namely theory of production and theory of costs. The theory of production shows the physical relationship between inputs and outputs. The theory of costs represent the level of output and costs.

A production function is a technique which represents the technology involved in the process of production.

Production explains the relationship between the inputs and outputs. The output of a commodity depends on the employed inputs. Production function explains the physical relationship between the units of employed inputs and the output. The money price do not appear in a production function.

Mathematically $X = f(L, K, l, E)$

When $X =$ output, $L =$ labour, $K=$ capital, $l =$ land and $E =$ Enterprise

The output in the dependent variable and all others are independent variables.

If any firm wants to increase its production it can do in two ways. Either it can increase all the units of required inputs or can increase the units of some inputs white others remain constant.

Production not only depends on the quantity of inputs but also depend upon the technique of production. The improvement in technology results in increased production, without any increase in inputs. In this case production function itself changes. The production function depends upon two things i) The quantity of production and ii) The range of production techniques. The possibility of substituting different factors of production play an important role in the production process.

Production Analysis:- The production of certain commodities requires use of two or more factors of production. The amount of production of any commodity uniquely depends upon the amounts of different factors used under certain technical conditions.

Suppose the quantity x of commodity X is produced by using amounts $a_1 a_2$ of factors A_1 and A_2 . Then the production function is $x = f(a_1 a_2)$.

We assume that only factor A_1 is varied and A_2 is held constant. Then the ratio $\frac{x}{a_1}$ is termed as average product of factor A_1 . The partial derivative $\frac{\partial x}{\partial a_1}$ is termed as marginal production of factor A_1

Law of variable proportions

Relation ship between average product (AP), Marginal production (MP) and total product (TP) curves.

Total product, average product and marginal product are related different ways. Their relationship is different in different stages of production.

Stage I MP begins to decline but AP reaches its maximum and MP curve cuts AP curve at its maximum.

Stage II AP and MP both fall and MP = 0, then T.P is maximum.

Stage III MP is negative: TP and AP both fall.

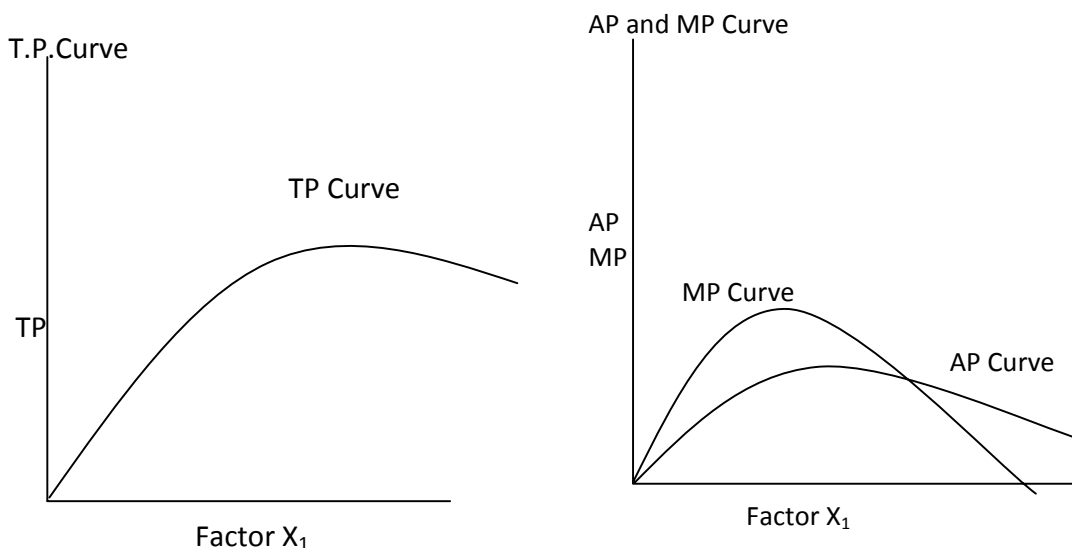
End of stage I is where AP is maximum and at this maximum AP = MP.

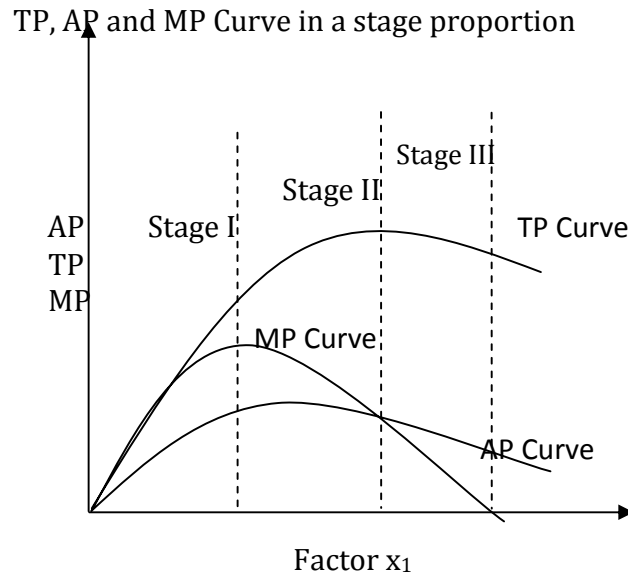
End of stage II is where MP = 0 and TP is maximum.

The three phases of law of variable proportions are

- i) Stage of increasing average returns
- ii) Stage of diminishing average returns
- iii) Stage of absolute diminishing returns.

