REAL ANALYSIS

STUDY MATERIAL

B.Sc. Mathematics

VI SEMESTER

CORE COURSE

(2011 Admission)

UNIVERSITY OF CALICUT

SCHOOL OF DISTANCE EDUCATION

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CONTINUOUS FUNCTIONS ON INTERVALS

Definitions A function $f : A \to \mathbb{R}$ is said to be **bounded on** A if there exists a constant $M > 0$ such that

$$|f(x)| \le M \qquad \text{for all } x \in A.$$

That is, a function $f : A \to \mathbb{R}$ is bounded on the set A if its range $f(A)$ is a bounded set in $\mathbb{R}$.

$f : A \to \mathbb{R}$ is **unbounded on** A, if f is not bounded on A.

i.e., f is unbounded on A if given any $M > 0$, there exists a point $x_M \in A$ such that

$$|f(x_M)| > M .$$

Example We now give an example of a continuous function that is not bounded. Consider the function f defined on the interval $A = (0, \infty)$ by

$$f(x) = \frac{1}{x}$$

is not bounded on A because for any $M > 0$ we can take the point $x_M = \dfrac{1}{M+1}$ in A to get $f(x_M) = \dfrac{1}{x_M} = M + 1 > M$. However, being the quotient two continuous functions 1 and x, f is continuous on A.

We now review some basic definitions and theorems.

Cluster Point: Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a **cluster point** of A if for every $¢ > 0$ there exists at least one point $x \in A$, $x \ne c$ such that $|x - c| < ¢$.

i.e., A point $c \in \mathbb{R}$ is a cluster point of $A \subseteq \mathbb{R}$ if for every $¢$-neighborhood $V_¢(c) = (c - ¢, c + ¢)$ of c contains at least one point of A distinct from c.

The cluster point c may or may note be a member of A, but even if it is in A, it is ignored when deciding whether it is a cluster point of A or not, since we explicitly require that there be points in $V_¢(c) \cap A$ distinct from c in order for c to be a cluster point of A

Example We now show that $A = \{1, 20, 323\}$, the set consisting of three elements 1, 20 and 323, has no cluster point.

The point 1 is not a cluster point of A, since choosing $¢ = \frac{1}{2}$ gives a neighborhood of 1 that contains no points of A distinct from 1. The same is true for the point 20 and 323. Also, a real number other than 1, 20 and 323 cannot be a cluster point. For, if $c \in \mathbb{R}$ with $c \ne 1$, $c \ne 20$, and $c \ne 323$, then choose $¢ = \min\{|c-1|, |c-20|, |c-323|\}$. Then $V_¢(c) \bigcap A = \varnothing$. Hence A has no cluster point.

Theorem A number $c \in \mathbb{R}$ is a cluster point of a subset A of $\mathbb{R}$ if and only if there exists a sequence (a_n) in A such that $\lim(a_n) = c$ and $a_n \ne c$ for all $n \in \mathbb{N}$.

Example Let A be the open interval $A=(2, 3)$. All the points of A are cluster points of A. Also the points 2, 3 are cluster points of A, though they do not belong to A. Hence every point of the closed interval $[0, 1]$ is a cluster point of A.

Example We now show that A finite set has no cluster point.

Let F be a finite set. No number $c \in \mathbb{R}$ is a cluster point of F as it is not possible to find a sequence (a_n) in F such that $\lim(a_n)=c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.

Example The infinite set $\mathbb{N}$ has no cluster point. No number $c \in \mathbb{R}$ is a cluster point of $\mathbb{N}$ as it is not possible to find a sequence (a_n) in $\mathbb{N}$ such that $\lim(a_n)=c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.

Example The set $B=\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ has only the point 0 as a cluster point.

Example If $I=[0, 1]$, then the set $C=I \cap \mathbb{Q}$ consists of all the rational numbers in I. It follows from Density Theorem ("If x and y are real numbers with $x<y$, then there exists a rational number r such that $x<r<y$") that every point in I is a cluster point of C.

Definition Let $A \subseteq \mathbb{R}$, and let c be a cluster point of A. For a function $f : A \to \mathbb{R}$, a real number L is said to be a **limit of** f at c if, for every number £ >0, there exists a corresponding number ¢ >0 such that if $x \in A$ and $0<|x-c|<$ ¢, then $|f(x)-L|<$ £. In that we write $\lim_{x \to c} f(x)=L$.

Since the value of ¢ usually depends on £, it may be denoted by ¢(£) instead of ¢ to emphasize this dependence.

The inequality $0<|x-c|$ is equivalent to saying $x \neq c$.

Theorem If $f : A \to \mathbb{R}$ and if c is a cluster point of A, then f can have at most one limit at c.

Theorem Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. Then the following statements are equivalent.

(i) $\lim_{x \to c} f(x)=L$.

(ii) Given any £ -neighborhood $V_{£}(L)$ of L, there exists a ¢ -neighborhood $V_{¢}(c)$ of c such that if $x \neq c$ is any point in $V_{¢}(c) \cap A$, then $f(x)$ belongs to $V_{£}(L)$.

Important Limits of Functions

(1) $\lim_{x \to c} k = k$. (2) $\lim_{x \to c} x = c$.

(3) $\lim_{x \to c} x^2 = c^2$. (4) $\lim_{x \to c} \frac{1}{x} = \frac{1}{c}$ if $c>0$.

The following important formulation of limit of a function is in terms of limits of sequences.

Sequential Criterion for Limits: Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. Then the following statements are equivalent.

(i) $\lim_{x \to c} f(x)=L$.

(ii) For every sequence (x_n) in A that converges to c such that $x_n \neq c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to $f(c)$.

Divergence Criterion: Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A.

(a) If $L \in \mathbb{R}$, then f does **not** have limit L at c if and only if there exists a sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence (x_n) converges to c but the sequence $(f(x_n))$ does **not** converge to $f(c)$.

(b) The function f does **not** have a limit at c if and only if there exists a sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence (x_n) converges to c but the sequence $(f(x_n))$ does **not** converge in $\mathbb{R}$.

Remark The assertion (a) in theorem enables us to show that a certain number L is *not* the limit of a function at a point, while (b) says that the function *does not have* a limit at a point.

We list some examples, which works on the basis of Divergence Criteria. The details are avoided as it is not mentioned in the syllabus.

Example If $f(x) = \dfrac{1}{x}$ $(x \neq 0)$, then $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \dfrac{1}{x}$ doesnot exist in $\mathbb{R}$.

The **signum function** sgn is defined by

$$\operatorname{sgn}(x) = \begin{cases} +1 & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -1 & \text{for } x < 0. \end{cases}$$

Then $\lim\limits_{x \to 0} \operatorname{sgn}(x)$ doesnot exist. We also note that $\operatorname{sgn}(x) = \dfrac{x}{|x|}$ for $x \neq 0$.

$\lim\limits_{x \to 0} \sin(1/x)$ doesnot exist in $\mathbb{R}$.

Properties of Limits

The following rules hold if $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ (L and M real numbers).

1. *Sum Rule*: $\lim\limits_{x \to c} [f(x) + g(x)] = L + M$

   i.e., the limit of the sum of two functions is the sum of their limits.
2. *Difference Rule*: $\lim\limits_{x \to c} [f(x) - g(x)] = L - M$

   i.e., the limit of the difference of two functions is the difference of their limits.
3. *Product Rule*: $\lim\limits_{x \to c} \left[f(x) \cdot g(x) \right] = L \cdot M$

   i.e., the limit of the product of two functions is the product of their limits.
4. *Constant Multiple Rule*: $\lim\limits_{x \to c} kf(x) = kL$ (any number k)

   i.e., the limit of a constant times a function is that constant times the limit of the function.
5. *Quotient Rule*: $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, $M \neq 0$

   i.e., the limit of the quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.
6. *Power Rule*: If m and n are integers, then

   $\lim\limits_{x \to c} [f(x)]^{\frac{m}{n}} = L^{\frac{m}{n}}$, provided $L^{\frac{m}{n}}$ is a real number.

   i.e., the limit of any rational power of a function is that power of the limit of the function, provided the latter is a real number.

Definition Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$, and let $c \in A$. We say that f is **continuous at** c if, for every number $£ > 0$ there exists $¢ > 0$ such that if x is any point of A satisfying $|x - c| < ¢$, then $|f(x) - f(c)| < £$.

Definition If f fails to be continuous at c, then f is **discontinuous at** c.

Similar to the definition of limit, the definition of continuity at a point can be formulated very nicely in terms of neighborhoods. .

Theorem A function $f: A \to \mathbb{R}$ is continuous at point $c \in A$ if and only if given any $£$-neighborhood $V_£(f(c))$ of $f(c)$ there exists a $¢$-neighborhood $V_¢(c)$ of c such that if x is any point of $A \cap V_¢(c)$, then $f(x)$ belongs to $V_£(f(c))$, that is

$$f(A \cap V_¢(c)) \subseteq V_£(f(c)).$$

If $c \in A$ is a cluster point of A, then a comparison of Definitions 2 and 3 show that f is continuous at c if and only if

$$f(c) = \lim_{x \to c} f(x). \qquad \ldots(1)$$

Thus, if c is a cluster point of A, then three conditions must hold for f to be continuous at c:

(i) f must be defined at c (so that $f(c)$ makes sense),

(ii) the limit of f at c must exist in $\mathbb{R}$ (so that $\lim_{x \to c} f(x)$ makes sense), and

(iii) $f(c) = \lim_{x \to c} f(x)$.

If $c \in A$ is not a cluster point of A, then there exists a neighborhood $V_¢(c)$ of c such that $A \cap V_¢(c) = \{c\}$. Thus we conclude that f is continuous at a point $c \in A$ that is not a cluster point of A. Such points are often called **isolated points** of A. They are of little practical interest to us, since they have no relation to a limiting process. Since continuity is automatic for such points, we generally test for continuity only at cluster points. Thus we regard condition (1) as being characteristic for continuity at c.

Similar to Sequential Criterion for Limits (Theorem 4), we have the Sequential Criterion for Continuity.

Sequential Criterion for Continuity: A function $f: A \to \mathbb{R}$ is continuous at the point $c \in A$ if and only if for every sequence (x_n) in A that converges to c, the sequence $(f(x_n))$ converges to $f(c)$.

Discontinuity Criterion: Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$, and let $c \in A$. Then f is discontinuous at c if and only if there exists a sequence (x_n) in A such that (x_n) converges to c, but the sequence $(f(x_n))$ does not converge to $f(c)$.

Example $f(x) = \dfrac{1}{x}$ is not continuous at $x = 0$.

The signum function sgn is *not* continuous at $x = 0$.

Definition Let $A \subseteq \mathbb{R}$, and let $f: A \to \mathbb{R}$. If B is a subset of A, we say that f is **continuous on the set** B if f is continuous at every point of B.

Example **13** The following functions are continuous on $\mathbb{R}$.

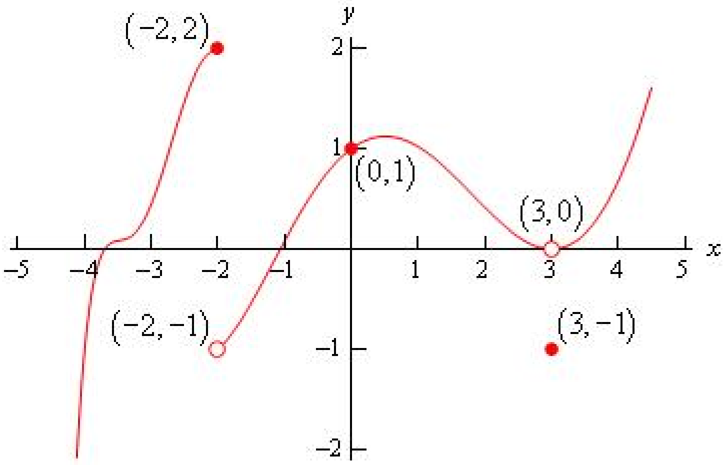
- The constant function $f(x) = b$.
- The identity function $g(x) = x$.
- $h(x) = x^2$.

Example $f(x) = \frac{1}{x}$ is continuous on $A = \{x \in \mathbb{R} : x > 0\}$.

Example The function $f(x) = \sin(1/x)$ for $x \neq 0$ is not continuous at 0 as $\lim_{x \to 0} \sin(1/x)$ does not exist.

Example The function $f(x) = x \sin(1/x)$ for $x \neq 0$ is not continuous at 0 as $f(x)$ is not defined at 0.

Example Given the graph of *f(x)*, shown below, determine if *f(x)* is continuous at $x = -2$, $x = 0$, $x = 3$.



Solution

To answer the question for each point we'll need to get both the limit at that point and the function value at that point. If they are equal the function is continuous at that point and if they aren't equal the function isn't continuous at that point.

First $x = -2$.

$$f(-2) = 2 \text{ and } \lim_{x \to -2} f(x) \text{ doesn't exist.}$$

The function value and the limit aren't the same and so the function is not continuous at this point. This kind of discontinuity in a graph is called a **jump discontinuity**. Jump discontinuities occur where the graph has a break in it as this graph does.

Now $x = 0$. .

$$f(0) = 1 \text{ and } \lim_{x \to 0} f(x) = 1 \text{ so that . } \lim_{x \to 0} f(x) = 1 = f(0).$$

The function is continuous at this point since the function and limit have the same value.

Finally $x = 3$. .

$$f(3) = -1 \text{ and } \lim_{x \to 3} f(x) = 0$$

The function is not continuous at this point. This kind of discontinuity is called a **removable discontinuity**. Removable discontinuities are those where there is a hole in the graph as there is in this case.

Example The function

$$F(x) = \begin{cases} x\sin(1/x) & \text{for } x \neq 0, \\ 0 & \text{for } x \neq 0, \end{cases}$$

is continuous at 0.

Combinations of Continuous Functions:

Let $A \subseteq \mathbb{R}$ and let f and g be functions that are defined on A to $\mathbb{R}$ and let $c \in A$. If k is a constant and f and g are continuous at c, then

(i) $f+g$, $f-g$, fg, and kf are continuous at c.

(ii) f/g is continuous at c, provided $g(x) \neq 0$ for all $x \in A$.

The next result is an immediate consequence of Theorem 10, applied to every point of A.

Theorem Let $A \subseteq \mathbb{R}$ and let f and g be continuous on A to $\mathbb{R}$. If k is a constant and f and g are continuous on A, then

(i) $f+g$, $f-g$, fg, and kf are continuous on A.

(ii) f/g is continuous on A, provided $g(x) \neq 0$ for all $x \in A$.

Examples A polynomial function is continuous on $\mathbb{R}$.

Rational function r defined by

$$r(x) = \frac{p(x)}{q(x)},$$

as the quotient of two polynomials $p(x)$ and $q(x)$, is continuous at every real number for which it is defined.

The trigonometric functions sine and cosine functions are continuous on $\mathbb{R}$.

The trigonometric functions tan, cot, sec, and csc are continuous where they are defined. For instance, the tangent function is defined by

$$\tan x = \frac{\sin x}{\cos x}$$

provided $\cos x \neq 0$ (that is, provided $x \neq (2n+1)$® $/2,\ n \in \mathbb{Z}$). Since sine and cosine functions are continuous on $\mathbb{R}$, it follows that the function tan is continuous on its domain.

Theorem Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $|f|$ be defined by

$$|f|(x) = |f(x)| \quad \text{for } x \in A.$$

(a) If f is continuous at point $c \in A$, then $|f|$ is continuous at c.

(a) If f is continuous on A, then $|f|$ is continuous on A.

Example Let $g(x) = |x|$ for $x \in \mathbb{R}$. Then g is continuous on $\mathbb{R}$.

Continuity of Composites:

If f is continuous at c, and g is continuous at $f(c)$, then $g \circ f$ is continuous at c

Example Let $g(x) = \sin x$ for $x \in \mathbb{R}$ and $f(x) = \dfrac{1}{x}$ for $x \neq 0$. Then, being the composition of the continuous functions g and f, $(g \circ f)(x) = \sin(1/x)$ is continuous at every point $c \neq 0$.

Theorem If $X = (x_n)$ is a convergent sequence and if $a \le x_n \le b$ for all $n \in \mathbb{N}$, then $a \le \lim(x_n) \le b$.

Sequential Criterion for Continuity A function $f : A \to \mathbb{R}$ is continuous at the point $c \in A$ if and only if for every sequence (x_n) in A that converges to *c*, the sequence $(f(x_n))$ converges to $f(c)$.

Hence if *f* is continuous at *x*, then

$$\lim_{n\to\infty} x_n = x \Rightarrow \lim_{n\to\infty} f(x_n) = f(x)$$

Equivalently,

$$\lim_{n\to\infty} x_n = x \Rightarrow \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$$

Squeeze Theorem: Suppose that $X = (x_n)$, $Y = (y_n)$, and $Z = (z_n)$ are sequences of real numbers such that

$$x_n \le y_n \le z_n \quad \text{for all} \quad n \in \mathbb{N},$$

and that $\lim(x_n) = \lim(z_n)$. Then $Y = (y_n)$ is convergent and

$$\lim(x_n) = \lim(y_n) = \lim(z_n).$$

Nested Intervals Property: If $I_n = [a_n, b_n]$, $n \in \mathbb{N}$, is a nested sequence of closed and bounded intervals, then there exists a number ¶ $\in \mathbb{R}$ such that ¶ $\in I_n$ for all $n \in \mathbb{N}$.

Characterization Theorem for Intervals: If S is a subset of $\mathbb{R}$ that contains at least two points and has the property

$$\text{if } x, y \in S \text{ and } x < y \text{ then } [x, y] \subseteq S, \qquad \ldots(1)$$

then S is an interval.

Discontinuity Criterion: Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in A$. Then f is discontinuous at *c* if and only if there exists a sequence (x_n) in A such that (x_n) converges to *c*, but the sequence $(f(x_n))$ does not converge to $f(c)$.

The Supremum Property (or Completeness Property): Every nonempty set of real numbers that has an upper bound has a supremum in $\mathbb{R}$.

The analogous property of infima can be deduced from the Supremum Property.

The Infimum Property of $\mathbb{R}$: Every nonempty set of real numbers that has a lower bound has an infimum in $\mathbb{R}$.

Theorem (Boundedness Theorem) Let $I = [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on *I*. Then *f* is bounded on *I*.

To prove this suppose that *f* is not bounded on *I*. Then, for any $n \in \mathbb{N}$ there is a number $x_n \in I$ such that $|f(x_n)| > n$. Since *I* is bounded, it follows that the sequence $X = (x_n)$ is bounded. Being a bounded sequence of real numbers, by the Bolzano-Weierstrass Theorem, X has a subsequence $X' = (x_{n_r})$ that converges to a number, say, *x*.

Since *I* is closed and since the elements of the subsequence X' belongs to *I*, it follows, from Theorem , that $x \in I$.

As f is continuous on I, $x \in I$ implies f is continuous at x. Now f is continuous at x, together with (x_{n_r}) converges to x implies, by Sequential Criterion for Continuity, that $\left(f(x_{n_r}) \right)$ converges to $f(x)$.

Since a convergent sequence of real numbers is bounded, the convergent sequence $\left(f(x_{n_r}) \right)$ must be a bounded sequence. But this is a contradiction since[1]

$$\left| f(x_{n_r}) \right| > n_r \geq r \text{ for } r \in \mathbb{N}.$$

Therefore the supposition that the continuous function f is not bounded on the closed bounded interval I leads to a contradiction. Hence f is bounded. This completes the proof.
The conclusion of the Boundedness Theorem fails if any one of the hypotheses is relaxed. The following examples illustrate this.

Example The function $f(x) = x$ for x in the unbounded, closed interval $A = [0, \infty)$ is continuous but *not* bounded on A.

Example The function $g(x) = \dfrac{1}{x}$ for x in the half-open interval $B = (0,1]$ is continuous but *not* bounded on B.

Example The function h defined on the closed interval $C = [0, 1]$ by

$$h(x) = \begin{cases} \dfrac{1}{x} & \text{when } 0 < x \leq 1 \\ 0 & \text{when } x = 0 \end{cases}$$

is discontinuous and unbounded on C.

Example Let $I = [a,b]$ and since $f : I \to \mathbb{R}$ be a continuous function such that $f(x) > 0$ for each $x \in I$. We prove that there exists a number $\sim\ > 0$ such that $f(x) \geq \sim$ for all $x \in I$.

To prove this, define $g : I \to \mathbb{R}$ by

$$g(x) = \frac{1}{f(x)} \qquad \text{for } x \in I.$$

Since $f(x) > 0$ for all $x \in I$, and since f is continuous on I, it follows that g is continuous on the closed bounded interval I. Hence, by Boundedness Theorem, g is bounded on I. Hence there exists an $\sim\ > 0$ such that

$$|g(x)| \leq \frac{1}{\sim} \qquad \text{for } x \in I.$$

As $f(x) > 0$, the above implies

$$g(x) = \frac{1}{f(x)} \leq \frac{1}{\sim} \qquad \text{for } x \in I.$$

Hence $\qquad f(x) \geq \sim \qquad$ for $x \in I$.

Absolute Maximum and Absolute Minimum

Definition Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. We say that f has an **absolute maximum on** $\boldsymbol{A}$ if there is a point $x^* \in A$ such that

$$f(x^*) \geq f(x) \text{ for all } x \in A.$$

We say that *f* has an **absolute minimum on** ***A*** if there is a point $x_* \in A$ such that

$$f(x_*) \leq f(x) \text{ for all } x \in A$$

We say that x^* is an **absolute maximum point** for *f* on *A*, and that x_* is an **absolute minimum point** for *f* on *A*, if they exist.

Example We now give an example to show that a continuous function on set *A* does not necessarily have an absolute maximum or an absolute minimum on the set.

The function $f(x) = \frac{1}{x}$ has neither an absolute maximum nor an absolute minimum on the set $A = (0, \infty)$ (Fig. 1). There can be no absolute maximum for *f* on *A* since *f* is not bounded above on *A*. *f* can have no absolute minimum as there is no point at which *f* attains the value $0 = \inf\{f(x) : x \in A\}$. The same function has neither an absolute maximum nor an absolute minimum when it is restricted to the set $(0, 1)$, while it has both an absolute maximum and an absolute minimum when it is restricted to the set $[1, 2]$. In addition, $f(x) = \frac{1}{x}$ has an absolute maximum but no absolute minimum when restricted to the set $[1, \infty)$, but no absolute maximum and no absolute minimum when restricted to the set $(1, \infty)$.

Example We now give an example to show that if a function has an absolute maximum point, then this point is not necessarily uniquely determined.

Consider the function $g(x) = x^2$ defined for $x \in A = [-1, 1]$ has two points $x = \pm 1$ giving the absolute maximum on *A*, and the single point $x = 0$ yielding its absolute minimum on *A*.

Example The constant function $h(x) = 1$ for $x \in \mathbb{R}$ is such that every point of $\mathbb{R}$ is both an absolute maximum and an absolute minimum point for *h*.

Theorem (Maximum-Minimum Theorem) Let $I = [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on *I*. Then *f* has an absolute maximum and an absolute minimum on *I*.

To prove the result, let

$$f(I) = \{f(x) : x \in I\}.$$

i.e., $f(I)$ is the nonempty set of values of *f* on *I*.

Since *f* is continuous and I is closed and bounded, using Boundedness Theorem, we have $f(I)$ is a bounded subset of $\mathbb{R}$. Then $f(I)$ is bounded above and bounded below. Hence by the completeness property of $\mathbb{R}$, supremum and infimum of the set exist.

Let

$$s^* = \sup f(I) \quad \text{and} \quad s_* = \inf f(I).$$

We claim that there exist points x^* and x_* in *I* such that $s^* = f(x^*)$ and $s_* = f(x_*)$.

We only establish the existence of the point x^*, as the proof of the existence of x_* is similar.

Since $s^* = \sup f(I)$, if $n \in \mathbb{N}$, then the number $s^* - \frac{1}{n}$ is not an upper bound of the set $f(I)$.

Consequently there exists a number $x_n \in I$ such that

$$s^* - \frac{1}{n} < f(x_n) \le s^* \text{ for all } n \in \mathbb{N}. \qquad \ldots (2)$$

By this way we obtain a sequence $X = (x_n)$ with members in *I*. Also, since *I* is bounded, the sequence $X = (x_n)$ is bounded. Therefore, by the Bolzano-Weierstrass Theorem, there is subsequence $X' = (x_{n_r})$ of *X* that converges to some number x^*. Since the elements of X' belong to $I = [a, b]$, we have $a \le x_{n_r} \le b$ and hence it follows from Theorem A that

$$a \le \lim x_{n_r} \le b.$$

That is,

$$a \le x^* \le b$$

and hence $x^* \in I$. Therefore *f* is continuous at x^* and, by Sequential Criterion for Continuity, it follows that

$$\lim f(x_{n_r}) = f(x^*).$$

Since it follows from (2) that

$$s^* - \frac{1}{n_r} < f(x_{n_r}) \le s^* \text{ for all } r \in \mathbb{N},$$

we conclude from the Squeeze Theorem that[2]

$$\lim(f(x_{n_r})) = s^*.$$

Therefore we have

$$f(x^*) = \lim\left(f(x_{n_r})\right) = s^* = \sup f(I).$$

We conclude that x^* is an absolute maximum point of *f* on *I*. This completes the proof.

Remark With the assumptions of the theorem, $s^* = f(x^*) \in f(I)$ and $s_* = f(x_*) \in f(I)$

Theorem (Location of Roots Theorem) Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be continuous on *I*. If $f(a) < 0 < f(b)$, or if $f(a) > 0 > f(b)$, then there exists a number $c \in (a,b)$ such that $f(c) = 0$.

To prove the result, we assume that $f(a) < 0 < f(b)$. We will generate a sequence of intervals by successive bisections.

Let $\qquad I_1 = [a_1, b_1],$

where $a_1 = a,\ b_1 = b$, and let p_1 be the midpoint

$$p_1 = \frac{1}{2}(a_1 + b_1).$$

If $f(p_1) = 0$, we take $c = p_1$ and we are done.

If $f(p_1) \ne 0$, then either $f(p_1) > 0$ or $f(p_1) < 0$.

If $f(p_1) > 0$, then we set $a_2 = a_1,\ b_2 = p_1$,

while if $f(p_1) < 0$, then we set $a_2 = p_1, b_2 = b_1$.

In either case, we let

$$I_2 = [a_2, b_2];$$

then we have

$$I_2 \subset I_1 \text{ and } f(a_2) < 0, f(b_2) > 0.$$

We continue the bisection process. Suppose that the intervals $I_1, I_2, \ldots, I_k$ have been obtained by successive bisection in the same manner. Then we have $f(a_k) < 0$ and $f(b_k) > 0$, and we set

$$p_k = \frac{a_k + b_k}{2}.$$

If $f(p_k) = 0$, we take $c = p_k$ and we are done.

If $f(p_k) > 0$, we set

$$a_{k+1} = a_k, b_{k+1} = p_k,$$

while if $f(p_k) < 0$, we set

$$a_{k+1} = p_k, b_{k+1} = b_k.$$

In either case, we let

$$I_{k+1} = [a_{k+1}, b_{k+1}];$$

then $\quad I_{k+1} \subset I_k$ and $f(a_{k+1}) < 0, \ f(b_{k+1}) > 0$.

If the process terminates by locating a point p_n such that $f(p_n) = 0$, then we are done. If the process does not terminate, then we obtain a nested sequence of closed bounded intervals $I_n = [a_n, b_n]$ such that for every $n \in \mathbb{N}$ we have

$$f(a_n) < 0 \quad \text{and} \quad f(b_n) > 0$$

Furthermore, since the intervals are obtained by repeated bisections, the length of I_n is equal to $b_n - a_n = (b-a)/2^{n-1}$. It follows from the Nested Intervals Property that there exists a point *c* that belongs to I_n for all $n \in \mathbb{N}$. Now $c \in I_n$ for all $n \in \mathbb{N}$ implies $a_n \le c \le b_n$ for all $n \in \mathbb{N}$, and hence we have

$$0 \le c - a_n \le b_n - a_n = \frac{(b-a)}{2^{n-1}},$$

and $\quad 0 \le b_n - c \le b_n - a_n = (b-a)/2^{n-1}$.

Hence, it follows, by Squeeze Theorem, that

$$0 \le \lim(c - a_n) \le (b-a)\lim\frac{1}{2^{n-1}}$$

and $\quad 0 \le \lim(b_n - c) \le (b-a)\lim\dfrac{1}{2^{n-1}}$.

That is,

$$0 \le c - \lim(a_n) \le 0$$

and $\quad 0 \le \lim(b_n) - c \le 0$.

Hence $\quad \lim(a_n) = c = \lim(b_n)$.

Since *f* is continuous at c, the fact that $(a_n) \to c$ and $(b_n) \to c$ implies (by Sequential Criterion for Continuity) that

$$\lim\big(f(a_n)\big) = f(c) = \lim\big(f(b_n)\big).$$

The fact that $f(a_n) < 0$ for all $n \in \mathbb{N}$ implies that

$$f(c) = \lim\big(f(a_n)\big) \le 0.$$

Also, the fact that $f(b_n) \geq 0$ for all $n \in \mathbb{N}$, implies that

$$f(c) = \lim\big(f(b_n)\big) \geq 0.$$

Thus, we conclude that $f(c) = 0$. Consequently, c is a root of f. This completes the proof.

Remark The above theorem can be restated as follows:

"Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be continuous on I. If $f(a)f(b) < 0$, then there exists a number $c \in (a,b)$ such that $f(c) = 0$."

RemarkThe above result is called 'location of roots theorem' as it is the theoretical basis for locating roots of a continuous function by means of sign changes of the function. The proof of the theorem provides an algorithm, known as the Bisection Method, for finding solutions of equations of the form $f(x) = 0$, where f is a continuous function. (This method is discussed in details in the text "*Numerical Analysis*" Sixth semester core).

Now using location of roots theorem , we examine whether there is a real number that is one less than its fifth power?

The number we seek must satisfy the equation

$$x = x^5 - 1.$$

i.e., $$x^5 - x - 1 = 0.$$

In other words we are looking for a zero of the function

$$f(x) = x^5 - x - 1$$

By trial, we have

$$f(1) = 1 - 1 - 1 = -1$$

and $$f(2) = 32 - 1 - 1 = 30.$$

Thus $$f(1) < 0 \quad \text{and} \quad f(2) > 0.$$

Hence, by location of roots theorem, there exists a number $c \in (1,2)$ i.e., $1 < c < 2$ such that

$$f(c) = c^5 - c - 1 = 0$$

Hence $$c = c^5 - 1.$$

Therefore c is the required number which is one less than its fifth power.

Assignments

1. Show that every polynomial of odd degree with real coefficients has at least one real root.
2. Let $I = [a,b]$ and let $f : I \to \mathbb{R}$ be a continuous function on I such that for each x in I there exists y in I such that $|f(y)| \leq \frac{1}{2}|f(x)|$. Prove that there exists a point c in I such that $f(c) = 0$.
3. Show that the polynomial $p(x) = x^3 - x - 1$ has at least two real roots. Use a calculator to locate these roots to within two decimal places.
4. Show that the polynomial $p(x) = x^4 + 7x^3 - 9$ has at least two real roots. Use a calculator to locate these roots to within two decimal places.
5. Let f be continuous on the interval $[0,1]$ to $\mathbb{R}$ and such that $f(0) = f(1)$. Prove that there exists a point c in $[0, \frac{1}{2}]$ such that $f(c) = f\left(c + \frac{1}{2}\right)$. [Hint: Consider $g(x) = f(x) - f(x + \frac{1}{2})$.] Conclude that there are, at any time, antipodal points on the earth's equator that have the same temperature.

6. Show that the function $f(x) = 2\ln x + \sqrt{x} - 2$ has root in the interval $[1, 2]$. Use the Bisection Method and a calculator to find the root with error less than 10^{-2}.
7. Show that the equation $x = \cos x$ has a solution in the interval $[0, \frac{\pi}{2}]$. Use the Bisection Method and a calculator to find an approximate solution of this equation, with error less than 10^{-3}.
8. The function $f(x) = (x-1)(x-2)(x-3)(x-4)(x-5)$ has five roots in the interval [0, 7]. If the Bisection Method is applied on this interval, which of the roots is located?
9. The function for $g(x) = (x-2)(x-3)(x-4)(x-5)(x-6)$ has five roots on the interval [0, 7]. If the Bisection Method is applied on this interval, which of the roots is located?
10. If the Bisection Method is used on an interval of length 1 to find p_n with error $|p_n - c| < 10^{-5}$, determine the least value of n that will assure this accuracy.

2

BOLZANO'S INTERMEDIATE VALUE THEOREM

The next result is a generalization of the Location of Roots Theorem. It assures us that a continuous function on an interval takes on (at least once) any number that lies between two of its values.

Bolzano's Intermediate Value Theorem: Let I be an interval and let $f: I \to \mathbb{R}$ be continuous on I. If $a, b \in I$ and if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$, then there exists a point $c \in I$ between a and b such that $f(c) = k$.

To prove the result, suppose that $a < b$ and let $g(x) = f(x) - k$. Then $f(a) < k < f(b)$ implies $g(a) < 0 < g(b)$. By the Location of Roots Theorem there exists a point c with $a < c < b$ such that

$$0 = g(c).$$

i.e., such that $$0 = f(c) - k.$$

Therefore $$f(c) = k.$$

If $b < a$, let $h(x) = k - f(x)$ so that $h(b) < 0 < h(a)$. Therefore there exists a point c with $b < c < a$ such that $0 = h(c) = k - f(c)$, and hence $f(c) = k$. This completes the proof.

Corollary Let $I = [a,b]$ be a closed, bounded interval and let $f: I \to \mathbb{R}$ be continuous on I. If $k \in \mathbb{R}$ is any number satisfying

$$\inf f(I) \le k \le \sup f(I)$$

then there exists a number $c \in I$ such that $f(c) = k$.

It follows from the Maximum-Minimum Theorem that there are points c_* and c^* in I such that

$$\inf f(I) = f(c_*) \text{ and } f(c^*) = \sup f(I).$$

Hence by the assumption

$$\inf f(I) = f(c_*) \le k \le f(c^*) = \sup f(I).$$

Hence by Bolzano's Intermediate Value Theorem, there exists a point $c \in I$ between c_* and c^* such that $f(c) = k$. This completes the proof.

Theorem Let I be a closed bounded interval and let $f: I \to \mathbb{R}$ be continuous on I. Then the set $f(I) = \{f(x) : x \in I\}$ is a closed bounded interval.

To prove this, we let $m = \inf f(I)$ and $M = \sup f(I)$, then by the Maximum–Minimum Theorem, m and M belong to $f(I)$. Moreover, we have

$$f(I) \subseteq [m, M]. \qquad \ldots (3)$$

If k is any element of $[m, M]$, then it follows from Corollary to Bolzano's Intermediate Value Theorem that there exists a point $c \in I$ such that $k = f(c)$. Hence, $k \in f(I)$ and we conclude that

$$[m, M] \subseteq f(I). \qquad \ldots (4)$$

From (3) and (4), we have $f(I) = [m, M]$.Since $[m, M]$ is a closed and bounded interval, $f(I)$ is also a closed and bounded interval. (Also, we note that the endpoints of the image interval are m and M, resp., the absolute minimum and absolute maximum values of the function). This completes the proof.

The previous theorem states that the image of closed bounded interval under a continuous function is also a closed bounded interval. The endpoints of the **image interval** are the absolute minimum and absolute maximum values of the function, and all values between the absolute minimum and the absolute maximum values belong to the image of the function.

If $I=[a,b]$ is an interval and $f = I \to \mathbb{R}$ is continuous on *I*, we have proved that $f(I)$ is the interval $[m, M]$. We have not proved (and it is not always true) that $f(I)$ is the interval $[f(a), f(b)]$.

The preceding theorem is a preservation theorem in the sense that it states that the continuous image of a closed bounded interval is a set of the same type. The next theorem extends this result to general intervals. However, it should be noted that although the continuous image of an interval is shown to be an interval, it is *not* true that the image interval necessarily has the same form as the domain interval.

Example We now give examples to show that the continuous image of an open interval need *not* be an open interval, and the continuous image of an unbounded closed interval need not be a closed interval.

If

$$f(x)=\frac{1}{x^2+1}$$

for $x \in \mathbb{R}$, then f is continuous on $\mathbb{R}$. It is easy to see that if $I_1=(-1,1)$, then $f(I_1)=\left(\frac{1}{2},1\right]$, which is not an open interval. Also, if I_2 is the closed interval given by $I_2=\left[0,\infty\right)$, then $f(I_2)=\left(0,1\right]$, which is not a closed interval (Fig. 4).

Preservation of Intervals Theorem Let *I* be an interval and let $f: I \to \mathbb{R}$ be continuous on *I*. Then the set $f(I)$ is an interval.

To prove the result, let $\sim,\ \ \in f(I)$ with $\sim <\ \ $. Then there exist points $a, b \in I$ such that $\sim = f(a)$ and $\ \ = f(b)$. Further, it follows from Bolzano's Intermediate Value Theorem that if $k \in (\sim,\ \ )$ then there exists a number $c \in I$ with $k = f(c) \in f(I)$.

Hence, $[\sim,\ \ ] \subseteq f(I)$, showing that $f(I)$ possesses property (1) of Characterization Theorem for Intervals . Therefore $f(I)$ is an interval and this completes the proof.

Example Let $I=[a,b]$ and let $f: I \to \mathbb{R}$ and $g: I \to \mathbb{R}$ be continuous functions on *I*. We show that the set $E=\{x \in I: f(x)=g(x)\}$ is with the property that if $(x_n) \subseteq E$ and $x_n \to x_0$, then $x_0 \in E$.

To prove this, we note that since *f* and *g* are continuous, $f-g: I \to \mathbb{R}$ is also continuous.

Also, since $\qquad E=\{x \in I: (f-g)(x)=0\}$,

we have $\qquad E=(f-g)^{-1}\{0\}$.

As $\{0\}$ is a closed set and $f-g$ is continuous, $(f-g)^{-1}\{0\}$ is a closed subset of *I*. i.e., *E* is a closed subset of *I*.

Hence if we take a sequence (x_n) in *E* with limit x_0, then its limit x_0 must belongs to the set *E*.

Solved Examples

Verify the following two important limits:

(a) $\lim_{x \to x_0} x = x_0$ (b) $\lim_{x \to x_0} k = k$ (k constant).

Answers

a) Let $£ > 0$ be given. We must find $¢ > 0$ such that for all x

$0 < |x - x_0| < ¢$ implies

$|x - x_0| < £$.

The implication will hold if $¢ = £$ or any smaller positive number. This proves that $\lim_{x \to x_0} x = x_0$

b) Let $£ > 0$ be given. We must find $¢ > 0$ such that for all x,

$0 < |x - x_0| < ¢$ implies $|k - k| < £$.

Since $k - k = 0$, we can use any positive number for $¢$ and the implication will hold. This proves that $\lim_{x \to x_0} k = k$.

Example Evaluate the following limit. $\lim_{x \to 0} e^{\sin x}$

Since we know that exponentials are continuous everywhere we can use the fact above.

$$\lim_{x \to 0} e^{\sin x} = e^{\lim_{x \to 0} \sin x} = e^0 = 1$$

Example For the limit $\lim_{x \to 5} \sqrt{x-1} = 2$, find a $¢ > 0$ that works for $£ = 1$.