Law of variable proportions is based on technological facts. The law of variable proportions operates by assuming that technological production remain unchanged.



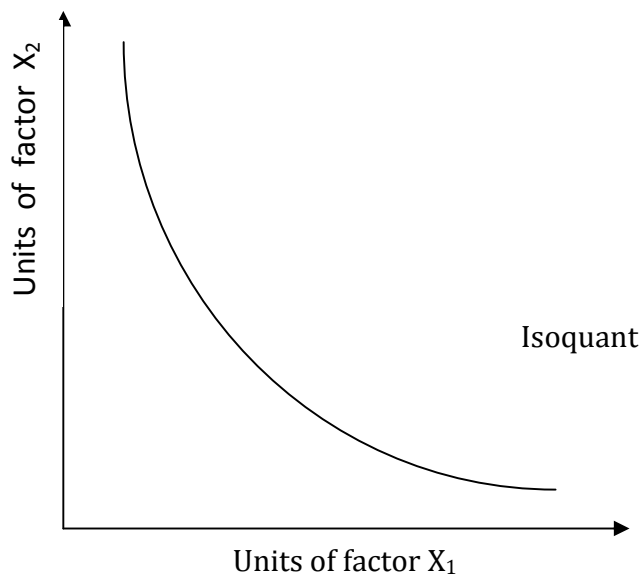


Isoquants

Isoquants or equal product curves are also known as production indifference curves. It is a curve which represents the different combinations of two factors of production which are capable of producing the same output. These curves may be infinite numbers. Different isoquants share different levels of output that can be obtained from different combinations of factors of production. The isoquant curve have the following properties.

- i) The isoquants slope down from left to right
- ii) Isoquants cannot cut each other.
- iii) Isoquants are convex to origin.

ISOQUANT



Producers Equilibrium

In the consumers theory, the consumer is in equilibrium when he gets maximum satisfaction for his fixed income. Likewise a producer also tried to maximize his output at minimum cost. If he attains such a point then he has no tendency to move further. Thus, producers equilibrium is that situation where he has no tendency to move further or where producers gain is maximum. The maximum gain can be achieved through two methods i) Maximisation of production and ii) Minimisation of production cost.

Maximisation of Production

Let the production function be $q = f(x_1, x_2)$ and cost function be $TC = x_1p_1 + x_2p_2$, where p_1 and p_2 be the unit prices of factors x_1 and x_2 respectively. The main aim of a producer is to attain the maximum output for the given level of inputs.

The constrained objective function is

$$Z = f(x_1, x_2) + \lambda [TC - x_1p_1 - x_2p_2]$$

Where λ is the Lagrange multiplier

Let $q = f(x_1, x_2)$ be the production function

Then $dq = \frac{\partial q}{\partial x_1} dx_1 + \frac{\partial q}{\partial x_2} dx_2$ is the rate of change in total product q with respect to change in inputs.

Since on being the same isoquant, output remains unchanged.

$$\therefore \quad = \frac{\partial q}{\partial x_1} dx_1 + \frac{\partial q}{\partial x_2} dx_2 = 0$$

$$f_1 dx_1 + f_2 dx_2 = 0$$

$$\therefore \quad \frac{dx_2}{dx_1} = -\frac{f_1}{f_2}$$

$\frac{dx_2}{dx_1}$ is called the rate of technological substitution. $\frac{dx_2}{dx_1}$ represents the marginal rate at which the producer can substitute input X_2 for X_1 or X_1 for X_2 , in order to maintain specific output level q .

The marginal rate of technical substitution

$$MRTS = \frac{dx_2}{dx_1} = -\frac{f_1}{f_2} = \frac{\frac{\partial q}{\partial x_1}}{\frac{\partial q}{\partial x_2}} = \frac{-M.P. \text{ of } x_1}{M.P. \text{ of } x_2}$$

$\frac{dx_2}{dx_1}$ represent the slope of isoquant having a negative sign.

MRTS of X_2 for x_1 is that amount of factor X_2 which the producer is ready to sacrifice to get an additional amount of factor X_1 .

On differentiating Z Partially, we get

$$\frac{\partial z}{\partial x_1} = f_1 - \lambda p_1, \quad \frac{\partial z}{\partial x_2} = f_2 - \lambda p_2$$

$$\text{And } \frac{\partial z}{\partial \lambda} = TC - x_1 p_1 - x_2 p_2$$

By applying the conditions for maximum

$$\text{We get } \frac{f_1}{f_2} = \frac{p_1}{p_2} \quad \text{where } f_1 = \frac{\partial q}{\partial x_1} \quad \text{and } f_2 = \frac{\partial q}{\partial x_2}$$

$$\text{i.e., } \frac{\text{M.P. of factor } x_1}{\text{M.P. of factor } x_2} = \frac{\text{Price of factor } x_1}{\text{Price of factor } x_2}$$

Minimumisation of Cost

Here, the objective is to minimize the cost at optimum output

Here the objective function is

$$Z = x_1 p_1 + x_2 p_2 + \lambda [q - f(x_1, x_2)]$$

Then the minimising condition are

$$\frac{\partial z}{\partial x_1} = p_1 - \lambda f_1 = 0$$

$$\frac{\partial z}{\partial x_2} = p_2 - \lambda f_2 = 0$$

$$\text{And } \frac{\partial z}{\partial \lambda} = q - f(x_1, x_2) = 0$$

Using these conditions we get

$$\frac{p_1}{p_2} = \frac{f_1}{f_2} \quad \text{OR} \quad \frac{\text{Price of factor } x_1}{\text{Price of factor } x_2} = \frac{\text{Marginal Product of } x_1}{\text{Marginal Product of } x_2}$$

Expansion Path

In the producer's equilibrium condition it is assumed that the producer's cost remains fixed. But if the change in the cost takes place while the prices of factors of production remain the same, then what will be the effect in the equilibrium point? When the isocost line changes, the producer's equilibrium will change. If we connect all the producer's equilibrium points of different isocost lines, we get the expansion path of a producer. The size of the expansion path depends upon the nature of the production function.

Problem: Find the firms expansion path if the production function is $q = 12 \log x_1 + 30 \log x_2$ and the input price of unit values of x_1 and x_2 respectively are P_1 and P_2 and $P_1 = 2, P_2 = 5$

Solution

Using Lagranges multiplication, the objective function be

$$Z = 12 \log x_1 + 30 \log x_2 + \lambda[TC - 2x_1 - 5x_2]$$

Then the maximising conditions are

$$\frac{\partial z}{\partial x_1} = 0 \quad \frac{\partial z}{\partial x_2} = 0 \quad \frac{\partial z}{\partial \lambda} = 0$$

$$\Rightarrow \frac{12}{x_1} - 2 \lambda = 0, \quad \frac{30}{x_2} - 5 \lambda = 0$$

$$\text{And } TC - 2x_1 - 5x_2 = 0$$

Using these conditions, we get

$$\frac{12}{x_1} = 2 \lambda \text{ and } \frac{30}{x_2} = 5 \lambda$$

On division, we get $\frac{\frac{12}{x_1}}{\frac{30}{x_2}} = \frac{2}{5}$

On simplification we get $x_1 = x_2$

Hence $TC = 2x_1 + 5x_1$

or $TC = 2x_2 + 5x_2$

$$\therefore x_1 = \frac{TC}{7}, \quad x_2 = \frac{TC}{7} \text{ and hence } \lambda = \frac{42}{TC}$$

These value constitutes the required expansion path

Elasticity of Substitution

The elasticity of substitution, σ , is defined as

$$\sigma = \frac{\text{Percentage change in factor quantity ratio}}{\text{Percentage change in factor Price ratio}}$$

$$\frac{\Delta \left(\frac{x_1}{x_2}\right) / \frac{x_1}{x_2}}{\Delta \left(\frac{p_2}{p_1}\right) / \frac{p_2}{p_1}} = \frac{\partial \log\left(\frac{x_1}{x_2}\right)}{\partial \log\left(\frac{p_2}{p_1}\right)}$$

If i) $\sigma = \infty$, then both factors x_1 and x_2 are perfect substitutes for each other.

ii) $\sigma > 0$ ie, $0 < \sigma < \infty$ then both factors x_1 and x_2 are close substitutes for each other.

If iii) $\sigma = 0$, then there is no possibility of any substitution between them.

Problem: If the profit remains unaltered, the ratio of amount of capital employed per unit of labour shifts from 10:10 to 12:11, given that rise in wages is 25%.

Determine the elasticity of substitution.

Solution:-

$$\sigma = \frac{\partial(K/L)/K/L}{\partial(P_L/P_K)/P_L/P_K} \quad \text{Where K = Capital, L = Labour}$$

$$\frac{K}{L} = \frac{10}{10} \text{ and changed to } \frac{K}{L} = \frac{12}{11}$$

$$\therefore \partial\left(\frac{K}{L}\right) = \frac{12}{11} - \frac{10}{10} = \frac{1}{11}$$

$$\therefore \frac{\partial(K/L)}{K/L} = \frac{1}{11} \div \frac{10}{10} = \frac{1}{11}$$

The wages are increased by 25%

The initial ratio of prices of labour and capital be $\frac{P_L}{P_K} = 1 : 1$

The wages are increased by 25%. So the ratio becomes $1.25 : 1 = \frac{1.25}{1} = \frac{5}{4}$

$$\therefore \text{change in } \frac{P_L}{P_K} = \partial\left(\frac{P_L}{P_K}\right) = \frac{5}{4} - \frac{1}{1} = \frac{1}{4}$$

$$\therefore \frac{\partial\left(\frac{P_L}{P_K}\right)}{\frac{P_L}{P_K}} = \frac{1}{4} \div \frac{1}{1} = \frac{1}{4}$$

$$\therefore \sigma = \frac{1}{11} \div \frac{1}{4} = \frac{4}{11}$$

i.e, $\sigma > 0$

MODULE III

EULER'S THEOREM

Euler's theorem states that when all factors of production are increased in a given proportion, the resulting output will also increase in a given proportion, provided each factor of production is paid the value of its marginal product and the total output is just exhausted.

If the production function $q = f(x_1, x_2)$ is a linear homogeneous function, then

$$q = x_1 \cdot \frac{\partial q}{\partial x_1} + x_2 \cdot \frac{\partial q}{\partial x_2}$$

Where q = total output, x_1 = unit of one input, x_2 = unit of second input

$$\frac{\partial q}{\partial x_1} = MP_{x_1} \quad \text{and} \quad \frac{\partial q}{\partial x_2} = MP_{x_2}$$

It tells us that factor x_1 will get its award equal to $\frac{x_1 \partial q}{\partial x_1}$ and x_2 will get $\frac{x_2 \partial q}{\partial x_2}$

Since if a firm pays the suppliers of an input its award equal to marginal physical product, the total output is just exhausted.