Answers We have to find a $¢ > 0$ such that for all x

$$0 < |x - 5| < ¢ \Rightarrow |\sqrt{x-1} - 2| < 1.$$

<u>Step 1 :</u> We solve the inequality $|\sqrt{x-1} - 2| < 1$ to find an interval about $x_0 = 5$ on which the inequality holds for all $x \neq x_0$.

$$|\sqrt{x-1} - 2| < 1$$
$$-1 < \sqrt{x-1} - 2 < 1$$
$$1 < \sqrt{x-1} < 3$$
$$1 < x - 1 < 9$$
$$2 < x < 10$$

The inequality holds for all x in the open interval $(2, 10)$, so it holds for all $x \neq 5$ in this interval as well.

<u>Step 2 :</u> We find a value of $¢ > 0$ that places the centered interval $5 - ¢ < x < 5 + ¢$ inside the interval $(2, 10)$. The distance from 5 to the nearer endpoint of (2, 10) is 3. If we take $¢ = 3$ or any smaller positive number, then the inequality $0 < |x - 5| < ¢$ will automatically place x between 2 and 10 to make $|\sqrt{x-1} - 2| < 1$.

$$0 < |x - 5| < 3 \quad \Rightarrow \quad |\sqrt{x-1} - 2| < 1.$$

Example Prove that $\lim_{x \to 2} f(x) = 4$ if

$$f(x)=\begin{cases} x^2, & x\neq 2 \\ 1, & x=2. \end{cases}$$

Answers Our task is to show that given $£ > 0$ there exists $¢ > 0$ such that for all x

$$0<|x-2|<¢ \quad \Rightarrow \quad |f(x)-4|<£ .$$

<u>Step 1</u> : We solve the inequality $|f(x)-4|<£$ to find on open interval about $x_0 = 2$ on which the inequality holds for all $x \neq x_0$.

For $x \neq x_0 = 2,$ we have $f(x) = x^2$, and the inequality to solve is $|x^2-4|<£$:

$$|x^2-4|<£$$

$$-£ < x^2-4 < £$$

$$4-£ < x^2 < 4+£$$

$$\sqrt{4-£} < |x| < \sqrt{4+£} \quad \text{here we assume that } £<4$$

$$\sqrt{4-£} < x < \sqrt{4+£} \quad \text{this is an open interval about } x_0 = 2 \text{ that solves the inequality}$$

The inequality $|f(x)-4|<£$ holds for all $x\neq 2$ in the open interval $\left(\sqrt{4-£}, \sqrt{4+£}\right)$.

<u>Step 2 :</u> We find a value of $¢ > 0$ that places the centered interval $(2-¢, 2+¢)$ inside the interval $(\sqrt{4-£}, \sqrt{4+£})$:

Take $¢$ to be the distance from $x_0 = 2$ to the nearer endpoint of $\left(\sqrt{4-£}, \sqrt{4+£}\right)$. In other words, take

$$¢ = \min\left\{2-\sqrt{4-£}, \sqrt{4+£}-2\right\},$$

the *minimum* (the smaller) of the two numbers $2-\sqrt{4-£}$ and $\sqrt{4+£}-2$. If $¢$ has this or any smaller positive value, the inequality $0<|x-2|<¢$ will automatically place x between $\sqrt{4-£}$ and $\sqrt{4+£}$ to make $|f(x)-4|<£$. For all x,

$$0<|x-2|<¢ \quad \Rightarrow \quad |f(x)-4|<£ .$$

This completes the proof.

Example Show that $p(x) = 2x^3 - 5x^2 - 10x + 5$ has a root somewhere in the interval [-1,2].

What we're really asking here is whether or not the function will take on the value

$$p(x) = 0$$

somewhere between -1 and 2. In other words, we want to show that there is a number *c* such that $-1 < c < 2$ and $p(c) = 0$. However if we define $M = 0$ and acknowledge that $a = -1$ and $b = 2$ we can see that these two condition on *c* are exactly the conclusions of the Intermediate Value Theorem. So, this problem is set up to use the Intermediate Value Theorem and in fact, all we need to do is to show that the function is continuous and that $M = 0$ is between $p(-1)$ and $p(2)$ (*i.e.* $p(-1) < 0 < p(2)$ or $p(2) < 0 < p(-1)$ and we'll be done.

To do this all we need to do is compute,

$$p(-1) = 8 \qquad\qquad p(2) = -19$$

So we have, $-19 = p(2) < 0 < p(-1) = 8$

Therefore $M = 0$ is between $p(-1)$ and $p(2)$ and since $p(x)$ is a polynomial it's continuous everywhere and so in particular it's continuous on the interval [-1,2]. So by the Intermediate Value Theorem there must be a number $-1 < c < 2$ so that, $p(c) = 0$

Therefore the polynomial does have a root between -1 and 2.

Example If possible, determine if $f(x) = 20\sin(x+3)\cos\left(\frac{x^2}{2}\right)$ takes the following values in the interval [0,5].

(a) Does $f(x) = 10$? **(b)** Does $f(x) = -10$?

If possible, suppose the function takes on either of the two values above in the interval [0,5]. First, let's notice that this is a continuous function and so we know that we can use the Intermediate Value Theorem to do this problem.

Now, for each part we will let *M* be the given value for that part and then we'll need to show that *M* lives between $f(0)$ and $f(5)$. If it does then we can use the Intermediate Value Theorem to prove that the function will take the given value.

So, since we'll need the two function evaluations for each part let's give them here,

$$f(0) = 2.8224 \qquad\qquad f(5) = 19.7436$$

Now, let's take a look at each part.

(a) In this case we'll define $M = 10$ and we can see that,

$$f(0) = 2.8224 < 10 < 19.7436 = f(5)$$

So, by the Intermediate Value Theorem there must be a number $0 \le c \le 5$ such that

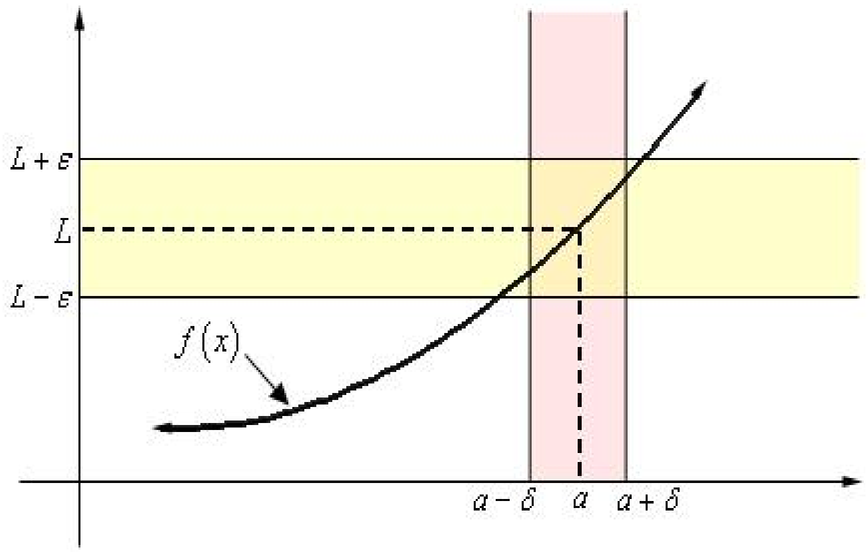
$$f(c) = 10$$

(b) In this part we'll define $M = -10$. We now have a problem. In this part *M* does not live between $f(0)$ and $f(5)$. So, what does this mean for us? Does this mean that $f(x) \neq -10$ in [0,5]?

Unfortunately for us, this doesn't mean anything. It is possible that $f(x) \neq -10$ in [0,5], but is it also possible that $f(x) = -10$ in [0,5]. The Intermediate Value Theorem will only tell us that c's will exist. The theorem will NOT tell us that c's don't exist.

In this case it is not possible to determine if $f(x) = -10$ in [0,5] using the Intermediate Value Theorem.

Theory: Let's take a look at the following graph and let's also assume that the limit does exist.



What the definition is telling us is that for **any** number $\varepsilon > 0$ that we pick we can go to our graph and sketch two horizontal lines at $L+\varepsilon$ and $L-\varepsilon$ as shown on the graph above. Then somewhere out there in the world is another number $\delta > 0$, which we will need to determine, that will allow us to add in two vertical lines to our graph at $a+\delta$ and $a-\delta$.

Now, if we take any x in the region, *i.e.* between $a+\delta$ and $a-\delta$, then this x will be closer to a than either of $a+\delta$ and $a-\delta$. Or,

$$x-a| < \delta$$

If we now identify the point on the graph that our choice of x gives then this point on the graph **will** lie in the intersection of the regions. This means that this function value *f(x)* will be closer to L than either of $L+\varepsilon$ and $L-\varepsilon$. Or,

$$f(x) - L| < \varepsilon$$

So, if we take any value of x in the region then the graph for those values of x will lie in the region.

Notice that there are actually an infinite number of possible δ's that we can choose. In fact, if we go back and look at the graph above it looks like we could have taken a slightly larger δ and still gotten the graph from that region to be completely contained in the region.

Also, notice that as the definition points out we only need to make sure that the function is defined in some interval around $x = a$ but we don't really care if it is defined at $x = a$. Remember that limits do not care what is happening at the point, they only care what is happening around the point in question.

Okay, now that we've gotten the definition out of the way and made an attempt to understand it let's see how it's actually used in practice.

These are a little tricky sometimes and it can take a lot of practice to get good at these so don't feel too bad if you don't pick up on this stuff right away. We're going to be looking at a couple of examples that work out fairly easily.

Example Use the definition of the limit to prove the following limit.

$$\lim_{x\to 0} x^2 = 0$$

In this case both *L* and *a* are zero. So, let $\varepsilon > 0$ be any number. Don't worry about what the number is, ε is just some arbitrary number. Now according to the definition of the limit, if this limit is to be true we will need to find some other number $\delta > 0$ so that the following will be true.

$$\left|x^2 - 0\right| < \varepsilon \qquad \text{whenever} \qquad 0 < \left|x - 0\right| < \delta$$

Or upon simplifying things we need,

$$\left|x^2\right| < \varepsilon \qquad \text{whenever} \qquad 0 < \left|x\right| < \delta$$

Often the way to go through these is to start with the left inequality and do a little simplification and see if that suggests a choice for δ. We'll start by bringing the exponent out of the absolute value bars and then taking the square root of both sides.

$$\left|x\right|^2 < \varepsilon \qquad \Rightarrow \qquad \left|x\right| < \sqrt{\varepsilon}$$

Now, the results of this simplification looks an awful lot like $0 < \left|x\right| < \delta$ with the exception of the "$0 <$" part. Missing that however isn't a problem, it is just telling us that we can't take $x = 0$. So, it looks like if we choose $\delta = \sqrt{\varepsilon}$ we should get what we want.

We'll next need to verify that our choice of δ will give us what we want, *i.e.*,

$$x^2\right| < \varepsilon \qquad \text{whenever} \qquad 0 < \left|x\right| < \sqrt{\varepsilon}$$

Verification is in fact pretty much the same work that we did to get our guess. First, let's again let $\varepsilon > 0$ be any number and then choose $\delta = \sqrt{\varepsilon}$. Now, assume that $0 < |x| < \sqrt{\varepsilon}$. We need to show that by choosing *x* to satisfy this we will get,

$$|x^2| < \varepsilon$$

To start the verification process we'll start with $|x^2|$ and then first strip out the exponent from the absolute values. Once this is done we'll use our assumption on *x*, namely that $|x| < \sqrt{\varepsilon}$. Doing all this gives,

$\lvert x^2 \rvert = \lvert x \rvert^2$	strip exponent out of absolute value bars
$< \left(\sqrt{\varepsilon}\right)^2$	use the assumption that $\lvert x \rvert < \sqrt{\varepsilon}$
$= \varepsilon$	simplify

Or, upon taking the middle terms out, if we assume that $0 < |x| < \sqrt{\varepsilon}$ then we will get,

$|x^2| < \varepsilon$

and this is exactly what we needed to show.

So, just what have we done? We've shown that if we choose $\varepsilon > 0$ then we can find a $\delta > 0$ so that we have,

$|x^2 - 0| < \varepsilon$ whenever $0 < |x - 0| < \sqrt{\varepsilon}$

and according to our definition this means that, $\lim_{x \to 0} x^2 = 0$

Assignments

1. Let $I = [a,b]$, let $f : I \to \mathbb{R}$ be continuous on *I*, and assume that $f(a) < 0, f(b) > 0$. Let $W = \{x \in I : f(x) < 0\}$, and let $w = \sup W$. Prove that $f(w) = 0$. (This provides an alternative proof of Location of Roots Theorem).
2. Let $I = [0,® /2]$ and let $f : I \to \mathbb{R}$ be defined by $f(x) = \sup\{x^2, \cos x\}$ for $x \in I$. Show there exists an absolute minimum point $x_0 \in I$ for *f* on *I*. Show that x_0 is a solution to the equation $\cos x = x^2$.
3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and that $\lim_{x \to -\infty} f = 0$ and $\lim_{x \to \infty} f = 0$. Prove that f is bounded on $\mathbb{R}$ and attains either a maximum or minimum on $\mathbb{R}$. Give an example to show that both a 4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let $\in \mathbb{R}$. Show that if $x_0 \in \mathbb{R}$ is such that $f(x_0) <$, then there exists a ¢ -neighborhood *U* of x_0 such that $f(x) <$ for all $x \in U$.

5. Examine which open [respectively, closed] intervals are mapped by $f(x) = x^2$ for $x \in \mathbb{R}$ onto open [respectively, closed] intervals.
6. Examine the mapping of open [respectively, closed] intervals under the functions $g(x) = 1/(x^2 + 1)$ and $h(x) = x^3$ for $x \in \mathbb{R}$.
7. Give an example of an open interval that is not mapped by $f(x) = x^2$ for $x \in \mathbb{R}$ onto open interval.
8. If $f[0,1] \to \mathbb{R}$ is continuous and has only rational [respectively, irrational] values, must *f* be constant? Prove your assertion.
9. Let $J = (a,b)$ and let $g : J \to \mathbb{R}$ be a continuous function with the property that for every $x \in J$, the function g is bounded on a neighborhood $V_{¢_x}(x)$ of x. Show by example that g is not necessarily bounded on *J*.
10. Let $I = [a,b]$ and let $f : I \to \mathbb{R}$ be a (not necessarily continuous) function with the property that for every $x \in I$, the function f is bounded on a neighborhood $V_{¢_x}(x)$ of x (in the sense of the Definition " Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of *A*. We say that *f* is bounded on a neighborhood of c if there exists a ¢ neighborhood $V_{¢}(c)$ of *c* and a constant $M > 0$ such that such that $|f(x)| \le M$ for all $x \in A \cap V_{¢}(c)$"). Prove that *f* is bounded on *I*.

3
UNIFORM CONTINUITY

A function f is uniformly continuous if it is possible to guarantee that $f(x)$ and $f(u)$ be as close to each other as we please by requiring only that x and u are sufficiently close to each other; unlike ordinary continuity, the maximum distance between x and u cannot depend on x and u themselves.

Every uniformly continuous function is continuous. Uniform continuity, unlike continuity, relies on the ability to compare the sizes of neighbourhoods of distinct points of a given space. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. By the definition of continuity at a point it follows that the following statements are equivalent.

(i) f is continuous at every point $u \in A$.

(ii) Given $£ > 0$ and $u \in A$, there is a $¢(£, u) > 0$ such that for all x such that $x \in A$ and $|x - u| < ¢(£, u)$, then $|f(x) - f(u)| < £$.

In the above note that $¢$ depends, in general, on both $£ > 0$ and $u \in A$. For example, consider $f(x) = \sin(1/x)$ for $x > 0$. f change its values rapidly near certain points and slowly near other points. Hence $¢$ depends on u.

Example If $g(x) = 1/x$ for $x \in A = \{x \in \mathbb{R} : x > 0\}$, then

$$g(x) - g(u) = \frac{u - x}{ux} \qquad \dots (1)$$

If $u \in A$ is given and if we take

$$¢(£, u) = \inf\{\tfrac{1}{2}u, \tfrac{1}{2}u^2 £\}, \qquad \dots (2)$$

then if $|x - u| < ¢(£, u)$, we have $|x - u| < \frac{1}{2}u$ so that $-\frac{1}{2}u < x - u < \frac{1}{2}u$ so that $\frac{1}{2}u < x < \frac{3}{2}u$, and hence it follows that $\frac{1}{x} < \frac{2}{u}$. Thus, if $|x - u| < \frac{1}{2}u$, the equality (1) yields the inequality

$$|g(x) - g(u)| \le \left(\frac{2}{u^2}\right)|x - u| \ . \qquad \dots (3)$$

Hence if $|x - u| < ¢(£, u)$, then (2) and (3) imply that

$$|g(x) - g(u)| < \left(\frac{2}{u^2}\right)(\tfrac{1}{2}u^2 £) = £.$$

We have seen that the selection of $¢(£, u)$ by the formula (2) works in the sense that it enables us to give a value of $¢$ that will ensure that $|g(x) - g(u)| < £$ when $|x - u| < ¢$ and $x,\ u \in A$. We note that the value of $¢(£, u)$ given in (2) certainly depends on the point $u \in A$. If we wish to consider all $u \in A$, formula (2) does not lead to one value $¢(£) > 0$ that will work simultaneously for all $u > 0$, since $\inf\{¢(£, u) : u > 0\} = 0$.

There are other selections that can be made for ¢ . (For example we could also take $¢_1(£,u) = \inf\{\frac{1}{3}u, \frac{2}{3}u^2£\}$; however, we still have $\inf\{¢_1(£,u) : u > 0\} = 0$.) In fact, there is no way of choosing one value of ¢ that will "work" for all $u > 0$ for the function $g(x) = 1/x$, as we shall see.

Uniform Continuity

Now it often happens that the function f is such that the number ¢ can be chosen to be independent of the point $u \in A$ and to depend only on £ . For example, if $f(x) = 2x$ for all $x \in \mathbb{R}$, then

$$|f(x) - f(u)| = 2|x - u|,$$

and so we can choose $¢(£,u) = £/2$ for all $£ > 0,\ u \in \mathbb{R}$. Here ¢ is independent of *u*.

Definition Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. We say that *f* is **uniformly continuous on** A if for each $£ > 0$ there is a $¢(£) > 0$ such that if $x,\ u \in A$ are any numbers satisfying $|x - u| < ¢(£)$, then $|f(x) - f(u)| < £$.

We note that $¢(£)$ in the definition above is independent of *u*. Also, it is clear that if *f* is uniformly continuous on *A*, then it is continuous at every point of *A*. In general, however, the converse does not hold, as is shown by the function $g(x) = 1/x$ on the set $A = \{x \in \mathbb{R} : x > 0\}$.

Example Show that the function $f(x) = x^2$ is uniformly continuous on $[-1,1]$.

Let *x, y* be any two points of $[-1,1]$. Then

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y|\,|x + y|$$

Now we note that

$$|x + y| \le |x| + |y| \le 2, \text{ since } x, y \in [-1,1] .$$

Hence, for $x,\ y \in [-1, 1]$

$$|f(x) - f(y)| \le 2|x - y|.$$

Hence for any $£ > 0$,

$$|f(x) - f(y)| < £$$

whenever $$|x - y| < \frac{£}{2}.$$

Hence taking $¢ = \frac{£}{2}$, we have for any $£ > 0$ and any $x, y \in [-1,1]$

$$|f(x) - f(y)| < £ \text{ whenever } |x - y| < ¢ .$$

This shows that the given function is uniformly continuous.

Example Show that the function $f(x) = 1/x$ is not uniformly continuous on $A = \{x \in \mathbb{R} :\ 0 < x \le 1\}$.