Euler's theorem plays an important part in the economic field especially in the field of distribution. The theorem suggests to a firm how the inputs should be employed. The input should be employed to that extent at which the price of factor is equal to its marginal revenue product. Thus it also suggests the determination of price of a factor of production.

Euler's theorem is based on following assumption.

- a) The law of constant returns to scale is being applied. It holds good only in the case of a linearly homogeneous production function of degree one.
- b) There is perfect competition in the market.
- c) It assumes perfect divisibility of the factors of production.
- d) The technology remains constant in the given time period.

Cobb – Douglas Production Function

The general form of Cobb – Douglas Production Function is

$$Q = AK^\alpha L^\beta \quad \text{where } Q = \text{output, } K = \text{capital, } L = \text{labour}$$

A is a constant α and β are positive parameters

K and L are the capital and labour inputs.

$$Q = AK^\alpha L^\beta$$

If $\alpha + \beta = 1$, the production function will operate under constant return to scale.

If $\alpha + \beta > 1$, there will be increasing returns to scale.

If $\alpha + \beta < 1$, there will be decreasing returns to scale.

Properties

a) If the inputs are increased by t times, the total output will also increase by t times

Proof

$$Q = AK^\alpha L^\beta$$

If K and L are increased by t times

$$\begin{aligned} Q^1 &= A (tK)^\alpha (tL)^\beta = t^{\alpha+\beta} \cdot AK^\alpha L^\beta \\ &= t \cdot AK^\alpha L^\beta \quad (\text{if } \alpha + \beta = 1) \\ &= tQ \end{aligned}$$

b) The production function is homogeneous of degree one.

Proof

If the production function is homogenous of degree one then the production is operating under constant returns to scale. i.e $\alpha + \beta = 1$

$$Q = AK^\alpha L^\beta$$

$$\text{i.e, } \log Q = \log A + \alpha \log K + \beta \log L$$

On differentiation partially with respect to K and L , respectively we get

$$\frac{\partial Q}{\partial K} = \frac{\alpha Q}{K} \quad \text{and} \quad \frac{\partial Q}{\partial L} = \frac{\beta Q}{L}$$

$$\begin{aligned} \text{Then } K \cdot \frac{\partial Q}{\partial K} + L \cdot \frac{\partial Q}{\partial L} &= \alpha Q + \beta Q = (\alpha + \beta)Q \\ &= Q \quad \text{if } \alpha + \beta = 1 \end{aligned}$$

c) If the production function is linear and homogeneous, then elasticity of substitution equal to unity (if $\alpha + \beta = 1$).

Proof

The elasticity of substitution σ ,

$$\sigma = \frac{\% \text{ change in factor quantity ratio}}{\% \text{ change in factor price ratio}}$$

$$= \frac{\partial \log(K/L)}{\partial \log\left(\frac{P_L}{P_K}\right)} = \frac{\frac{\partial(K/L)}{K/L}}{\frac{\partial(P_L/P_K)}{\frac{P_L}{P_K}}}$$

$$Q = AK^\alpha L^\beta$$

Differentiating with respect to K and L respectively,

We have

$$\frac{\partial Q}{\partial K} = A \cdot \alpha K^{\alpha-1} \cdot L^\beta \quad \text{and} \quad \frac{\partial Q}{\partial L} = A \cdot K^\alpha \beta \cdot L^{\beta-1}$$

On substitution we get

$$\frac{\partial Q/\partial L}{\partial Q/\partial K} = \frac{K}{L} \cdot \frac{\beta}{\alpha}$$

$$\text{Then } \sigma = \frac{\partial (K/L)/K/L}{\frac{\frac{\beta}{\alpha} \partial(K/L)}{\frac{\beta}{\alpha} \frac{K}{L}}} = 1$$

Hence the elasticity of substitution is unity.

d) α and β represents the capital share and labour share to the total output respectively.

Proof

$$Q = AK^\alpha L^\beta$$

$$\text{Then } \log Q = \log A + \alpha \log K + \beta \log L$$

Differentiating partially with respect to K, we get

$$\frac{1}{Q} \cdot \frac{\partial Q}{\partial K} = \frac{\alpha}{K}$$

$$\text{i.e., } \frac{K}{Q} \cdot \frac{\partial Q}{\partial K} = \alpha$$

$$\text{i.e., } \alpha = \frac{\text{Capital}}{\text{Output}} \times \text{M. P of capital}$$

$$= \frac{\text{Capital share}}{\text{Output}}$$

Similarly, differentiating partially with respect to L, we get

$$\beta = \frac{\text{Labour}}{\text{Output}} \times \text{Marginal Productivity of labour}$$

$$\text{i.e., } \beta = \frac{\text{Labour share}}{\text{Output}}$$

- e) α and β represent the elasticities of output with respect to capital and labour respectively.

Proof

$$Q = AK^\alpha L^\beta$$

$$\log Q = \log A + \alpha \log K + \beta \log L$$

Differentiating partially with respect to K, we get

$$\frac{1}{Q} \cdot \frac{\partial Q}{\partial K} = \frac{\alpha}{K}$$

$$\text{i.e., } \alpha = \frac{K}{Q} \cdot \frac{\partial Q}{\partial K}$$

$$\text{Similarly we get } \beta = \frac{L}{Q} \cdot \frac{\partial Q}{\partial L}$$

$$\begin{aligned} \text{Then } \alpha &= \frac{\partial Q / \partial K}{Q / K} \\ &= \frac{\% \text{ Change in total output}}{\% \text{ Change in capital input}} \end{aligned}$$

$$\begin{aligned} \text{Similarly } \beta &= \frac{\partial Q / \partial L}{Q / L} \\ &= \frac{\% \text{ Change in total output}}{\% \text{ Change in labour input}} \end{aligned}$$

- f) The expansion path of Cobb – Douglas production function is linear homogeneous and passes through origin.

Proof

Proceeding as in the earlier properties, we get

$$\frac{\partial Q}{\partial K} = \frac{\alpha Q}{K} \quad \text{and} \quad \frac{\partial Q}{\partial L} = \frac{\beta Q}{L}$$

$$\text{i.e., } MP_K = \frac{\alpha Q}{K} \quad \text{and} \quad MP_L = \frac{\beta Q}{L}$$

Using the condition for optimization, we get

$$\frac{MP_K}{MP_L} = \frac{P_K}{P_L} \quad \text{i.e., } \frac{\alpha Q}{K} \cdot \frac{L}{\beta Q} = \frac{P_K}{P_L}$$

$$\text{i.e., } \frac{\alpha}{\beta} \cdot \frac{L}{K} = \frac{P_K}{P_L}$$

$$\text{i.e., } \alpha \cdot L \cdot P_L = \beta \cdot K \cdot P_K$$

$\alpha \cdot L \cdot P_L - \beta \cdot K \cdot P_K = 0$, Which is the expansion path of Cobb – Douglas production function which is linear homogeneous and Passes through origin.

Importance of Cobb – Douglas production function

Cobb – Douglas production function is most commonly used function in Economics. This function is helpful in wage determination principles. This function is helpful in explaining marginal productivity principles and used to explain product exhaustion theorem. The parameter α and β represent elasticity coefficients. They are helpful in comparing international levels. This function is helpful in the study of different laws of returns to scale. This function plays an important role in economic field. It is used to determine wage policies under sector comparisons, substitutability and the degree of homogeneity.

Limitations: Cobb – Douglas production function has certain draws backs.

- a) If contains only two inputs, capital and labour. There are several other factors which are equally important in production.
- b) The production function operates under constant returns to scale. The conditions of increasing returns and diminishing returns are ignored.
- c) The function assumes that technological conditions remains constant. But it is not time. The output conditions vary with the change in technological conditions.
- d) The function assumes that all inputs are homogeneous. In practice, all factors are not equally efficient.
- e) It assumes perfect competition. But if there is imperfect competition this function does not hold good.
- f) This function ignores the negative marginal productivity of a factor.

Constant Elasticity Production Function (C.E.S. function)

Cobb – Douglas production function is a function whose elasticity of substitution is unity every where. Any production whose elasticity of substitution is constant but other than unity is called Constant Elasticity substitution production function or C.E.S. Production function.

C.E.S Production function represents the more general form of production technique than Cobb – Douglas production function. In C.E.S. Production function, $\sigma \neq 1$, but a constant C.E.S. Production function has wider scope, substitutability and efficiency.

C.E.S. Production function removes all the difficulties and unrealistic assumptions of Cobb-Douglas production function. Cobb – Douglas production function is a special case of C.E.S. production function.

A.C.E.S. Production function is expressed as

$$Q = r [\delta c^{-\alpha} + (1 - \delta)N^{-\alpha}]^{-\frac{V}{\alpha}}, \quad (r > 0 \quad 0 < \delta < 1 \quad \text{and } \sigma > -1)$$

Where Q = output, C = Capital input

N = labour input, α = Substitution parameter

$1 - \delta$ = labour intensity coefficient and

V = degree of freedom

C.E.S. Production function has several advantages

It has some limitations

- a) It consider only two factor N and C.
- b) C.E.S. function contains only one parameter, V, which is affected by the scale of operation and technological change, There two factors may affect the degree of returns to scale but cannot distinguish them separately.
- c) It is assumed that in this function that ' σ ' changes in response to technology only and factor proportions do not affect it.

While the empirical study shows that the elasticity of substitution also changes due to changed factor proportions. The function has ignored this important fact.

Returns to scale

A production function exhibits constant returns to scale if when all inputs are increased by a given proportion, K, output increases by the same proportion. If output increase by a proportion greater than K, there are increasing returns to scale. If the output increase by a proportion smaller than K, there are diminishing returns to scale. If a production function is homogeneous of degree greater than, equal to, or less than one return to scale are increasing, constant or diminishing. Economic analysis employs the Cobb – Douglas production function $Q = AK^\alpha L^\beta$, where Q is the quantity output, K is the capital input. L is labour input. α , β and A are parameters. If $\alpha + \beta = 1$, Cobb – Douglas function shows constant returns to scale.

If $\alpha + \beta > 1$, increasing returns to scale and if $\alpha + \beta < 1$, diminishing returns to scale. Optimisation of Cobb – Douglas production function is performed subject to a budget constraint.