Since x is continuous on $A = (0,1]$ and $x \ne 0$ for all $x \in A$, $1/x$ is continuous on *A*. If possible, let $f(x) = 1/x$ is uniformly continuous on *A*. Then corresponding to $£ = 1$, there exists a $¢ > 0$ such that

$$\left|\frac{1}{x}-\frac{1}{y}\right|<1 \quad \text{whenever } |x-y|<¢$$

By Archimedean Property of real numbers ["If $x\in\mathbb{R}$, then there exists $n_x\in\mathbb{N}$ such that $x<n_x$"], corresponding to this $¢>0$, we can find a natural number $m\in\mathbb{N}$ such that $\frac{1}{¢}<m$. Then $\frac{1}{m}<¢$.

Since $m\in\mathbb{N}$, both $\frac{1}{m}$ and $\frac{1}{2m}$ are elements of $A=(0,1]$ and

$$\left|\frac{1}{m}-\frac{1}{2m}\right|=\frac{1}{2m}<\frac{1}{m}<¢ \ .$$

But

$$\left|f\left(\frac{1}{m}\right)-f\left(\frac{1}{2m}\right)\right|=\left|\frac{1}{1/m}-\frac{1}{1/2m}\right|=|m-2m|=m\geq 1.$$

This is a contradiction to the inequality. Hence our assumption is wrong. i.e., *f* is not uniformly continuous on *A*.

Example If $g(x)=1/x$ for $x\in B=[1/2,1]$, then

$$|g(x)-g(u)|=\frac{|u-x|}{|ux|} \qquad \ldots (1)$$

Since $x, u\in[1/2,1]$, $|xu|\geq(1/2)^2=1/4 \Rightarrow \frac{1}{|xu|}\leq 4.$

So if $|x-u|<¢$, $|f(x)-f(u)|=\frac{|x-u|}{|xu|}<4¢<£$ if $¢<£/4$.

Here $¢$ doesn't depend on x and u. It works for any pair of numbers in the interval $[1/2,1]$ with $|x-u|<¢$. Hence $g(x)=1/x$ is uniformly continuous on $B=[1/2,1]$.

Example Show that $f(x)=\cos x^2, x\in\mathbb{R}$ is not uniformly continuous on $\mathbb{R}$.

If possible let $f(x)=\cos x^2$, $x\in\mathbb{R}$ is uniformly continuous on $\mathbb{R}$. Then corresponding to $£=1$, there exists a $¢>0$, such that for any $x, y\in\mathbb{R}$,

$$|f(x)-f(y)|<1 \quad \text{whenever } |x-y|<¢$$

By Archimedean Property, corresponding to this $¢>0$ we can find natural number $n\in\mathbb{N}$ such that $\frac{1}{¢^2}<n$. Hence $\frac{1}{\sqrt{n}}<¢$.

Now let $x=\sqrt{(n+1)®}$ and $y=\sqrt{n®}$.

Then

$$x-y=\sqrt{(n+1)®}-\sqrt{n®}=\frac{(n+1)®-n®}{\sqrt{(n+1)®}+\sqrt{n®}}$$

$$=\frac{®}{\left[\sqrt{n+1}+\sqrt{n}\right]\sqrt{®}}<\frac{\sqrt{®}}{2\sqrt{n}}<\frac{1}{\sqrt{n}}<¢$$

But $|f(x)-f(y)|=\left|\cos x^2-\cos y^2\right|=|\cos(n+1)®-\cos n®|$

$$=\left|(-1)^{n+1}-(-1)^n\right|=2>1.$$

This is in contradiction with inequality (6). Hence our assumption is wrong i.e., f is not uniformly continuous on $\mathbb{R}$.

It is useful to formulate a condition equivalent to saying that *f* is not uniformly continuous on A. We give such criteria in the next result.

Nonuniform Continuity Criteria: Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Then the following statements are equivalent:

(i) f is not uniformly continuous on A.

(ii) There exists an $\varepsilon_0 > 0$ such that for every $\delta > 0$ there are points x_δ, u_δ in A such that $|x_\delta - u_\delta| < \delta$ and $|f(x_\delta) - f(u_\delta)| \geq \varepsilon_0$.

(iii) There exists an $\varepsilon_0 > 0$ and two sequences (x_n) and (u_n) in A such that $\lim(x_n - u_n) = 0$ and $f(x_n) - f(u_n) \geq \varepsilon_0$ for all $n \in \mathbb{N}$.

Example Show that $g(x) = 1/x$ is not uniformly continuous on $A = \{x \in \mathbb{R} : x > 0\}$.

We apply Nonuniform Continuity Criterion. For, if $x_n = 1/n$ and $u_n = 1/(n+1)$, then we have $\lim(x_n - u_n) = 0$, but

$$g(x_n) - g(u_n) = \left|\frac{1}{x_n} - \frac{1}{u_n}\right| = \left|\frac{1}{1/n} - \frac{1}{1/(n+1)}\right| = |n - (n+1)| = 1$$

for all $n \in \mathbb{N}$. Now choose $\varepsilon_0 = 1$. By the Nonuniform Continuity Criterion, g is not uniformly continuous.

We now present an important result that assures that a continuous function on a closed bounded interval *I* is uniformly continuous on *I*.

Uniform Continuity Theorem: Let I be closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on *I*. Then *f* is uniformly continuous on *I*.

To prove the theorem, we apply Contrapositive Method (which works on the logic $p \to q \equiv \neg q \to \neg p$) for proving the Theorem. If *f* is not uniformly continuous on I, then, by the Nonuniform Continuity Criterion, there exists $\varepsilon_0 > 0$ and two sequences (x_n) and (u_n) in *I* such that

$$|x_n - u_n| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(u_n)| \geq \varepsilon_0 \quad \text{for all } n \in \mathbb{N}.$$

Since I is bounded, the sequence (x_n) is bounded; and hence by the Bolzano-Weierstrass Theorem there is a subsequence (x_{n_k}) of (x_n) that converges to an element, say, z. Since I is closed, the limit z belongs to *I*, by Theorem A in chapter 2.

Since

$$|u_{n_k} - z| \leq |u_{n_k} - x_{n_k}| + |x_{n_k} - z| < \frac{1}{n_k} + |x_{n_k} - z|$$

and since $\frac{1}{n_k} \to 0$ and $|x_{n_k} - z| \to 0$ as $n_k \to \infty$, we have the subsequence (u_{n_k}) also converges to z.

Now if f is continuous at the point z, then, by the sequential criterion for continuity, as $(x_{n_k}) \to z$ and $(u_{n_k}) \to z$, we have both of the sequences $\left(f(x_{n_k})\right)$ and $\left(f(u_{n_k})\right)$ converge to $f(z)$. But this is not possible since

$$\left|f(x_n) - f(u_n)\right| \geq £_0$$

for all $n \in \mathbb{N}$. Thus the hypothesis that f is not uniformly continuous on the closed bounded interval I implies that f is not continuous at some point $z \in I$. Consequently, if f is continuous at every point of I, then f is uniformly continuous on I. This completes the proof.

Example Show that $g(x) = 1/x$ is uniformly continuous on $A = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$.

To show this, we note that since $A = [1, 2]$ is closed and bounded and g is continuous on A, by Uniform Continuity Theorem, g is Uniformly Continuous.

Lipchitz Functions

If a uniformly continuous function is given on a set that is not a closed bounded interval, then it is sometimes difficult to establish its uniform continuity. However, there is a condition that frequently occurs that is sufficient to guarantee uniform continuity.

Definition Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. If there exists a constant $K > 0$ such that

$$\left|f(x) - f(u)\right| \leq K\left|x - u\right| \qquad \ldots (**)$$

for all $x, u \in A$, then f is said to be a **Lipchitz function** (or to satisfy a **Lipchitz condition**) on A.

Geometric Interpretation

The condition (**) that a function $f : I \to \mathbb{R}$ on an interval I is a Lipchitz function can be interpreted geometrically as follows. If we write the condition as

$$\left|\frac{f(x) - f(u)}{x - u}\right| \leq K, \quad x, u \in I, \ x \neq u,$$

then the quantity inside the absolute value is the slope of a line segment joining the points $\left(x, f(x)\right)$ and $\left(u, f(u)\right)$. Thus a function f satisfies a Lipchitz condition if and only if the slopes of all line segments joining two points on the graph of $y = f(x)$ over I are bounded by some number K.

Theorem If $f : A \to \mathbb{R}$ is a Lipchitz function, then f is uniformly continuous on A.

To prove this we note that if f is a Lipchitz function, then there exists $K > 0$ such that

$$\left|\frac{f(x) - f(u)}{x - u}\right| \leq K, \ x, u \in I, \ x \neq u,$$

i.e., such that

$$\left|f(x) - f(u)\right| \leq K\left|x - u\right|, \quad x, u \in I, \ x \neq u.$$

If $£ > 0$ is given, we can take $¢ = £ / K$. If $x, u \in A$ satisfy $\left|x - u\right| < ¢$, then

$$\left|f(x) - f(u)\right| < K \cdot \frac{£}{K} = £$$

Therefore f is uniformly continuous on A. This completes the proof.

Example If $f(x) = x^2$ on $A = [0, b]$, where $b > 0$, then

$$\left|f(x) - f(u)\right| = \left|x + u\right|\left|x - u\right| \leq 2b\left|x - u\right|$$

for all x, u in $[0, b]$. Thus $\left|\dfrac{f(x) - f(u)}{x - u}\right| \leq 2b$ for all $x, u \in [0, b], \ x \neq u$.

Thus *f* satisfies the Lipchitz condition (4) with $K = 2b$ on *A*, and therefore *f* is uniformly continuous on *A*. Of course, since *f* is continuous and *A* is a closed bounded interval, this can also be deducted from the Uniform Continuity Theorem. (**Attention!** The function *f* in the above example *f* does not satisfy a Lipchitz condition on the interval $[0,\infty)$.)

Converse of the Theorem is not true, in general. Not every uniformly continuous function is a Lipchitz function. The following example shows this.

Example Show that $g(x) = \sqrt{x}$ on [0, 2] is *not* a Lipchitz function, but a uniformly continuous function.

If possible let $g(x) = \sqrt{x}$, for $x \in [0,2]$ is a Lipchitz function. Then there exists a constant $K > 0$ such that

$$|g(x) - g(u)| < K|x-u| \text{ for all } x,u \in [0,2]$$

i.e., $$\left|\sqrt{x} - \sqrt{u}\right| < k|x-u| \text{ for all } x,u \in [0,2]$$

Then, $$\sqrt{x} + \sqrt{u} = \sqrt{x} + \sqrt{u} \times \left|\frac{\sqrt{x} - \sqrt{u}}{\sqrt{x} - \sqrt{u}}\right|$$

$$= \frac{|x-u|}{\left|\sqrt{x} - \sqrt{u}\right|} > \frac{1}{K} \text{ for all } x, u \in A$$

By Archimedean Property, corresponding to the real number *K,* we can find a natural number *n*, such that $n > K$. Now consider $x = \frac{1}{4n^2}$ and $u = \frac{1}{16n^2}$. Clearly, both $x, u \in [0,2]$.

But

$$\sqrt{x} + \sqrt{u} = \frac{1}{2n} + \frac{1}{4n} = \frac{3}{4n} < \frac{1}{n} < \frac{1}{K}, \quad \text{since} \quad n > K$$

This is a contradiction. Hence our assumption is wrong. Hence $g(x)$ is *not* a Lipchitz function. Since [0, 2] is closed and bounded and since *g* is continuous, by Uniform Continuity Theorem *g* is uniformly continuous.

Alternatively, There is no number $K > 0$ such that $|g(x)| \le K|x|$ for all $x \in I$. The reason is this: If there is such a *K* then, in particular for $0 < x < 1$, $\frac{1}{x^{1/2}} < K$, which is not possible as the set $\{\frac{1}{x^{1/2}} : 0 < x < 1\}$ is unbounded. Therefore, *g* is not a Lipchitz function on *I*.

The following example illustrates that Uniform Continuity Theorem and Theorem 3 can sometimes be combined to establish the uniform continuity of a function on a set.

Example We now verify the uniform continuity of the function g defined by $g(x) = \sqrt{x}$ on the set $A = [0,\infty)$.

The uniform continuity of the continuous function *g* on the closed bounded interval $I = [0,2]$ follows from the Uniform Continuity Theorem. If $J = [1,\infty)$, then if both x,u are in *J*, we have

$$|g(x)-g(u)| = \left|\sqrt{x}-\sqrt{u}\right| = \frac{|x-u|}{\sqrt{x}+\sqrt{u}} \le \tfrac{1}{2}|x-u| .$$

Thus g is a Lipchitz function on J with constant $K=\frac{1}{2}$, and hence, by Theorem 3, g is uniformly continuous on $[1,\infty)$. Now $A = I \cup J$, and for a given $£ > 0$, uniform continuity of g on I gives a $¢$, denoted by $¢_I(£)$ and uniform continuity of g on J gives a $¢$, denoted by $¢_J(£)$.

Now take

$$¢(£) = \inf\{1, ¢_I(£), ¢_J(£)\} .$$

Then for a given $£ > 0$ there exists $¢ > 0$ such that

$$|g(x)-g(u)| < £ \text{ for } x,u \in A \text{ and } |x-u| < ¢.$$

Hence g is uniformly continuous on A.

The Continuous Extension Theorem

We have seen examples of functions that are continuous but not uniformly continuous on open intervals; for example, the function $f(x)=1/x$ on the interval (0, 1). On other hand, by the Uniformly Theorem, a function that is continuous on a closed bounded interval is always uniformly continuous. So the question arises: Under what conditions is a function uniformly continuous on a bounded **open** interval? The answer reveals the strength of uniform continuity, for it will be shown that a function on (*a*, *b*) is uniformly continuous if and only if it can be defined at the endpoints to produce a function that is continuous on the closed interval.

We first establish a result that is of interest in itself.

Theorem If $f : A \to \mathbb{R}$ is uniformly continuous on a set A of $\mathbb{R}$ and if (x_n) is a Cauchy sequence in A, then $\left(f(x_n)\right)$ is Cauchy sequence in $\mathbb{R}$.

To prove this we Let (x_n) be a Cauchy sequence in A and let $£ > 0$ be given. Choose $¢ > 0$ such that if x, u in A satisfy $|x-u| < ¢$, then $|f(x)-f(u)| < £$. Since (x_n) is a Cauchy sequence, by the definition[1] of Cauchy sequence, there exists $H(¢) \in \mathbb{N}$ such that

$$|x_n - x_m| < ¢ \text{ for all } n, m > H(¢).$$

By the choice of $¢$, this implies that for $n, m > H(¢)$, we have

$$|f(x_n) - f(x_m)| < £ .$$

Therefore, by the definition of Cauchy sequence, the sequence $\left(f(x_n)\right)$ is a Cauchy sequence. This completes the proof.

Definition A sequence $X=(x_n)$ of real numbers is said to be a **Cauchy sequence** if for every $£ > 0$ there is a natural number $H(£)$ such that for all natural numbers $n, m \ge H(£)$, the terms x_n, x_m satisfy $|x_n - x_m| < £$.

Example Using Theorem 4 show that $f(x)=1/x$ is not uniformly continuous on (0,1).

We note that the sequence given by $x_n = 1/n$ in (0,1) is Cauchy sequence, but the image sequence, where $f(x_n)=n$, is not a Cauchy sequence.

Sequential Criterion for Limits Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. Then the following statements are equivalent.

(i) $\lim_{x \to c} f(x) = L.$

(ii) For every sequence (x_n) in A that converges to c such that $x_n \neq c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to $f(c)$.

Cauchy Convergence Criterion A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Continuous Extension Theorem: A function *f* is uniformly continuous on the interval (a,b) if and only if it can be defined at the endpoints *a* and *b* such that the extended function is continuous on $[a,b]$.

Proof.

$(\Leftarrow)$ This direction is trival.

$(\Rightarrow)$ Suppose *f* is uniformly continuous on (a,b). We shall show how to extend *f* to *a*; the argument for *b* is similar. This is done by showing that $\lim_{x \to a} f(x) = L$ exists, and this is accomplished by using the sequential criterion for limits. If (x_n) is a sequence in (a,b) with $\lim(x_n) = a$, then it is a Cauchy sequence, and by the preceding theorem, the sequence $\left(f(x_n)\right)$ is also a Cauchy sequence, and so is convergent by Cauchy Convergence Criterion. Thus the limit $\lim\left(f(x_n)\right) = L$ exists. If (u_n) is any other sequence in (a,b) that converges to *a*, then $\lim(u_n - x_n) = a - a = 0$, so by the uniform continuity of *f* we have

$$\lim\left(f(u_n)\right) = \lim\left(f(u_n) - f(x_n)\right) + \lim\left(f(x_n)\right)$$

$$= 0 + L = L.$$

Since we get the same value *L* for every sequence converging to *a*, we infer from the sequential criterion for limits that *f* has limit *L* at *a*. If we define $f(a) = L$, then *f* is continuous at *a*. The same argument applies to *b*, so we conclude that *f* has a continuous extension to the interval [*a*, *b*]. This completes the proof.

Example Examine the uniform convergence of the functions

$$f(x) = \sin(1/x)$$

and

$$g(x) = x\sin(1/x)$$

on $(0,b]$ for all $b > 0$.

Since the limit of $f(x) = \sin(1/x)$ at 0 does not exist (Fig. *B*), we infer from the Continuous Extension Theorem that the function is not uniformly continuous on $(0,b]$ for any $b > 0$. On the other hand, since $\lim_{x \to 0} x\sin(1/x) = 0$ exists, the function $g(x) = x\sin(1/x)$ is uniformly continuous on $(0,b]$ for all $b > 0$.

Approximation

In this section we describe the way of approximating continuous functions by functions of an elementary nature.

Definition Let $I \subseteq \mathbb{R}$ be an interval and let $s: I \to \mathbb{R}$. This *s* is called a **step function** if it has only a finite number of distinct values, each value being assumed on one or more intervals in *I*.

Example The function $s: [-2,4] \to \mathbb{R}$ defined by

$$s(x)=\begin{cases} 0, & -2 \le x < -1, \\ 1, & -1 \le x \le 0, \\ \frac{1}{2}, & 0 < x < \frac{1}{2}, \\ 3, & \frac{1}{2} \le x < 1 \\ -2, & 1 \le x \le 3, \\ 2, & 3 < x \le 4, \end{cases}$$

is a step function.

We will now show that a continuous function on a closed bounded interval I can be approximated arbitrarily closely by step functions.

Theorem Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. If $£ > 0$, then there exists a step function $s_£ : I \to \mathbb{R}$ such that $|f(x) - s_£(x)| < £$ for all $x \in I$.

Proof. Since f is continuous on the closed bounded interval I, by the Uniform Continuity Theorem, the function f is uniformly continuous on I. Hence it follows that, given $£ > 0$ there is a number $¢(£) > 0$ such that if $x, y \in I$ and $|x - y| < ¢(£)$, then $|f(x) - f(y)| < £$.

Let $I = [a, b]$ and let $m \in \mathbb{N}$ be sufficiently large so that $h = \dfrac{b-a}{m} < ¢(£)$. We now divide $I = [a, b]$ into m disjoint intervals of length h; namely,

$I_1 = [a, a+h]$ and $I_k = (a + (k-1)h, a + kh]$ for $k = 2, \dots, m$.

Since the length of each subinterval I_k is $h < ¢(£)$, the difference between any two values of f in I_k is less than $£$. We now define

$$s_£(x) = f(a + kh) \text{ for } x \in I_k,\ k = 1, \dots, m,$$

so that $s_£$ is constant on each interval I_k. (In fact the value of $s_£$ on I_k is the value of f at the right endpoint of I_k. Consequently if $x \in I_k$, then

$$|f(x) - s_£(x)| = |f(x) - f(a + kh)| < £.$$

Therefore we have $|f(x) - s_£(x)| < £$ for all $x \in I$. This completes the proof.

The proof of the preceding theorem also establishes the following result.

Corollary Let $I = [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. If $£ > 0$, there exists a natural m such that if we divide I into m disjoint intervals I_k having length $h = (b - a)/m$, then the step function $s_£$ defined by

$$s_£(x) = f(a + kh) \text{ for } x \in I_k,\ k = 1, \dots, m,$$

satisfies

$$|f(x) - s_£(x)| < £ \text{ for all } x \in I.$$

Step functions are extremely elementary in character, but they are not continuous (except in trivial cases). Since it is often desirable to approximate continuous functions by elementary continuous functions, we now shall show that we can approximate continuous functions by continuous piecewise linear functions.

Definition Let $I=[a,b]$ be an interval. Then a function $g:I\to\mathbb{R}$ is said to be **piecewise linear** on *I* if *I* is the union of a finite number of disjoint intervals $I_1,...,I_m$, such that the restriction of *g* to each interval I_k is a linear function.

Remark It is evident that in order for a piecewise linear function *g* to be continuous on *I*, the line segments that form the graph of *g* must meet at the endpoints of adjacent subintervals $I_k, I_{k+1} (k=1, \dots, m-1)$.

Theorem Let *I* be a closed bounded interval and let $f:I\to\mathbb{R}$ be continuous on *I*. If $£>0$, then there exists a continuous piecewise linear function $g_£ : I\to\mathbb{R}$ such that $|f(x)-g_£(x)|<£$ for all $x\in I$.

To prove this we note that being continuous on a closed bounded interval, *f* is uniformly continuous on $I=[a,b]$. Hence corresponding to $£>0$, there is a number $¢(£)>0$ such that if $x,y\in I$ and $|x-y|<¢(£)$, then $|f(x)-f(y)|<£$. Let $m\in\mathbb{N}$ be sufficiently large so that $h=(b-a)/m<¢(£)$. Divide $I=[a,b]$ into *m* disjoint intervals of length *h*; namely let $I_1=[a,a+h]$, and let $I_k=[a+(k-1)h, a+kh]$ for $k=2,\dots,m$.

On each interval I_k we define $g_£$ to be linear function joining the points

$$(a+(k-1)h, f(a+(k-1)h)) \text{ and } (a+kh, f(a+kh)).$$

Then $g_£$ is a continuous piecewise linear function on *I*.

If $x\in I_k$ then $|x-(a+(k-1)h)|<h<¢(£)$, so that $|f(x)-f(a+(k-1)h)|<£$. Similarly, $|f(x)-f(a+kh)|<£$. i.e., for $x\in I_k$ the value $f(x)$ is within £ of $f(a+(k-1)h)$ and $f(a+kh)$. Hence

for $x\in I_k$,

$$g_£(x)=f(a+(k-1)h)$$
$$+\frac{f(a+kh)-f(a+(k-1)h)}{h}\cdot(x-(a+(k-1)h)).$$

Hence for $x\in I_k$,

$$|f(x)-g_£(x)|\le|f(x)-f(a+(k-1)h)|+$$
$$\frac{|f(a+kh)-f(a+(k-1)h)|}{h}\cdot h,$$

as $|x-(a+(k-1)h)|<h$

$$\le|f(x)-f(a+(k-1)h)|+|f(a+kh)-f(a+(k-1)h)|$$
$$\le|f(a+kh)-f(a+(k-1)h)|+|f(a+(k-1)h)-f(x)|$$
$$\le|f(a)+kh)-f(x)|, \text{ by Triangle Inequality } <£ .$$

That is, for $x\in I_k$, $|f(x)-g_£(x)|<£$ for all $x\in I_k$; therefore this inequality holds for all $x\in I$. This completes the proof.

Next is the important theorem of Weierstrass concerning the approximation of continuous functions by polynomial functions.

Weierstrass Approximation Theorem

Let $I=[a,b]$ and let $f: I \to \mathbb{R}$ be a continuous function. If $\varepsilon > 0$ is given, then there exists a polynomial function p_ε such that

$$|f(x) - p_\varepsilon(x)| < \varepsilon \text{ for all } x \in I.$$

Remark In order to obtain an approximation within an arbitrarily pre-assigned $\varepsilon > 0$, we have to choose polynomials of arbitrarily high degree.

There are a number of proofs of Weierstrass Approximation Theorem. One of the most elementary proofs is based on the following theorem, due to Serge Bernstein, for continuous functions on [0,1]. Given $f:[0,1] \to \mathbb{R}$, Bernstein defined the sequence of polynomials:

$$B_n(x) = \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

The polynomial function B_n is called the nth **Bernstein polynomial** for *f*; it is a polynomial of degree at most *n* and its coefficients depend on the values of the function *f* at the $n+1$ equally spaced points $0, 1/n, 2/n, \ldots, k/n, \ldots, 1$ and on the binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)} = \frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdots k}$$

Example If $f(x) = x^2$, for $x \in [0,1]$, calculate the first few Bernstein polynomials for *f*. Show that $B_n(x) = \left(1 - \frac{1}{n}\right)x^2 + \frac{1}{n}x$.

Here $f(x) = x^2$ for $x \in [0,1]$

Bernstein polynomials for *f* are given by

$$B_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) C(n,k)\, x^k (1-x)^{n-k},\ n = 1, 2, \ldots$$

Then $B_1(x) = \sum_{k=0}^{1} f\left(\frac{k}{1}\right) C(1,k)\, x^k (1-x)^{1-k}$

$$= \sum_{k=0}^{1} k^2\, C(1,k)\, x^k (1-x)^{1-k} = 0 + x = x.$$

$$B_2(x) = \sum_{k=0}^{2} f\left(\frac{k}{2}\right) C(2,k)\, x^k (1-x)^{2-k}$$

$$= \sum_{k=0}^{2} \frac{k^2}{4} C(2,k)\, x^k (1-x)^{2-k}$$

$$= 0 + \tfrac{1}{4}\cdot 2\cdot x(1-x) + x^2 = \tfrac{1}{2}x^2 + \tfrac{1}{2}x$$

$$B_3(x) = \sum_{k=0}^{3} f\left(\frac{k}{3}\right) C(3,k)\, x^k (1-x)^{3-k}$$

$$= \sum_{k=0}^{3} \frac{k^2}{9} C(3,k)\, x^k (1-x)^{3-k}$$

$$= 0 + \tfrac{1}{9}\cdot 3\cdot x(1-x)^2 + \tfrac{4}{9}\cdot 3\cdot x^2(1-x) + \tfrac{9}{9}\cdot 1\cdot x^3$$

$$= \tfrac{1}{3}[x - 2x^2 + x^3] + \tfrac{4}{3}[x^2 - x^3] + x^3 = \tfrac{2}{3}x^2 + \tfrac{1}{3}x .$$

In general for any $n = 1, 2, ...$

$$B_n(x) = \sum_{k=0}^{n} \frac{k^2}{n^2} C(n,k)\, x^k\, (1-x)^{\,n-k}$$

$$= \frac{1}{n^2} \sum_{k=0}^{n} [k(k-1) + k] \frac{n!}{k!(n-k)!} x^k\, (1-x)^{n-k}$$

$$= \frac{1}{n^2} \left[\sum_{k=2}^{n} \frac{n!}{(k-2)!(n-k)!} x^k\, (1-x)^{n-k} + \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} x^k\, (1-x)^{n-k} \right]$$

$$= \frac{1}{n^2} \left[n(n-1)x^2 \sum_{k=1}^{n} \frac{(n-2)!}{(k-2)!(n-k)!} x^{k-2}\, (1-x)^{n-k} + nx \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} x^{k-1}\, (1-x)^{\,n-k} \right]$$

$$= \frac{1}{n^2} \left\{ n(n-1)\, x^2 [x + (1-x)]^{n-2} + nx[x + 1(1-x)]^{n-1} \right\}$$

$$= \frac{1}{n^2} \left\{ n(n-1)x^2 + nx \right\} = \left(1 - \frac{1}{n} \right) x^2 + \frac{1}{n} x .$$

Bernstein's Approximation Theorem

Let $f : [0,1] \to \mathbb{R}$ be continuous and let $£ > 0$. There exists an $n_£ \in \mathbb{N}$ such that if $n \geq n_£$, then we have $\left| f(x) - B_n(x) \right| < £$ for all $x \in [0,1]$.

The Weierstrass Approximation Theorem can be derived from the Bernstein Approximation Theorem by a change of variable. Specifically, we replace $f : [a,b] \to \mathbb{R}$ by a function $F : [0,1] \to \mathbb{R}$, defined by

$$F(t) = f(a + (b-a)t) \text{ for } t \in [0,1].$$

The function *F* can be approximated by Bernstein polynomials for *F* on the interval [0, 1], which can then yield polynomials on [*a*, *b*] that approximate *f*.

Assignments

1. Show that the function $f(x) = \dfrac{1}{x^2}$ is uniformly continuous on $A = [1, \infty)$, but that it is not uniformly continuous on $B = (0, \infty)$.
2. Show that the fun*ction* $f(x) = \dfrac{1}{x}$ is uniformly continuous on the set $A = [a, \infty)$, where *a* is a positive constant.
3. Use the Nonuniform Continuity Criterion to show that the following functions are not uniformly continuous on the given sets.
   a) $f(x) = x^2, \quad A = [0, \infty)$
   b) $g(x) = \sin(1/x), \quad B = [0, \infty)$

4. Show that the function $f(x)=1/(1+x^2)$ for $x\in\mathbb{R}$ is uniformly continuous on $\mathbb{R}$.
5. Show that if *f* and *g* are uniformly continuous on a subset *A* of $\mathbb{R}$, then $f+g$ is uniformly continuous on *A*.
6. Show that if *f* and *g* are uniformly continuous on $A\subseteq\mathbb{R}$ and if they are both bounded on *A*, then their product *fg* is uniformly continuous on *A*.
7. If $f(x)=x$ and $g(x)=\sin x$, show that both *f* and *g* are uniformly continuous on $\mathbb{R}$, but that their product fg is not uniformly continuous on $\mathbb{R}$.
8. Give an example to show that boundedness of *f* and *g* is a necessary condition for the uniform continuity of the product.
9. Prove that if *f* and *g* are each uniformly continuous on $\mathbb{R}$, then the composite function $f\circ g$ is uniformly continuous on $\mathbb{R}$.
10. If *f* is uniformly continuous on $A\subseteq\mathbb{R}$, and $|f(x)|\geq k>0$ for all $x\in A$, show that $1/f$ is uniformly continuous on *A*.

11. Prove that if *f* is uniformly continuous on a bounded subset *A* of $\mathbb{R}$, then *f* is bounded on *A*.
12. If $g(x)=\sqrt{x}$ for $x\in[0,1]$, show that there does not exist a constant *K* such that $|g(x)|\leq K|x|$ for all $x\in[0,1]$. Conclude that the uniformly continuous *g* is not a Lipchitz function on [0,1].
13. Show that if *f* is continuous on $[0,\infty)$ and uniformly continuous on $[a,\infty)$ for some positive constant *a*, then *f* is uniformly continuous on $[0,\infty)$.
14. Let $A\subseteq\mathbb{R}$ and suppose that $f:A\to\mathbb{R}$ has the following property: for each $£>0$ there exists a function $g_£:A\to\mathbb{R}$ such that $g_£$ is uniformly continuous on *A* and $|f(x)-g_£(x)|<£$ for all $x\in A$. Prove that *f* is uniformly continuous on *A*.
15. A function $f:\mathbb{R}\to\mathbb{R}$ is said to be **periodic** on $\mathbb{R}$ if there exists a number $p>0$ such that $f(x+p)=f(x)$ for all $x\in\mathbb{R}$. Prove that a continuous periodic function on $\mathbb{R}$ is bounded and uniformly continuous on $\mathbb{R}$.
16. If $f_0(x)=1$ for $x\in[0,1]$, calculate the first few Bernstein polynomials for f_0. Show that they coincide with f_0. [*Hint*: Binomial Theorem asserts that

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}. \qquad]$$

17. Let $f:[0.1]\to\mathbb{R}$ is continuous and has only rational values. Show that *f* must be constant.
18. If $f_1(x)=x$ for $x\in[0,1]$, calculate the first few Bernstein polynomials for f_1. Show that they coincide with f_1.

4
THE RIEMANN INTEGRAL - PART I

The Riemann integral, created by Bernhard Riemann, was the first rigorous definition of the integral of a function on an interval. For many functions and practical applications, the Riemann integral can be evaluated by using the fundamental theorem of calculus or (approximately) by numerical integration. The Riemann integral is unsuitable for many theoretical purposes. Some of the technical deficiencies in Riemann integration can be remedied with the Riemann–Stieltjes integral, and most disappear with the Lebesgue integral.

Partition and Tagged Partitions

If $I=[a,b]$ is a closed bounded interval in $\mathbb{R}$, then a partition of I is a finite, ordered set $P=(x_0, x_1, ..., x_{n-1}, x_n)$ of points in I such that

$$a = x_0 < x_1 < ... < x_{n-1} < x_n = b$$

The points of P are used to divide $I=[a,b]$ into non-overlapping subintervals

$$I_1 = [x_0, x_1],\ I_2 = [x_1, x_2], ...,\ I_n = [x_{n-1}, x_n].$$

Often we will denote the partition P by the notation $P = \{[x_{i-1}, x_i]\}_{i=1}^n$. We define the **norm** (or **mesh**) of P to be the number

$$\|P\| = \max\{x_1 - x_0, x_2 - x_1, ..., x_n - x_{n-1}\}.$$

Thus the norm of a partition is merely the length of the largest subinterval into which the partition divides $[a,b]$.

Clearly, many partitions have the same norm, so the partition is *not* a function of the norm.

If a point t_i has been selected from each subinterval $I_i = [x_{i-1}, x_i]$, for $i = 1, 2, ..., n,$ then the points are called **tags** of the sub-intervals I_i. A set of ordered pairs

$$\dot{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$$

of subintervals and corresponding tags is called a **tagged partition** of I. (The dot over P indicates that a tag has been chosen for each subinterval.) The tags can be chosen in a wholly arbitrary fashion; for example, we can choose the tags to be the left endpoints, or the midpoints of the subintervals, etc. Note that an endpoint of a subinterval can be used as a tag for two consecutive subintervals. Since each tag can be chosen in infinitely many ways, each partition can be tagged in infinitely many ways.

The *norm of a tagged partition* is defined as for an ordinary partition and does not depend on the choice of tags.

Definition If $\dot{P}$ is the tagged partition given above, we define the **Riemann sum** of a function $f:[a,b] \to \mathbb{R}$ corresponding to $\dot{P}$ to be the number

$$S(f;\dot{P}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) .$$

Notation We will also use this notation when $\dot{P}$ denotes a *subset* of a partition, and not the entire partition.

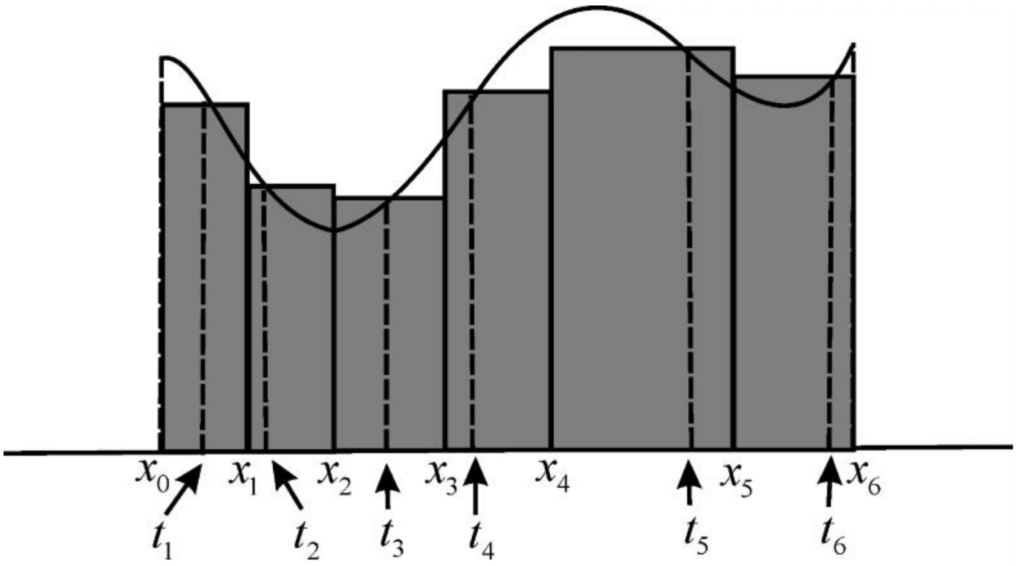


Fig. 1 Riemann sum of a positive function.

Definition If the function *f* is **positive** on $[a,b]$, then the Riemann sum (2) is the sum of the areas of *n* rectangles whose bases are the subintervals $I_i = [x_{i-1}, x_i]$ and whose heights are $f(t_i)$ (Fig.1).

Riemann Integral

We now define the Riemann integral of a function *f* on an interval $[a,b]$.

Definition A function $f:[a,b] \to \mathbb{R}$ is said to be **Riemann integrable** on $[a,b]$ if there exists a number $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $\dot{P}$ is any tagged partition of $[a,b]$ with $\|\dot{P}\| < \delta_\varepsilon$, then

$$|S(f;\dot{P}) - L| < \varepsilon .$$

The set of all Riemann integrable functions on $[a,b]$ will be denoted by $R\ [a,b]$.

Remark It is sometimes said that integral *L* is "the limit" of the Riemann sums $S(f;\dot{P})$ as the norm $\|\dot{P}\| \to 0$. However, since $S(f;\dot{P})$ is not a function of $\|\dot{P}\|$, this limit is not of the type that we have studied before.

First we will show that if $f \in R\ [a,b]$, then the number *L* is uniquely determined. It will be called the **Riemann integral of** *f* over $[a,b]$. Instead of *L*, we will usually write

$$L = \int_a^b f \quad \text{or} \quad \int_a^b f(x)dx.$$

It should be understood that any letter than x can be used in the latter expression, so long as it does not cause any ambiguity.

Theorem If $f \in R\ [a,b]$, then the value of the integral is uniquely determined.

Proof. Assume that L' and L'' both satisfy the definition of Riemann integral of *f* over $[a,b]$, and let $\pounds > 0$. Then there exists $¢'_{\pounds/2} > 0$ such that if $\dot{P}_1$ is any tagged partition with $\|\dot{P}_1\| < ¢'_{\pounds/2}$, then

$$\left|S(f;\dot{P}_1) - L'\right| < \pounds/2. \qquad \ldots (*)$$

Also there exists $¢''_{\pounds/2} > 0$ such that if $\dot{P}_2$ is any tagged partition with $\|\dot{P}_2\| < ¢''_{\pounds/2}$, then

$$\left|S(f;\dot{P}_2) - L''\right| < \pounds/2. \qquad \ldots (**)$$

Now let $¢_{\pounds} = \min\{¢'_{\pounds/2}, ¢''_{\pounds/2}\} > 0$ and let $\dot{P}$ be a tagged partition with $\|\dot{P}\| < ¢_{\pounds}$. Since both $\|\dot{P}\| < ¢'_{\pounds/2}$ and $\|\dot{P}\| < ¢''_{\pounds/2}$, then (*) and (**) gives

$$\left|S(f;\dot{P}) - L'\right| < \pounds/2 \text{ and } \left|S(f;\dot{P}) - L''\right| < \pounds/2,$$

and hence it follows from the Triangle Inequality that

$$\begin{aligned} |L' - L''| &= \left|L' - S(f;\dot{P}) + S(f;\dot{P}) - L''\right| \\ &\le \left|L' - S(f;\dot{P})\right| + \left|S(f;\dot{P}) - L''\right| \\ &< \pounds/2 + \pounds/2 = \pounds. \end{aligned}$$

Since $\pounds > 0$ is arbitrary, it follows that $L' = L''$.

Method of Verifying $f \in R\ [a,b]$ using Definition

If we use only the definition, in order to show that a function *f* is Riemann integrable we must (i) know (or guess correctly) the value L of the integral, and (ii) construct a $¢_{\pounds}$ that will suffice for an arbitrary $\pounds > 0$. The determination of L is sometimes done by calculating Riemann sums and guessing what L must be. The determination of $¢_{\pounds}$ is likely to be difficult.