For further reading refer to

- i) Edward.T. Dowling:
Introduction to Mathematical Economics (3rd edition)
- ii) S.P. Sing. A.K. Parashar, H.P. Singh
Econometrics and Mathematical Economics
- iii) C.R. Kothari.
An introduction to operation Research (3rd edition)

MODULE IV

INVESTMENT DECISIONS & ANALYSIS OF RISK/UNCERTAINTY

NATURE OF INVESTMENT DECISIONS

Investment decisions are also termed as capital investment decisions. They represent a long term commitment. Such decisions form the framework for future development of an organization. They have a profound effect on its future earnings and growth. They are a major determinant of efficiency and competitive strength of a business firm. To reach such decisions the following steps are to be followed.

- a. Formulation of objectives.
- b. Establishing existing demand and future demand for a service.
- c. Determine whether the demand can be met with existing productive capacity or whether new investment is required to increase supply.
- d. Consider alternative projects that would enable the objective to be achieved.
- e. Appraise the investment projects.

Appraisal Necessary

Investment appraisal is necessary in case of investment decisions. Investment appraisal happens to be an aid to decision making in the allocation of scarce resources to competing uses of both in public and private sectors. If the appraisal indicates that a project can be undertaken profitably, then the investment project that shows the biggest return on the investment capital should be chosen. But if it is a case of public sector investment projects which are to provide social services then the criterion adopted in selecting such project happens to be one of maximizing social benefit. Investment decisions in public sector are not only governed by an analysis of accounting costs and benefits but also influenced by several other factors such as political or social pressure.

For proper and appropriate investment decisions the information required are given below.

- a. Estimates of capital outlay for the project.
- b. Estimates of the future cash flows of the project
- c. Estimate of the availability of capital along with the cost of capital.
- d. Set of standards for the selection of projects for execution. (bigger benefits are preferable to smaller ones; early benefits are preferable to later benefits.)

There are several appraisal techniques for judging the profitability of new investments in assets.

They may be classified as Non-discounting and Discounting techniques.

Non-discounting techniques are

1. Payback method.
2. Average rate of return method (ARR Method)

Discounting techniques are

1. Net present value of NPV method
2. Profitability Index (P.I)
3. Net terminal value or NTV method.
4. Internal rate of return or IRR method.

PAYBACK METHOD

Under this method projects are assessed on the basis of the period that it would take to generate sufficient income to repay the original outlay. Payback is simply a measure of the time required for the cash income before charging depreciation but after payment of tax from a project to return the initial investment to the firms treasury. Payback occurs when the cumulative cash inflows from a project minus the initial investment is equal to zero.

The advantages of payback method are

1. It is the most simplest criterion for appraising investment proposals since shorter payback is preferred to longer payback.
2. It takes into account preference for an early repayment of capital invested, the capital is then available for other uses.
3. The method is preferred because of the fact that the returns beyond three-four years are quite uncertain.

The method has some drawbacks. It fails to allow returns after the end of the payback period. This method gives weights only to cash flows before the recovery of investment zero weights to all subsequent receipts. It over emphasizes liquidity aspect. It ignores time value of money. Projects with longer gestation periods can never be launched if we use this method since they begin to give return after a lapse of considerable period of time.

Payback method is popular method for evaluating investment proposals.

AVERAGE RATE OF RETURN or ARR method

This method of evaluating investment proposals is primarily based on accounting approach rather than cash flow approach. This method consists of aggregating all the

earnings from the project after depreciation and dividing them by the project's life to obtain the average annual earnings and then this average earning is divided by the averaged investment and the resulting figure is multiplied by 100 so as to obtain the desire ARR.

$$\text{ARR} = (\text{average annual earnings}) / (\text{average investment}) \times 100.$$

NET PRESENT VALUE or NPV method

Payback and ARR methods do not consider the time value of money which is a factor of basic importance in evaluating investment proposals. The money received today is worth more than the same amount to be received at a later time. It is preferable to have money now than the same amount in future. The future is uncertain and in mean time capital may be earning a return. Discounted cash flow (DCF) technique dos consider the time value of money and is considered more objective basis for evaluating investment proposals. Under DCF technique the cash flows are discounted at a certain rate. i.e; the rate must be earned so that the value remains constant over a period of time. DCF method is superior on another ground as well since it does consider all benefits and costs of the project during the entire period of its life. DCF methods are mainly of two types. Net present value method and the internal rate of return method. Two variants of net present value method happens to be 1. Net terminal value method and 2. The profitability index.

NPV is one of the DCF techniques of evaluating investment proposals. In applying NPV method the following steps are to be followed.

- a. First the annual net cash flow expected from a project is calculated by estimating all cash receipts and deducting from them all expenditure arising out of the project.
- b. The net cash flow is then discounted to give its present value. The rate used to discount the cash flow is the required rate of return. The minimum rate of return expected to be earned from the investment projects.
- c. The net present value of an investment proposal is then computed. It is equal to the sum of the present value of its annual net cash flows after tax less the investment's initial outlay.

$$\text{NPV} = \sum_{t=1}^n \frac{\text{CF}_t}{(1+r)^t} - I_0$$

where CF_t = net cash flow in time period t, r = rate of discount
 I_0 = Initial cash outlay, n = projects expected life

Accept project if $\text{NPV} > 0$, Reject if $\text{NPV} < 0$ If $\text{NPV} = 0$, a promotion of indifference

The advantage of NPV method are

- a. This method deals with cash flows rather than accounting profits. Thus, it is sensitive to true the timing on the benefits assuming from the investment.
- b. It recognizes the time value of money and allows comparison of benefits costs in a logical manner.
- c. This method is consistent with the goal of maximizing the shareholder's wealth since projects under this method are accepted only if they have positive NPV which infact increases the value of the firm.
- d. This method is specially useful in evaluating mutually exclusive projects.

The major limitation of this method is that it requires detailed long term estimates of the incremental benefits and costs. There may also arise difficulty in deciding the appropriate rate of discount for finding the present values of the cash flows coming in over project's life. The relative desirability of an investment proposal may change with a change in the discount rate. Another shortcoming of this method is that it may not give dependable results in case of projects involving different outlays or having different effective lives.

NPV method is theoretically most correct criterion and is frequently used in practice.

THE PROFITABILITY INDEX (P.I)

The profitability index is the ratio of the present value of the future net cash flows to the initial outlay of the project. Thus index provides a relative measure for judging desirability and evaluating the worth of an investment proposal

$$P.I = \frac{\sum_{t=1}^n \frac{CF_t}{(1+r)^t}}{I_0}$$

Where CF_t = net cash flow in time period t

r = rate of discount

n = The projects expected life

I_0 = Initial cash outlay

Accept the project if its P.I > 1

Reject the project if its P.I < 1

Remain indifferent between acceptance and rejection if PI = 1

The Advantages are

1. It uses cash flows.
2. It recognizes the time value of money.
3. It is consistent with the goal of maximizing the shareholders wealth.

The main disadvantage of it is that it requires detailed long-term forecasts of the incremental benefits and costs. It also poses difficulty in determining appropriate discount rate.

NET TERMINAL VALUE (NTV) method

Net terminal value method is an important variant of NPV method but separates the timings of the cash inflows and outflows more distinctly. The assumption underlying this method is that each cash inflow is re-invested in another asset at a certain rate of return from the moment it is received till the termination of project. Under this method we calculate the compounded re-investment cash flows and then take their total and of this total and present value is determined at a given rate of discount. From this present value we subtract the original outlay so as to obtain the net terminal value, NTV.

Accept the project if NTV is positive

Reject the project if NTV is negative

Remain indifferent between acceptance and rejection of the project if NTV is zero

This method has several disadvantages. It evaluates the investment worth in a better manner. This method is easier to understand. The main drawback of this method lies in determining the future rates of interest at which the cash inflows received are to be reinvested.

INTERNAL RATE OF RETURN (I.R.R) METHOD

Internal rate of return is the discount rate corresponding to which the NPV of a project happens to be zero. Internal rate of return may be defined as the discount rate that equates the present value of project's future net cash flow with the project's initial cash outlay.

Accept investment proposal if its IRR is greater than the required rate of return by investors. Reject the investment proposal if its IRR is less than the required rate of return by investors.

IRR being one of the discounted cash flow criteria is consistent and provides the same accept-reject decision for an investment proposal as is being provided by either NPV method or PI.

A project have only one NPV and one PI. But under certain situations it can have more than one IRR. This is the problem of multiple IRRs.

IRR method has certain advantages.

- a. It takes into account the total cash inflow and outflows.
- b. It recognizes the time value of money.
- c. It is consistent with the objective of maximizing shareholder's wealth.
- d. IRR method is easy to understand.

IRR method has some limitations.

- a. It involves tedious calculations.
- b. It requires detailed long-term forecasts of incremental benefits and costs.
- c. Possibility of multiple IRRs remains in some situations which reduces the utility of this method.

ANALYSIS OF RISK/UNCERTAINTY

While making investment decisions we make an estimate of benefits, in terms of cash flows, to be derived from the project on the basis of certain assumptions. But the actual flow depends on several other factors such as price of product, volume of sales, the rate of inflation etc. The actual cash flows may not correspond to the estimated cash flow. The variation between actual and estimated cash flows of the investment project is termed as risk. The risk of a project depends upon the degree of variability between the two. If the variability happens to be greater, the project is considered more risky.

Risk refers to the set of unique outcomes for a given event which can be assigned probabilities while uncertainty refers to the outcomes of a given event which are unsure to be assigns probabilities. In risk situation the probability distribution is known to the decision maker. When uncertainty exists, the decision maker has to compute subjective probability on the basis of his experience. Variability is less in the situation of risk compared to variability in uncertain situations.

The important risk handling approaches are

- a. Risk-adjusted discount rate approach (RAD approach)
- b. Certainty-equivalent approach

- c. Probability distribution approach (Frederick Hillier models)
- d. Decision –trees approach
- e. Simulation approach
- f. Sensitivity analysis.

Risk-adjusted discount rate approach

It is the most simple and widely used approach for incorporating risk into investment decisions. Under this approach the discount rate used in the present value calculation is so adjusted as to incorporate the amount of risk inherent in a project. For more risky projects the higher discount rates and for less risky projects lower discount rates are used for present value calculations. RAD approach can be used with NPV and IRR techniques of evaluating investment proposals. In case of NPV method, the NPV will be calculated using the risk-adjusted rate and the project would be accepted in NPV is positive. In case of IRR, the internal rate of return would be compared with the risk-adjusted rate of return. If IRR exceeds the risk-adjusted rate, the project would be accepted.

RAD approach is simple and easy to understand and possesses the merit of operational feasibility. Thus it is frequently used in practice. RAD approach is a crude method of incorporating risk into investment decisions.