In actual practice, we usually show that $f \in R\ [a,b]$ by making use of some of the theorems that will be discussed after considering some examples based on the Definition.

Example Every constant function on $[a,b]$ is in $R\ [a,b]$.

Let $f(x) = k$ for all $x \in [a,b]$. If $\dot{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is any tagged partition of [*a*, *b*], then it is clear that

$$\begin{aligned} S(f;\dot{P}) &= \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) = \sum_{i=1}^n k(x_i - x_{i-1}) = k\sum_{i=1}^n (x_i - x_{i-1}) \\ &= k(x_n - x_{n-1} + x_{n-1} + x_{n-1} - x_{n-2} + \cdots - x_0) \\ &= k(x_n - x_0) \end{aligned}$$

$= k(b-a)$.

Hence, for any $£ > 0$, we can choose $¢_£ = 1$ so that if $\|\dot{P}\| < ¢_£$, then

$$\left|S(f;\dot{P}) - k(b-a)\right| = 0 < £.$$

Since $£ > 0$ is arbitrary, we conclude that $f \in R\ [a,b]$ and $\int_a^b f = k(b-a)$.

Example Let $g:[0,3] \to \mathbb{R}$ be defined by $g(x) = 2$ for $0 \le x \le 1$ and $g(x) = 3$ for $1 < x \le 3$. Find $\int_0^3 g$.

A preliminary investigation, based on the graph of *g*, suggests that we might expect that $\int_0^3 g = 2\times 1 + 2\times 3 = 8$.

Let $\dot{P}$ be a tagged partition of [0,3] with norm $< ¢$; we will show how to determine $¢$ in order to ensure that $\left|S(g;\dot{P}) - 8\right| < £$. Let $\dot{P}_1$ be the subset of $\dot{P}$ having its tags in [0,1] where $g(x) = 2$, and the let $\dot{P}_2$ be the subset of $\dot{P}$ with its tags in (1,3] where $g(x) = 3$. It is obvious that we have

$$S(g;\dot{P}) = S(g;\dot{P}_1) + S(g;\dot{P}_2).$$

Since $\|\dot{P}\| < ¢$, if $u \in [0, 1-¢]$ and $u \in [x_{i-1}, x]$, then $x_{i-1} \le 1-¢$ so that $x_i < x_{i-1} + ¢ \le 1$, and hence the tag $t_i \in [0,1]$. Therefore, the interval $[0, 1-¢]$ is contained in the union of all subintervals in $\dot{P}$ with tags $t_i \in [0,1]$. Similarly, this union is contained in $[0, 1+¢]$. Since $g(t_i) = 2$ for these tags, we have

$$2(1-¢) \le S(g;\dot{P}_1) \le 2(1+¢).$$

A similar argument shows that the union of all subintervals with tags $t_i \in (1,3]$ contains the interval $[1+¢, 3]$ of length $2-¢$, and is contained in $[1-¢, 3]$ of length $2+¢$. Therefore,

$$3(2-¢) \le S(g;\dot{P}_2) \le 3(2+¢).$$

Adding these inequalities and using equation (3) we have

$$8 - 5¢ \le S(g;\dot{P}) = S(g;\dot{P}_1) + S(g;\dot{P}_2) \le 8 + 5¢$$

and hence it follows that

$$\left|S(g;\dot{P}) - 8\right| \le 5¢.$$

To have this final term $< £$, we are led to take $¢_£ = £/5$.

Making such a choice (for example, if we take $¢_£ = £/10$), we can retrace the argument and see that $\left|S(g;\dot{P}) - 8\right| < £$ when $\|\dot{P}\| < ¢_£$. Since $£ > 0$ is arbitrary, we have proved that $g \in R[0,3]$ and $\int_0^3 g = 8$, as expected.

Example Let $h(x) = x$ for $x \in [0,1]$. Show that $h \in R[0,1]$.

We will employ a trick that enables us to guess the value of the integral by considering a particular choice of the tag points. Indeed, if $\{I_i\}_{i=1}^n$ is any partition of [0, 1] and we choose the tag of the interval $I_i = [x_{i-1}, x_i]$ to be the midpoint $q_i = \frac{1}{2}(x_{i-1} + x_i)$, then the contribution of this term to the Riemann sum corresponding to the tagged partition $\dot{Q} = \{(I_i, q_i)\}_{i=1}^n$ is

$$h(q_i)(x_i - x_{i-1}) = \tfrac{1}{2}(x_i + x_{i-1})(x_i - x_{i-1}) = \tfrac{1}{2}(x_i^2 - x_{i-1}^2).$$

If we add these terms and note that the sum telescopes, we obtain

$$S(h, \dot{Q}) = \sum_{i=1}^{n} \tfrac{1}{2}(x_i^2 - x_{i-1}^2) = \tfrac{1}{2}(1^2 - 0^2) = \tfrac{1}{2}.$$

Now let $\dot{P} = \{(I_i, t_i)\}_{i=1}^n$ be an arbitrary tagged partition of [0,1] with $\|\dot{P}\| < ¢$ so that $x_i - x_{i-1} < ¢$ for $i = 1,...,n$. Also let $\dot{Q}$ have the same partition points, but where we choose the tag q_i to be the midpoint of the interval I_i. Since both t_i and q_i belong to this interval, we have $|t_i - q_i| < ¢$. Using the Triangle Inequality, we deduce

$$\begin{aligned} |S(h;\dot{P}) - S(h;\dot{Q})| &= \left| \sum_{i=1}^{n} t_i(x_i - x_{i-1}) - \sum_{i-1}^{n} q_i(x_i - x_{i-1}) \right| \\ &\le \sum_{i=1}^{n} |t_i - q_i|(x_i - x_{i-1}) \\ &< ¢ \sum_{i=1}^{n} (x_i - x_{i-1}) \\ &= ¢\,(x_n - x_0) = ¢ . \end{aligned}$$

Since $S(h;\dot{Q}) = \frac{1}{2}$, we infer that if $\dot{P}$ is any tagged partition with $\|\dot{P}\| < ¢$, then

$$\left| S(h;\dot{P}) - \tfrac{1}{2} \right| < ¢ .$$

Therefore we are led to take $¢_£ \le £$. If we choose $¢_£ = £$, we can retrace the argument to conclude that $h \in R$ [0,1] and $\int_0^1 h = \int_0^1 x dx = \frac{1}{2}$.

Example Let $F(x) = 1$ for $x = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$, and $F(x) = 0$ elsewhere on [0, 1]. Show that $F \in R$ [0,1] and $\int_0^1 F = 0$.

Here there are four points where F is not 0, each of which can belong to two sub-intervals in a given tagged partition $\dot{P}$. Only these terms will make a non zero contribution to $S(F;\dot{P})$. Therefore we choose $¢_£ = £/8$.

If $\|\dot{P}\| < ¢_£$, let $\dot{P}_0$ be the subset of $\dot{P}$ with tags different from $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$, and let $\dot{P}_1$ be the subset of $\dot{P}$ with tags at these points. Since $S(F;\dot{P}_0) = 0$, it is seen that

$S(F;\dot{P}) = S(F;\dot{P}_0) + S(F;\dot{P}_1) \;\; = S(F;\dot{P}_1)$. Since there are at most 8 terms in $S(F;\dot{P}_1)$ and each term is $< 1 \cdot ¢_£$, we conclude that $0 \le S(F;\dot{P}) = S(F;\dot{P}_1) < 8¢_£ = £$. Thus $F \in R\ [0,1]$ and $\int_0^1 F = 0$.

Example Let $G(x) = 1/n \;\; (n \in \mathbb{N})$, and $G(x) = 0$ elsewhere on [0,1]. Show that $G \in R\ [0,1]$ and $\int_0^1 G = 0$.

Given $£ > 0$, let $E_£$ be the (finite) set of points where $G(x) \ge £$, let $n_£$ be the number of points in $E_£$, and let $¢ = £/(2n_£)$. Let $\dot{P}$ be a tagged partition such that $\|\dot{P}\| < ¢_£$. Let $\dot{P}_0$ be the subset of $\dot{P}$ with tags outside of $E_£$ and let $\dot{P}_1$ be the subset of $\dot{P}$ with tags in $E_£$. Then, as in the previous example, we have

$$0 \le S(G;\dot{P}) = S(G;\dot{P}_1) < (2n_£)¢_£ = £.$$

Since $£ > 0$ is arbitrary, we conclude that $G \in R[0,1]$ and $\int_0^1 G = 0$.

Example Suppose that $c \le d$ are points in $[a,b]$. If $¨ : [a,b] \to \mathbb{R}$ satisfies $¨(x) = ~ > 0$ for $x \in [c,d]$ and $¨(x) = 0$ elsewhere in $[a,b]$, prove that $¨ \in R\ [a,b]$ and that $\int_a^b ¨ = ~(d-c)$.

For a tagged partition $\dot{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$, Riemann sum is giving by

$$S(¨, \dot{P}) = \sum_{i=1}^{n} ¨(t_i)(x_i - x_{i-1}).$$

Given $£ > 0$, let $¢_£ = £/4~$. Then if $\|\dot{P}\| < ¢_£$, then $\|\dot{P}\| < £/4~$, and then the union of the subintervals in $\dot{P}$ with tags in $[c,d]$ contains the interval $[c + ¢_£, d - ¢_£]$ and is contained in $[c - ¢_£, d + ¢_£]$. Therefore

$$~(d - c - 2¢_£) \;\le S(¨;\dot{P}) \;\le ~(d - c + 2¢_£),$$

and hence

$$\left| S(¨;\dot{P}) - ~(d-c) \right| \le 2~¢_£ < 4~¢_£ = £.$$

Hence by the Definition of Riemann integrability, $¨ \in R\ [a,b]$ and that $\int_a^b ¨ = ~(d-c)$.

Some Properties of the Integral

The difficulties involved in determining the value of the integral and of $¢_£$ suggest that it would be very useful to have some general theorems. The first result in this direction enables us to form certain algebraic combinations of integrable functions.

Theorem Suppose that *f* and *g* are in $R\ [a,b]$. Then:

a) If $k \in \mathbb{R}$, the function kf is in $R\ [a,b]$ and

$$\int_a^b kf = k\int_a^b f$$

b) The function $f + g$ is in $R\ [a,b]$ and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

c) If $f(x) \le g(x)$ for all $x \in [a,b]$, then

$$\int_a^b f \le \int_a^b g$$

To prove (b), we note that if $\dot{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a tagged partition of $[a,b]$, then

$$S(kf;\dot{P}) = \sum_{i=1}^n (kf)(t_i)(x_i - x_{i-1}) = \sum_{i=1}^n k \cdot f(t_i)(x_i - x_{i-1})$$

$$= k\sum_{i=1}^n f(t_i)(x_i - x_{i-1}) = kS(f;\dot{P}).$$

Now the assertion (a) follows.

$S(f+g;\dot{P}) = S(f;\dot{P}) + S(g;\dot{P})$.

Given $£ > 0$, we can use the argument in the proof of the Uniqueness Theorem to construct a number $¢_£ > 0$ such that if $\dot{P}$ is any tagged partition with $\|\dot{P}\| < ¢_£$, then both

Hence

$$\left|S(f;\dot{P}) - \int_a^b f\right| < £/2 \quad \text{and} \quad \left|S(g;\dot{P}) - \int_a^b g\right| < £/2 \qquad \dots (4)$$

$$\left|S(f+g;\dot{P}) - \left(\int_a^b f + \int_a^b g\right)\right| = \left|S(f;\dot{P}) + S(g;\dot{P}) - \int_a^b f - \int_a^b g\right|$$

$$\le \left|S(f;\dot{P}) - \int_a^b f\right| + \left|S(g;\dot{P}) - \int_a^b g\right|$$

$$< £/2 + £/2 = £.$$

Since $£ > 0$ is arbitrary, we conclude that $f + g \in R[a,b]$ and that its integral is the sum of the integrals of *f* and *g*.

To prove (c), we note that $S(f;\dot{P}) \le S(g;\dot{P})$ and also (4) implies

$$\int_a^b f - \frac{£}{2} < S(f;\dot{P}) \text{ and } S(g;\dot{P}) < \int_a^b g + \frac{£}{2}.$$

If we use the fact that $S(f;\dot{P}) \le S(g;\dot{P})$, we have

$$\int_a^b f \le \int_a^b g + £.$$

But since $£ > 0$ is arbitrary, we conclude that $\int_a^b f \le \int_a^b g$. This completes the proof.

Theorem If $f_1, \dots, f_n$ are in $R[a,b]$ and if $k_1, \dots, k_n \in \mathbb{R}$, then the linear combination $f = \sum_{i=1}^n k_i f_i$ belongs to $R\ [a,b]$ and

$$\int_a^b f = \sum_{i=1}^n k_i \int_a^b f_i$$

Hint for the Proof. Use mathematical Induction and part (a) and (b) of Theorem 2 above.

Theorem If $f \in R[a,b]$ and $|f(x)| \le M$ for all $x \in [a,b]$, then $\left|\int_a^b f\right| \le M(b-a)$.

Take

$$g(x) = M \text{ for } x \in [a,b]$$

and use part (c) of Theorem 2.

Boundedness Theorem If $f \in R\,[a,b]$, then f is bounded on $[a,b]$.

Proof Assume that f is an unbounded function in $R[a,b]$ with integral L. Then there exists $¢ > 0$ such that if $\dot{P}$ is any tagged partition of $[a,b]$ with $\|\dot{P}\| < ¢$, then we have

$$|S(f;\dot{P}) - L| < 1,$$

which implies that

$$|S(f;\dot{P})| < |L| + 1. \qquad (***)$$

Now let $Q = \{[x_{i-1}, x_i]\}_{n=1}^{n}$ be a partition of $[a,b]$ with $\|Q\| < ¢$. Since $|f|$ is not bounded on $[a,b]$, then there exists at least one subinterval in Q, say $[x_{k-1}, x_k]$, on which $|f|$ is not bounded $-$ for, if $|f|$ is bounded on each subinterval $[x_{i-1}, x_i]$ by M_i, then it is bounded on $[a,b]$ by max $\{M_1, ..., M_n\}$.

We will now pick tags for Q that will provide a contradiction to (***). We tag Q by $t_i = x_i$ for $i \neq k$ and we pick $t_k \in [x_{k-1}, x_k]$ such that

$$|f(t_k)(x_k - x_{k-1})| > |L| + 1 + \left|\sum_{i \neq k} f(t_i)(x_i - x_{i-1})\right|.$$

From the Triangle Inequality (in the form $|A + B| \geq |A| - |B|$), we have

$$|S(f;\dot{Q})| \geq |f(t_k)(x_k - x_{k-1})| - \left|\sum_{i \neq k} f(t_i)(x_i - x_{i-1})\right| > |L| + 1,$$

which contradicts (***).

Remark In view of the above theorem we have the following:
An unbounded function cannot be Riemann integrable.

We now consider an example of a function that is discontinuous at every rational number and is not monotone, but is Riemann integrable.

Example We consider Thomae's function $h : [0,1] \to \mathbb{R}$ defined by $h(x) = 0$ if $x \in [0,1]$ is irrational, $h(0) = 1$ and by $h(x) = 1/n$ if $x \in [0,1]$ is the rational number $x = m/n$ where $m, n \in \mathbb{N}$ have no common integer factors except 1.

We claim that h is continuous at every irrational number and discontinuous at every rational number in $[0,1]$. If $a > 0$ is rational, let (x_n) be a sequence of irrational numbers in A that converges to a. Then $\lim(h(x_n)) = \lim(0) = 0$, while $h(a) > 0$. Hence, in view of Sequential Criterion for Continuity, h is discontinuous at the rational point a.

If b is an irrational number and $£ > 0$, then (by the Archimedean Property) there is a natural number n_0 such that $\frac{1}{£} < n_0$. i.e., such that $\frac{1}{n_0} < £$. There are only a finite number of rationals with denominator less than n_0 in the interval $(b-1, b+1)$. Hence $¢ > 0$ can be chosen so small that the neighborhood $(b-¢, b+¢)$ contains no rational numbers with denominator less than n_0. It then follows that for $|x-b| < ¢$ and $x \in (0,\infty)$, we have $|h(x)-h(b)| = |h(x)| \le \frac{1}{n_0} < £$. Then h is continuous at the irrational number a.

We will now show that $h \in R[0,1]$.

Let $£ > 0$. Then the set $E_£ = \{x \in [0,1] : h(x) \ge \frac{£}{2}\}$ is a finite set. We let $n_£$ be the number of elements in $E_£$ and let $¢_£ = £/(4n_£)$. If $\dot{P}$ is a tagged partition with $\|\dot{P}\| < ¢_£$, let $\dot{P}_1$ be the subset of $\dot{P}$ having tags in $E_£$ and $\dot{P}_2$ be the subset of $\dot{P}$ having tags elsewhere in $[0,1]$. We observe that $\dot{P}_1$ has at most $2n_£$ intervals whose total length is $< 2n_£ ¢_£ = £/2$ and that $0 < h(t_i) \le 1$ for every tag in $\dot{P}_1$. Also the total lengths of the subintervals in $\dot{P}_2$ is ≤ 1 and $h(t_i) < £/2$ for every tag in $\dot{P}_2$. Therefore we have

$$|S(h;\dot{P})| = S(h;\dot{P}_1) + S(h;\dot{P}_2) < 1 \cdot 2n_£ ¢_£ + (£/2) \cdot 1 = £ .$$

Since $£ > 0$ is arbitrary, we infer that $h \in R[0,1]$ with integral 0.

Assignments

1. If $I = [1, 8]$, calculate the norms of the following partitions:
   a) $P_1 = (1,2,3,4,5,6,7,8)$ b) $P_2 = (1,3,4,6,8)$
   c) $P_3 = (1,3,6,8)$ d) $P_4 = (1,5,8)$
2. If $f(x) = x^2$ for $x \in [1,8]$, calculate the following Riemann sums where $\dot{P}_i$ has the same partition points as in Assignment 1, and the tags are selected as indicated.
   a) P_1 with the tags at the left endpoints of the subintervals.
   b) P_1 with the tags at the right endpoints of the subintervals.
   c) P_2 with the tags at the left endpoints of the subintervals.
   d) P_2 with the tags at the right endpoints of the subintervals.

3. Show that $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ if and only if there exists $L \in \mathbb{R}$ such that for every $£ > 0$ there exists $¢_£ > 0$ such that if $\dot{P}$ is any tagged partition with norm $\|\dot{P}\| \le ¢_£$, then $|S(f;\dot{P}) - L| \le £$.
4. Let $\dot{P}$ be a tagged partition of $[0,3]$.

a) Show that the union U_1 of all subintervals in $\dot{P}$ with tags in $[0,1]$ satisfies $[0, 1-\|\dot{P}\|] \subseteq U_1 \subseteq [0, 1+\|\dot{P}\|]$

b) Show that the union U_2 of all subintervals in $\dot{P}$ with tags in $[1,2]$ satisfies $[1+\|\dot{P}\|, 2-\|\dot{P}\|] \subseteq U_2 \subseteq [1-\|\dot{P}\|, 2+\|\dot{P}\|]$.

5. Let $\dot{P} = \{(I_i, t_i)\}_{i=1}^n$ be a tagged partition of $[a,b]$ and let $c_1 < c_2$.

a) If u belongs to a subinterval I_i whose tag satisfies $c_1 \le t_i \le c_2$, show that $c_1 - \|\dot{P}\| \le u \le c_2 + \|\dot{P}\|$.

b) If $v \in [a,b]$ and satisfies $c_1 + \|\dot{P}\| \le v \le c_2 - \|\dot{P}\|$, then the tag t_i of any subinterval I_i that contains v satisfies $t_i \in [c_1, c_2]$.