Certainty-Equivalent Approach

Under this approach of incorporating risk in evaluating investment proposals, the expected cash flows are so adjusted as to take care of the inherent risk of the project. The following steps are involved under this approach.

- a. As the expected cash flow are to be modified, the first step happens to be the determination of basis for modifying the cash flows to adjust for risk. This is done by computing the certainty-equivalent coefficient which shows the relationship between certain and uncertain cash flows. This coefficient is calculated when we divide riskless cash flow by risky cash flow.
Certainty-equivalent coefficient = Riskless cash flow/risky cash flow
- b. With the help of certainty = equivalent coefficients the expected cash flows are converted to certainty equivalents.
- c. The present values of cash flows calculated are worked out using the risk-free discount rate or the rate which appropriately reflects the time value of money.
- d. For accepting the project or not accepting it, either NPV or IRR method may be used.

For the purpose of computing NPV under certainty-equivalent approach we use

$$NPV = \sum_{t=0}^n \frac{a_t CFAT_t}{(1+i)^t} - C$$

Where a_t = Certainty equivalent coefficient for the year t.

i = riskless interest rate

C = Cash out flow (or the initial investment)

This approach is simple and easy to understand. It is superior to time-adjusted discount rate approach.

Probability Distribution Approach

This approach is also known as statistical distribution approach. Under this approach, the degree of risk associated with a project is measured in terms of the variance of NPV distribution. Investment decisions are taken on the basis of mean and standard deviation values of the net present value distribution. In this method net cash flow from an investment in each period is viewed as a random variable which can assume any one of the possible values. It requires that probability distribution of cash flows for each of the years be obtained and made use of it. The expected value and variance of NPV distribution are calculated as

Expected value of NPV

$$E(NPV) = \mu = \sum_{t=0}^n \frac{a_t \bar{C}_t}{(1+i)^t}$$

Where $E(NPV)$ = expected NPV of the project

\bar{C}_t = mean of the cash flows in year t

i = rate of discount

Variance of NPV (Hillier Model)

- i. Independent cash flows: - When cash flows of one period are not related to cash flows of another period.

$$\text{Variance of NPV; } V(NPV) = \sigma^2 = \sum_{t=0}^n \frac{\sigma_t^2}{(1+i)^{2t}}$$

Where σ_t^2 = Variance of cash flows in year t, i = rate of discount

- ii. Perfectly correlated cash flows : when cash flows are perfectly correlated,

$$V(NPV) = \sigma^2 = \left[\sum_{t=0}^n \frac{\sigma_t}{(1+i)^t} \right]^2$$

Where σ_t = Standard deviation of cash flows in year t. i = rate of discount

- iii. Mixed cash flows: (where a part of cash flows are perfectly correlated and part are independent)

$$V(\text{NPV}) = \sigma^2 = \sum_{t=0}^n \frac{V(C_t^1)}{(1+i)^{2t}} + \left[\sum_{t=0}^n \frac{\sqrt{V(C_t^{11})}}{(1+i)^t} \right]^2$$

C_t^1 = independent cash flows in year t, i = rate of discount

C_t^{11} = Correlated cash flows in year t, correlated with any other period.

V = Variance cash flow streams

$$\text{Coefficient of Variation} = \frac{\text{Standard deviation}}{E(\text{NPV})} \times 100$$

A higher coefficient of variation indicates higher risk in the concerned project

Decision Trees approach

Decision trees approach is considered very useful and convenient approach for incorporating risk into investment decisions. Under this approach the problems of investment decisions are depicted in the form of diagrams on the basis of which the best course of action is suggested. The main feature of this approach is that it takes into account the impact of all probabilistic estimates of potential outcomes. This approach particularly suits those situations where-in decisions at one point of time also affected decisions at some later date. Projects requiring sequence of decisions over time and having several possible outcomes are most suitable to be tackled by decision tree approach.

Simulation approach

Simulation approach may be used in investment decisions especially when due to uncertainties a satisfactory mathematical model cannot be used. Simulation technique can be used to approximate NPV or IRR and its dispersion about the expected value. The following steps are involved in simulation approach.

- The various factors influencing cash and NPV of an investment proposal are identified.
- The probability distributions of each of identified factors are obtained either using past data or by an experience of decision maker.
- Then the values of each of variables are simulated with the help of random numbers. A value for each of the factors is then selected at random.

- d. On the basis of the set of values so obtained NPV or IRR is computed for each combination of identified factors.
- e. Repeat the above procedure to develop probability distribution of NPV or IRR and calculate mean standard deviation and statistical parameters. Then a decision is taken on the related proposal.

Sensitivity analysis

Sensitivity analysis is also a technique of incorporating risk in investment decisions.

Following steps are involved in using sensitivity analysis in investment decisions.

- a. Data relating to cash flows of an investment proposal are broken down into some major variables like initial outlay required, wage rates, selling prices, interest rates, etc. A model of the project is drawn up on the basis of some estimated values of related variables.
- b. Each variable is considered in turn and its value is changed both upward and downward from its original estimate.
- c. The NPV or IRR values recomputed by taking each new estimate in turn.
- d. With recomputed NPV or IRR values an effort is made to determine as to with which variable fluctuations the given project is particularly sensitive.

Sensitivity analysis serves as a guide and indicate areas where additional effort to produce better estimates may be made. In view of this sensitivity analysis is widely used in practice.

For further reading and reference:

An introduction to operations research by KOTHARI. C.R.