6. a) Let $f(x) = 2$ if $0 \le x < 1$ and $f(x) = 1$ if $1 \le x \le 2$. Show that $f \in R[0,2]$ and evaluate its integral.

b) Let $h(x) = 2$ if $0 \le x < 1$, $h(1) = 3$ and $h(x) = 1$ if $1 < x \le 2$. Show that $h \in R[0,2]$ and evaluate its integral.

7. If $f \in R[a,b]$ and if $(\dot{P}_n)$ is any sequence of tagged partitions of $[a,b]$ such that $\|\dot{P}_n\| \to 0$, prove that $\int_a^b f = \lim_n S(f; \dot{P}_n)$.

8. Let $g(x) = 0$ if $x \in [0,1]$ is rational and $g(x) = 1/x$ if $x \in [0,1]$ is irrational. Explain why $g \notin R[0,1]$. However, show that there exists a sequence $(\dot{P}_n)$ of tagged partitions of $[a,b]$ such that $\|\dot{P}_n\| \to 0$ and $\lim_n S(g; \dot{P}_n)$ exists.

9. Suppose that *f* is bounded on $[a,b]$ and that there exists two sequences of tagged partitions of $[a,b]$ such that $\|\dot{P}\| \to 0$ and $\|\dot{Q}_n\| \to 0$, but such that $\lim_n S(f; \dot{P}_n) \ne \lim_n S(f; \dot{Q}_n)$. Show that *f* is not in $R[a,b]$.

10. Consider the Dirichlet function, defined by $f(x) = 1$ for $x \in [0,1]$ rational and $f(x) = 0$ for $x \in [0,1]$ irrational. Use the preceding exercise to show that *f* is not Riemann integrable on $[0,1]$.

11. Suppose that $f : [a,b] \to \mathbb{R}$ and that $f(x) = 0$ except for a finite number of points $c_1, ..., c_n$ in $[a,b]$. Prove that $f \in R[a,b]$ and that $\int_a^b f = 0$.

12. If $g \in R[a,b]$ and if $f(x) = g(x)$ except for a finite number of points in $[a,b]$, prove that $f \in R[a,b]$ and that $\int_a^b f = \int_a^b g$.

13. Let $0 \le a < b$, let $Q(x) = x^2$ for $x \in [a,b]$ and let $P = \{[x_{i-1}, x_i]\}_{i=1}^n$ be a partition of $[a,b]$. For each *i*, let q_i be the positive *square root* of

$$\tfrac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2)$$

a) Show that q_i satisfies $0 \le x_{i-1} \le q_i \le x_i$

b) Show that $Q(q_i)(x_i - x_{i-1}) = \frac{1}{3}(x_i^3 - x_{i-1}^3)$

c) If $\dot{Q}$ is the tagged partition with the same subintervals as P and the tags q_i, show that $S(Q,\dot{Q}) = \frac{1}{3}(b^3 - a^2)$

d) Show that $Q \in R\,[a,b]$ and

$$\int_a^b Q = \int_a^b x^2 dx = \tfrac{1}{3}(b^3 - a^3).$$

14. If $f \in R\,[a,b]$ and $c \in \mathbb{R}$, we define g on $[a+c, b+c]$ by $g(y) = f(y-c)$. Prove that $g \in R\,[a+c, b+c]$ and that $\int_{a+c}^{b+c} g = \int_a^b f$. The function g is called the c-**translate** of f.

15. Let $0 \le a < b$ and $m \in \mathbb{N}$, let $M(x) = x^m$ for $x \in [a,b]$ and let $P = \{[x_{i-1}, x_i]\}_{i=1}^n$ be a partition of $[a,b]$. For each i, let q_i be the positive mth root of

$$\frac{1}{m+1}(x_i^m + x_i^{m-1}x_{i-1} + \cdots + x_i x_{i-1}^{m-1} + x_{i-1}^m).$$

a) Show that q_i satisfies $0 \le x_{i-1} \le q_i \le x_i$

b) Show that $M(q_i)(x_i - x_{i-1}) = \frac{1}{m+1}(x_i^{m+1} - x_{i-1}^{m+1})$

c) If $\dot{Q}$ is the tagged partition with the same subintervals as P and the tags q_i, show that $S(M;\dot{Q}) = \dfrac{1}{m+1}(b^{m+1} - a^{m+1})$

d) Show that $M \in R\,[a,b]$ and

$$\int_a^b M = \int_a^b x^m dx = \frac{1}{m+1}(b^{m+1} - a^{m+1}).$$

5
THE RIEMANN INTEGRAL - PART II

Cauchy Convergence Criterion for Sequences: A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Cauchy Criterion for Integrability: A function $f:[a,b]\to\mathbb{R}$ belongs to $R\,[a,b]$ if and only if for every $\varepsilon>0$ there exists $\delta_\varepsilon>0$ such that if $\dot{P}$ and $\dot{Q}$ are any tagged partitions of $[a,b]$ with $\|\dot{P}\|<\delta_\varepsilon$ and $\|\dot{Q}\|<\delta_\varepsilon$, then

$$\left|S(f;\dot{P})-S(f;\dot{Q})\right|<\varepsilon.$$

Proof.

$(\Rightarrow)$ If $f\in R\,[a,b]$ with integral L, let $\delta_\varepsilon=\frac{\gamma_\varepsilon}{2}>0$ be such that if $\dot{P}$ and $\dot{Q}$ are any tagged partitions of $[a,b]$ with $\|\dot{P}\|<\delta_\varepsilon$ and $\|\dot{Q}\|<\delta_\varepsilon$, then

$$\left|S(f;\dot{P})-L\right|<\varepsilon/2 \quad\text{and}\quad \left|S(f;\dot{Q})-L\right|<\varepsilon/2.$$

Therefore we have

$$\begin{aligned}\left|S(f;\dot{P})-S(f;\dot{Q})\right| &\le \left|S(f;\dot{P})-L+L-S(f;\dot{Q})\right|\\ &\le \left|S(f;\dot{P})-L\right|+\left|L-S(f;\dot{Q})\right|\\ &\le \varepsilon/2+\varepsilon/2=\varepsilon.\end{aligned}$$

$(\Leftarrow)$ For each $n\in\mathbb{N}$, let $\gamma_n>0$ be such that if $\dot{P}$ and $\dot{Q}$ are tagged partitions with norms $<\gamma_n$, then

$$\left|S(f;\dot{P})-S(f;\dot{Q})\right|<1/n.$$

Evidently we may assume that $\gamma_n\ge\gamma_{n+1}$ for $n\in\mathbb{N}$; otherwise, we replace γ_n by $\gamma_n'=\min\{\gamma_1,\ldots,\gamma_n\}$.

For each $n\in\mathbb{N}$, let $\dot{P}_n$ be a tagged partition with $\|\dot{P}_n\|<\gamma_n$. Clearly, if $m>n$ then both $\dot{P}_m$ and $\dot{P}_n$ have norms $<\gamma_n$, so that

$$\left|S(f;\dot{P}_n)-S(f;\dot{P}_m)\right|<\frac{1}{n} \qquad \text{for} \qquad m>n. \qquad \ldots (*)$$

Consequently, the sequence $\left(S(f;\dot{P}_m)\right)_{m=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. Therefore, by Cauchy Convergence Criterion for Sequences[1] this sequence converges in $\mathbb{R}$ and we let $A=\lim_m S(f;\dot{P}_m)$.

Passing to the limit in (1) as $m\to\infty$, we have

$$\left|S(f;\dot{P}_n)-A\right|\le 1/n \text{ for all } n\in\mathbb{N}.$$

To see that A is the Riemann integral of *f*, given $£ > 0$, let $K \in \mathbb{N}$ satisfy $K > 2/£$. If $\dot{Q}$ is any tagged partition with $\|\dot{Q}\| < ¢_K$, then

$$|S(f;\dot{Q}) - A| \le |S(f;\dot{Q}) - S(f;\dot{P}_K)| + |S(f;\dot{P}_k) - A|$$

$$\le 1/K + 1/K < £ .$$

Since $£ > 0$ is arbitrary, then $f \in R\ [a,b]$ with integral A. This completes the proof.

Example Let $g:[0,3] \to \mathbb{R}$ be defined by $g(x) = 2$ for $0 \le x \le 1$, and $g(x) = 3$ for $1 < x \le 3$. We have seen that if $\dot{P}$ is a tagged partition of $[0,3]$ with norm $\|\dot{P}\| < ¢$, then

$$8 - 5¢ \le S(g;\dot{P}) \le 8 + 5¢ .$$

Hence if $\dot{Q}$ is another tagged partition with $\|\dot{Q}\| < ¢$, then

$$8 - 5¢ \le S(g;\dot{Q}) \le 8 + 5¢ .$$

If we subtract these two inequalities, we obtain

$$|S(g;\dot{P}) - S(g;\dot{Q})| \le 10¢ .$$

In order to make this final term $< £$, we are led to employ the Cauchy Criterion with $¦_£ = £/20$.

The Cauchy Criterion can be used to show that a function $f:[a,b] \to \mathbb{R}$ is not Riemann integrable. To do this we need to show that: There exists $£_0 > 0$ such that for any $¦ > 0$ there exists tagged partitions $\dot{P}$ and $\dot{Q}$ with $\|\dot{P}\| < ¦$ and $\|\dot{Q}\| < ¦$ such that

$$|S(f;\dot{P}) - S(f;\dot{Q})| \ge £_0 .$$

Example Show that the Dirichlet function, defined by $f(x) = 1$ if $x \in [0,1]$ is rational and $f(x) = 0$ if $x \in [0,1]$ is irrational is not Riemann integrable.

Here we take $£_0 = \frac{1}{2}$. If $\dot{P}$ is any partition all of whose tags are rational numbers then $S(f;\dot{P}) = 1$, while if $\dot{Q}$ is any tagged partition all of whose tags are irrational numbers then $S(f;\dot{Q}) = 0$. Since we are able to take such tagged partitions with arbitrarily small norms, we conclude that the Dirichlet function is not Riemann integrable.

The next result will be used to establish the Riemann integrability of some important classes of functions.

Squeeze Theorem: Let $f:[a,b] \to \mathbb{R}$. Then $f \in R\ [a,b]$ if and only if for every $£ > 0$ there exist functions $\sim_£$ and $µ_£$ in $R[a,b]$ with

$$\sim_£(x) \le f(x) \le µ_£(x) \text{ for all } x \in [a,b], \qquad \dots (**)$$

and such that

$$\int_a^b (µ_£ - \sim_£) < £ \qquad \dots (***)$$

Proof.

$(\Rightarrow)$ Take $\sim_£ = µ_£ = f$ for all $£ > 0$.

$(\Leftarrow)$ Let $£ > 0$. Since $\sim_{£}$ and $µ_{£}$ belong to $R\ [a,b]$, there exists $¢_{£} > 0$ such that if $\dot{P}$ is any tagged partition with $\|\dot{P}\| < ¢_{£}$ then

$$\left|S(\sim_{£};\dot{P}) - \int_a^b \sim_{£}\right| < £ \quad \text{and} \quad \left|S(µ_{£};\dot{P}) - \int_a^b µ_{£}\right| < £ .$$

It follows from these inequalities that

$$\int_a^b \sim_{£} - £ < S(\sim_{£};\dot{P}) \text{ and } S(µ_{£};\dot{P}) < \int_a^b µ_{£} + £$$

In view of inequality (**), we have

$$S(\sim_{£};\dot{P}) \le S(f;\dot{P}) \le S(µ_{£};\dot{P}),$$

and hence

$$\int_a^b \sim_{£} - £ < S(f;\dot{P}) < \int_a^b µ_{£} + £$$

If $\dot{Q}$ is another tagged partition with $\|\dot{Q}\| < ¢_{£}$, then we also have

$$\int_a^b \sim_{£} - £ < S(f;\dot{Q}) < \int_a^b µ_{£} + £$$

If we subtract these two inequalities and use (***), we conclude that

$$\left|S(f;\dot{P}) - S(f;\dot{Q})\right| < \int_a^b µ_{£} - \int_a^b \sim_{£} + 2£$$
$$= \int_a^b (µ_{£} - \sim_{£}) + 2£ < 3£ .$$

Since $£ > 0$ is arbitrary, the Cauchy Criterion implies that $f \in R\ [a,b]$. This completes the proof.

Classes of Riemann Integrable Functions

A function $¨ : [a,b] \to \mathbb{R}$ is a **step function** if it has only a finite number of distinct values, each being assumed on one or more subintervals of $[a,b]$.

We begin with a result seen in the previous chapter.

Lemma 1 Suppose that $c \le d$ are points in $[a,b]$. If $¨ : [a,b] \to \mathbb{R}$ satisfies $¨(x) = \sim > 0$ for $x \in [c,d]$ and $¨(x) = 0$ elsewhere in $[a,b]$, prove that $¨ \in R\ [a,b]$ and that $\int_a^b ¨ = \sim(d-c)$.

Proof.

For a tagged partition $\dot{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$, Riemann sum is giving by

$$S(¨,\dot{P}) = \sum_{i=1}^{n} ¨(t_i)(x_i - x_{i-1}).$$

Given $£ > 0$, let $¢_{£} = £/4\sim$. Then if $\|\dot{P}\| < ¢_{£}$, then $\|\dot{P}\| < £/4\sim$, and then the union of the subintervals in $\dot{P}$ with tags in $[c,d]$ contains the interval $[c+¢_{£}, d-¢_{£}]$ and is contained in $[c-¢_{£}, d+¢_{£}]$. Therefore

$$\sim(d-c-2¢_{£}) \le S(¨;\dot{P}) \le \sim(d-c+2¢_{£}),$$

and hence

$$\left|S(¨;\dot{P}) - \sim(d-c)\right| \le 2\sim¢_{£} < 4\sim¢_{£} = £ .$$

Hence by the Definition of Riemann integrability, ¨ $\in R\ [a,b]$ and that $\int_a^b$ ¨ $= \sim (d-c)$.

Lemma 2 If J is a subinterval of $[a,b]$ having endpoints $c<d$ and if ¨$_J(x)=1$ for $x \in J$ and ¨$_J(x)=0$ elsewhere in $[a,b]$, then ¨$_J \in R[a,b]$ and $\int_a^b$ ¨$_J = d-c$.

Proof. If $J=[c,d]$ with $c \le d$, then the proof of Lemma 1, we can choose ¢$_£$ $=$ £ $/4$.

A similar proof can be given for the three other subintervals having these endpoints. Alternatively, we observe that we can write

¨$_{[c,d)}$ $=$ ¨$_{[c,d]}$ $-$ ¨$_{[d,d]}$, ¨$_{(c,d]}$ $=$ ¨$_{[c,d]}$ $-$ ¨$_{[c,c]}$ and ¨$_{(c,d)}$ $=$ ¨$_{[c,d)}$ $-$ ¨$_{[c,c]}$.

Since $\int_a^b$ ¨$_{[c,c]}$ $=0$, all four of these functions have integral equal to $d-c$. This completes the proof.

Riemann Integrability of Step Functions

Step functions of the type appearing in the previous lemma are called "elementary step functions". An arbitrary step function ¨ can be expressed as a linear combination of elementary step functions:

$$\text{¨} = \sum_{j=1}^{m} k_j \text{¨}_{J_j}$$

where J_j has endpoints $c_j < d_j$.

Theorem If ¨ $:[a,b] \to \mathbb{R}$ is a step function, then ¨ $\in R[a,b]$.

Proof . An arbitrary step function ¨ can be expressed as a linear combination of elementary step functions:

$$\text{¨} = \sum_{j=1}^{m} k_j \text{¨}_{J_j} \qquad \ldots (4)$$

where J_j has endpoints $c_j < d_j$.

Then

$$\int_a^b \text{¨} = \int_a^b \left(\sum_{j=1}^{m} k_j \text{¨}_{J_j} \right)$$

$$= \sum_{j=1}^{n} \left(\int_a^b k_j \text{¨}_{J_j} \right), \text{ using part (b) of Theorem 2 of the pervious chapter.}$$

$$= \sum_{j=1}^{n} k_j \left(\int_a^b \text{¨}_{J_j} \right), \text{ using part (a) of Theorem 2 of the pervious chapter.}$$

$$= \sum_{j=1}^{n} k_j \left(d_j - c_j \right), \text{ using Lemma 2}$$

Hence ¨ $\in R\ [a,b]$.

We will now use the Squeeze Theorem to show that an arbitrary continuous function is Riemann integrable.

Theorem If $f:[a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then

$f \in R\ [a,b]$.

Proof. Being continuous function on a closed and bounded interval, it follows that f is uniformly continuous on $[a,b]$. Therefore, given $£ > 0$ there exists $¢_£ > 0$ such that if $u,v \in [a,b]$ and $|u-v| < ¢_£$, then we have

$$|f(u) - f(v)| < \frac{£}{b-a}.$$

Let $P = \{I_i\}_{i=1}^{n}$ be a partition such that $\|P\| < ¢_£$, let $u_i \in I_i$ be a point where f attains its minimum value on I_i and let $v_i \in I_i$ be a point where *f* attains maximum value on I_i.

Let $\sim_£$ be the step function defined by $\sim_£(x) = f(u_i)$ for $x \in [x_{i-1}, x_i)$ $(i = 1,\dots,n-1)$ and $\sim_£(x) = f(u_n)$ for $x \in [x_{n-1}, x_n]$. Let $μ_£$ be defined similarly using the points v_i instead of the u_i. i.e., $μ_£(x) = f(v_i)$ for $x \in [x_{i-1}, x_i)(i = 1,\dots,n-1)$ and $μ_£(x) = f(v_n)$ for $x \in [x_{n-1}, x_n]$. Then we obtain

$$\sim_£(x) \le f(x) \le μ_£(x) \text{ for all } x \in [a,b].$$

Moreover, it is clear that

$$0 \le \int (μ_£ - \sim_£) = \sum_{i=1}^{n}\big((f(v_i) - f(u_j)\big)(x_i - x_{i-1})$$

$$< \sum_{i=1}^{n}\left(\frac{£}{b-a}\right)(x_i - x_{i-1}) = £\ .$$

Therefore it follows from the Squeeze Theorem that $f \in R[a,b]$. This completes the proof.

Monotone functions are not necessarily continuous at every point, but they are also Riemann integrable. Before proving this result, we recall Characterization Theorem for Intervals.

Characterization Theorem for Integrals: If S is a subset of $\mathbb{R}$ that contains at least two points and has the property

if $x,\ y \in S$ and $x < y$ then $[x,\ y] \subseteq S$,

then S is an interval.

Theorem If $f:[a,b] \to \mathbb{R}$ is monotone on $[a,b]$, then $f \in R\ [a,b]$.

Proof. Suppose that *f* is increasing on the interval $[a,b]$, $a < b$. If $£ > 0$ is given, we let $q \in \mathbb{N}$ be such that

$$h = \frac{f(b) - f(a)}{q} < \frac{£}{b-a}.$$

Let $y_k = f(a) + kh$ for $k = 0,1,\dots,q$ and consider sets $A_k = f^{-1}([y_{k-1}, y_k))$ for $k = 1,\dots,q-1$ and $A_q = f^{-1}([y_{q-1}, y_q])$. The sets $\{A_k\}$ are pairwise disjoint and have union $[a,b]$. The Characterization Theorem implies that each A_k is either (i) empty, (ii) contains a single point, or (iii) is a nondegenerate interval (not necessarily closed) in [*a, b*]. We discard the sets for which (i) holds and relabel the remaining ones. If we adjoin the endpoints to the remaining intervals

$\{A_k\}$, we obtain closed intervals $\{I_k\}$. It can be shown that relabeled intervals $\{A_k\}_{k=1}^{q}$ are pair wise disjoint, satisfy $[a,b]=\bigcup_{k=1}^{q} A_k$ and that $f(x)\in[y_{k-1},y_k]$ for $x\in A_k$.

We now define step functions $\sim_{£}$ and $\mu_{£}$ on $[a,b]$ by setting

$\sim_{£}(x)=y_{k-1}$ and $\mu_{£}(x)=y_k$ for $x\in A_k$ for $x\in A_k$

It is clear that $\sim_{£}(x)\le f(x)\le \mu_{£}(x)$ for all $x\in[a,b]$ and that

$$\int_a^b(\mu_{£}-\sim_{£})=\sum_{k=1}^{q}(y_k-y_{k-1})(x_k-x_{k-1}).$$

$$=\sum_{k=1}^{q}h\cdot(x_k-x_{k-1})=h\cdot(b-a)<£.$$

Since $£>0$ is arbitrary, the Squeeze Theorem implies that $f\in R[a,b]$. This completes the proof.

Additivity Theorem: Let $f:[a,b]\to\mathbb{R}$ and let $c\in(a,b)$. Then $f\in R[a,b]$ if and only if its restrictions to $[a,c]$ and $[c,b]$ are both Riemann integrable. In this case

$$\int_a^b f=\int_a^c f+\int_c^b f \qquad \ldots (*)$$

Proof. ($\Leftarrow$) Suppose that the restriction f_1 of f to $[a,c]$, and the restriction f_2 of f to $[c,b]$ are Riemann integrable to L_1 and L_2, respectively. Then, given $£>0$ there exists $¢'>0$ such that if $\dot{P}_1$ is a tagged partition of $[a,c]$ with $\|\dot{P}_1\|<¢'$, then $|S(f_1;\dot{P}_1)-L_1|<£/3$.

Also there exists $¢''>0$ such that if $\dot{P}_2$ is a tagged partition of $[c,b]$ with $\|\dot{P}_2\|<¢''$ then $|S(f_2;\dot{P}_2)-L_2|<£/3$. If M is a bound for $|f|$, we define $¢_{£}=\min\{¢',¢'',£/6M\}$ and let $\dot{P}$ be a tagged partition of $[a,b]$ with $\|\dot{Q}\|<¢$. We will prove that

$$|S(f;\dot{Q})-(L_1+L_2)|<£. \qquad \ldots (**)$$

(i) If c is a partition point of $\dot{Q}$, we split $\dot{Q}$ into a partition $\dot{Q}_1$ of $[a,c]$ and a partition $\dot{Q}_2$ of $[c,b]$. Since

$$S(f;\dot{Q})=S(f;\dot{Q}_1)+S(f;\dot{Q}_2),$$

and since $\dot{Q}_1$ has norm $<¢'$ and $\dot{Q}_2$ has norm $<¢''$, the inequality (**) is clear.

(ii) If c is not a partition point in $\dot{Q}=\{(I_k,t_k)\}_{k=1}^{m}$, there exists $k\le m$ such that $c\in(x_{k-1},x_k)$. We let $\dot{Q}_1$ be the tagged partition of $[a,c]$ defined by

$$\dot{Q}_1=\{(I_1,t_1),\ldots,(I_{k-1},t_{k-1}),([x_{k-1},c],c)\}$$

and $\dot{Q}_2$ be the tagged partition of $[c,b]$ defined by

$$\dot{Q}_2=\{([c,x_k],c),(I_{k+1},t_{k+1}),\ldots,(I_m,t_m)\}$$

A straightforward calculation yields

$$S(f;\dot{Q})-S(f;\dot{Q}_1)-S(f;\dot{Q}_2)=f(t_k)(x_k-x_{k-1})-f(c)(x_k-x_{k-1})$$
$$=(f(t_k)-f(c))\cdot(x_k-x_{k-1}),$$

and hence it follows that

$$\left|S(f;\dot{Q})-S(f;\dot{Q}_1)-S(f;\dot{Q}_2)\right|\le 2M(x_k-x_{k-1})<\frac{£}{3}.$$

But since $\|\dot{Q}_1\|<$ ¢ $\le$ ¢$'$ and $\|\dot{Q}_2\|<$ ¢ $\le$ ¢$''$, it follows that

$$\left|S(f;\dot{Q}_1)-L_1\right|<\frac{£}{3} \text{ and } \left|S(f;\dot{Q}_2)-L_2\right|<\frac{£}{3},$$

from which we obtain (**). Since £ >0 is arbitrary, we infer that $f\in R\,[a,b]$ and that (6) holds.

($\Rightarrow$) We suppose that $f\in R\,[a,b]$ and, given £ >0, we let ¦$_£>0$ satisfy the Cauchy Criterion. Let f_1 be the restriction of *f* to $[a,c]$ and let $\dot{P}_1$, $\dot{Q}_1$, be tagged partitions of $[a,c]$ with $\|\dot{P}_1\|<$ ¦$_£$ and $\|\dot{Q}_1\|<$ ¦$_£$. By adding additional partition points and tags from $[c,b]$, we can extend $\dot{P}_1$ and $\dot{Q}_1$ to tagged partitions $\dot{P}$ and $\dot{Q}$ of $[a,b]$ that satisfy $\|\dot{P}\|<$ ¦$_£$ and $\|\dot{Q}\|<$ ¦$_£$. If we use the same additional points and tags in $[c,b]$ for both $\dot{P}$ and $\dot{Q}$, then

$$S(f_1;\dot{P}_1)-S(f_1;\dot{Q}_1)=S(f;\dot{P})-S(f;\dot{Q}).$$

Since both $\dot{P}$ and $\dot{Q}$ have norm $<$ ¦$_£$, then $\left|S(f_1;\dot{P}_1)-S(f;\dot{Q}_1)\right|<$ £. Therefore the Cauchy Condition shows that the restriction f_1 of *f* to $[a,c]$ is in $R\,[a,c]$. In the same way, we see that the restriction f_2 of *f* to $[c,b]$ is in $R\,[c,b]$.

The equality (*) now follows from the first part of the theorem. This completes the proof.

Corollary If $f\in R\,[a,b]$, and if $[c,d]\subseteq[a,b]$, then the restriction of *f* to $[c,d]$ is in $R\,[c,d]$.

Proof. Since $f\in R\,[a,b]$ and $c\in[a,b]$, it follows from the theorem that its restriction to $[c,b]$ is in $R\,[c,b]$. But if $d\in[c,b]$, then another application of the theorem shows that the restriction of *f* to $[c,d]$ is in $R\,[c,d]$. This completes the proof.

Corollary If $f\in R\,[a,b]$ and if $a=c_0<c_1<\cdots<c_m=b$, then the restrictions of *f* to each of the subintervals $[c_{i-1},c_i]$ are Riemann integrable and

$$\int_a^b f=\sum_{i=1}^{m}\int_{c_{i-1}}^{c_i} f\,.$$

Definition If $f\in R\,[a,b]$ and if ~, $\in[a,b]$ with ~ $<$, we define

$$\int^{\sim} f=-\int_{\sim} f \quad\text{and}\quad \int_{\sim}^{\sim} f=0.$$

Theorem 7 If $f\in R\,[a,b]$ and if ~, , ¥ are any numbers in $[a,b]$, then

$$\int_{\sim} f=\int_{\sim}^{¥} f+\int_{¥} f, \qquad \dots (***)$$

in the sense that the existence of any two of these integrals implies the existence of the third integral and the equality (***).

Proof. If any two of the numbers ~, ,¥ are equal, then (***) holds. Thus we may suppose that all three of these numbers are distinct.

For the sake of symmetry, we introduce the expression

$$L(\sim, \ ,¥) = \int_{\sim} f + \int^{¥} + \int_{¥}^{\sim} f .$$

It is clear that (***) holds if and only if $L(\sim, \ ,¥) = 0$. Therefore, to establish the assertion, we need to show that $L = 0$ for all six permutations of the arguments ~, and ¥ .

We note that the Additivity Theorem implies that

$$L(\sim, \ ,¥) = 0 \text{ when } \sim\ < ¥ < \ .$$

But it is easily seen that both $L(\ ,¥,\sim)$ and $L(¥,\sim, \ )$ equal $L(\sim, \ ,¥)$. Moreover, the numbers

$$L(\ ,\sim,¥),\ L(\sim,¥, \ ) \text{ and } L(¥, \ ,\sim)$$

are all equal to $-L(\sim, \ ,¥)$. Therefore, L vanishes for all possible configurations of these three points. This completes the proof.

Exercises

1. Consider the function h defined by $h(x) = x+1$ for $x \in [0,1]$ rational, and $h(x) = 0$ for $x \in [0,1]$ irrational. Show that h is not Riemann integrable.
2. Let $f : [a,b] \to \mathbb{R}$. Show that $f \notin R\,[a,b]$ if and only if there exists $£_0 > 0$ such that for every $n \in \mathbb{N}$ there exist tagged partitions $\dot{P}_n$ and $\dot{Q}_n$ with $\|\dot{P}_n\| < 1/n$ and $\|\dot{Q}_n\| < 1/n$ such that $\left|S(f;\dot{P}_n) - S(f;\dot{Q}_n)\right| \ge £_0$.
3. Let $H(x) = k$ for $x = 1/k\,(k \in \mathbb{N})$ and $H(x) = 0$ elsewhere on [0,1]. Use Exercise 1 to show that H is not Riemann integrable.
4. If $\sim(x) = -x$ and $\mu(x) = x$ and if $\sim(x) \le f(x) \le \mu(x)$ for all $x \in [0,1]$, does it follow from the Squeeze Theorem that $f \in R[0,1]$?
5. If J is any subinterval of $[a,b]$ and if $¨_J(x) = 1$ for $x \in J$ and $¨_J(x) = 0$ elsewhere on $[a,b]$, we say that $¨_J$ is an **elementary step function** on $[a,b]$. Show that every step function is a linear combination of elementary step functions.
6. If $\cdot : [a,b] \to \mathbb{R}$ takes on only a finite number of distinct values, is $\cdot$ a step function?
7. If $S(f;\dot{P})$ is any Riemann sum of $f : [a,b] \to \mathbb{R}$, show that there exists a step function $¨ : [a,b] \to \mathbb{R}$ such that $\int_a^b ¨ = S(f;\dot{P})$.
8. Suppose that f is continuous on $[a,b]$, that $f(x) \ge 0$ for all $x \in [a,b]$ and that $\int_a^b f = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$.
9. Show that the continuity hypothesis in the preceding exercise cannot be dropped.
10. If f and g are continuous on $[a,b]$ and if $\int_a^b f = \int_a^b g$, prove that there exists $c \in [a,b]$ such that $f(c) = g(c)$.

11. If f is bounded by M on $[a,b]$ and if the restriction of f to every interval $[c,b]$ where $c \in (a,b)$ is Riemann integrable, show that $f \in R\ [a,b]$ and that $\int_c^b f \to \int_a^b f$ as $c \to a+$.
12. Show that $g(x) = \sin(1/x)$ for $x \in (0,1]$ and $g(0) = 0$ belongs to $R\ [0,1]$.
13. Give an example of a function $f:[a,b] \to \mathbb{R}$ that is in $R\ [c,b]$ for every $c \in (a,b)$ but which is not in $R\ [a,b]$.
14. Suppose that $f:[a,b] \to \mathbb{R}$, that $a = c_0 < c_1 < ... < c_m = b$ and that the restrictions of f to $[c_{i-1}, c_i]$ belong to $R\ [c_{i-1}, c_i]$ for $i = 1,...,m$. Prove that $f \in R\ [a,b]$ and that the formula $\int_a^b f = \sum_{i=1}^{m} \int_{c_{i-1}}^{c_i} f$ in Corollary holds.
15. If f is bounded and there is a finite set E such that f is continuous at every point of $[a,b] \setminus E$, show that $f \in R\ [a,b]$.
16. If f is continuous on $[a,b]$, $a < b$, show that there exists $c \in [a,b]$ such that we have $\int_a^b f = f(c)(b-a)$. This result is sometimes called the **Mean Value Theorem for Integrals**.
17. If f and g are continuous on $[a,b]$ and $g(x) > 0$ for all $x \in [a,b]$, show that there exists $c \in [a,b]$ such that $\int_a^b f\, g = f(c) \int_a^b g$. Show that this conclusion fails if we do not have $g(x) > 0$. (Note that this result is an extension of the preceding exercise.)
18. Let f be continuous on $[a,b]$, let $f(x) \geq 0$ for all $x \in [a,b]$, and let $M_n = \left(\int_a^b f^n \right)^{1/n}$. Show that $\lim(M_n) = \sup\{ f(x) : x \in [a,b]\}$.
19. Suppose that $a > 0$ and that $f \in R[-a,a]$.

    a) If f is **even** (that is, if $f(-x) = f(x)$ for all $x \in [0,a]$), show that $\int_{-a}^{a} f = 2\int_0^a f$.

    b) If f is **odd** (that is, if $f(-x) = -f(x)$ for all $x \in [0,a]$), show that $\int_{-a}^{a} f = 0$.
20. Suppose that $f:[a,b] \to \mathbb{R}$ and that $n \in \mathbb{N}$. Let $\dot{P}_n$ be the partition of $[a,b]$ into n subintervals having equal lengths, so that $x_i ;= a + i(b-a)/n$ for $i = 0,1,...,n$. Let $L_n(f) = S(f; \dot{P}_{n,l})$ and $R_n(f) = S(f; \dot{P}_{n,r})$, where $\dot{P}_{n,l}$ has its tags at the left endpoints, and $\dot{P}_{n,r}$ has its tags at the right endpoints of the subintervals $[x_{i-1}, x_i]$.

    a) If f is increasing on $[a,b]$, show that $L_n(f) \leq R_n(f)$ and that

    $$0 \leq R_n(f) - L_n(f) = \big((f(b) - f(a)\big) \cdot \frac{(b-a)}{n}$$

    b) Show that $f(a)(b-a) \leq L_n(f) \leq \int_a^b f \leq R_n(f) \leq f(b)(b-a)$.

    c) If f is decreasing on $[a,b]$, obtain an inequality similar to that in (a).

    d) If $f \in R[a,b]$ is not monotone, show that $\int_a^b f$ is not necessarily between $L_n(f)$ and $R_n(f)$.

21. If f is continuous on $[-1,1]$, show that $\int_0^{\pi/2} f(\cos x)dx = \int_0^{\pi/2} f(\sin x)dx = \frac{1}{2}\int_0^{\pi} f(\sin x)dx$. [Hint: Examine certain Riemann sums.]

22. If f is continuous on $[-a,a]$, show that $\int_{-a}^{a} f(x^2)dx = 2\int_0^{a} f(x^2)dx$.

Answers

1. If the tags are all rational, then $S(h;\dot{P}) \geq 1$, while if the tags are all irrational, then $S(h;\dot{P}) = 0$.
3. Let $\dot{P}_n$ be the partition of [0,1] into n equal subintervals with $t_1 = 1/n$ and $\dot{Q}_n$ be the same subintervals tagged by irrational points.
5. If $c_1,...,c_n$ are the distinct values taken by ¨ , then $¨^{-1}(c_j)$ is the union of a finite collection $\{J_{j1},...,J_{jr_j}\}$ of disjoint subintervals of $[a,b]$. We can write $¨ = \sum_{j=1}^{n}\sum_{k=1}^{r_j} c_j ¨_{j_{jk}}$.
6. Not necessarily.
8. If $f(c) > 0$ for some $c \in (a,b)$, there exists ¢ > 0 such that $f(x) > \frac{1}{2} f(c)$ for $|x-c| \leq$ ¢. Then $\int_a^b f \geq \int_{c-¢}^{c+¢} f \geq (2¢)\frac{1}{2} f(c) > 0$. If c is an endpoint, a similar argument applies.
10. Use Bolzano's Intermediate Value Theorem (stated in chapter 2).
11. Let $\alpha_c(x) = -M$ and $\omega_c(x) = M$ for $x \in [a,c)$ and $\sim_c(x) = w_c(x) = f(x)$ for $x \in [c,b]$. Apply the Squeeze Theorem for c sufficiently near a.]
12. Indeed, $|g(x)| \leq 1$ and is continuous on every interval $[c,1]$ where $0 < c < 1$. The preceding exercise applies.
13. Let $f(x) = 1/x$ for $x \in (0,1]$ and $f(0) = 0$.
16. Let $m = \inf f(x)$ and $M = \sup f$. Then we have $m(b-a) \leq \int_a^b f \leq M(b-a)$. By Bolzano's Intermediate Value Theorem, there exists $c \in [a, b]$ such that $f(c) = \dfrac{\int_a^b f}{b-a}$.
19. (a) Let $\dot{P}_n$ be a sequence of tagged partitions of $[0, a]$ with $\|\dot{P}_n\| \to 0$ and let $\dot{P}_n^*$ be the corresponding "symmetric" partition of $[-a, a]$. Show that $S(f;\dot{P}_n^*) = 2S(f;\dot{P}_n) \to 2\int_0^a f$.
21. Let $x_i = i(®/2)$ for $i = 0,1,...,n$. Then we have that
$$\frac{®}{2n} \cdot \sum_{i=0}^{n-1} f(\cos x_i) = \frac{®}{2n} \cdot \sum_{k=1}^{n} f(\sin x_k).$$
22. Note that $x \mapsto f(x^2)$ is an even continuous function.

6

FUNDAMENTAL THEOREMS OF CALCULUS

In this chapter we will explore the connection between the notions of derivative and integral.

The First form of Fundamental Theorem of Calculus: If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

The First Form of the Fundamental Theorem provides a theoretical basis for the method of calculating an integral. It asserts that if a function f is the derivative of a function F, and if f belongs to $R[a,b]$, then the integral $\int_a^b f$ can be calculated by means of the evaluation $F(b) - F(a)$. A function F such that $F'(x) = f(x)$ for all $x \in [a,b]$ is called an **antiderivative** or a **primitive** of f on $[a,b]$. Thus, when f has an antiderivative, it is a very simple matter to calculate its integral.

In practice, it is convenient to allow some exceptional points c where $F'(c)$ does not exist in $\mathbb{R}$, or where it does not equal $f(c)$. It turns o
ut that we can permit a finite number of such exceptional points.

We recall Mean Value Theorem.

Mean Value Theorem: Suppose that f is continuous on a closed interval $I = [a,b]$, and that f has a derivative in the open interval (a,b). Then there exists at least one point c in (a,b) such that

$$f(b) - f(a) = f'(c)(b-a).$$

The following is a result needed in the coming section.

Theorem If $f : I \to \mathbb{R}$ has a derivative at $c \in I$, then f is continuous at c.

Fundamental Theorem of Calculus (First Form): Suppose there is finite set E in $[a,b]$ and functions $f, F : [a,b] \to \mathbb{R}$ such that:

a) F is continuous on $[a,b]$,

b) $F'(x) = f(x)$ for all $x \in [a,b] \setminus E$,

Then we have

$$\int_a^b f = F(b) - F(a) \qquad \ldots (1)$$

Proof. We will prove the theorem in the case where $E = \{a,b\}$. The general case can be obtained by breaking the interval into the union of a finite number of intervals.

Let $\varepsilon > 0$ be given. By assumption (c), $f \in R[a,b]$. Then there exists $\delta_\varepsilon > 0$ such that if $\dot{P}$ is any tagged partition with $\|\dot{P}\| < \delta_\varepsilon$, then

$$\left| S(f;\dot{P}) - \int_a^b f \right| < \varepsilon. \qquad \ldots (2)$$

If the subintervals in $\dot{P}$ are $[x_{i-1}, x_i]$, then the Mean value Theorem applied to F on $[x_{i-1}, x_i]$ implies that there exists $u_i \in (x_{i-1}, x_i)$ such that

$$F(x_i) - F(x_{i-1}) = F'(u_i)\cdot(x_i - x_{i-1}) \text{ for } i = 1, ..., n.$$

If we add these terms, note the telescoping of the sum, and use the fact that $F'(u_i) = f(u_i)$, and obtain

$$F(b) - F(a) = \sum_{i=1}^{n}(F(x_i) - F(x_{i-1})) = \sum_{i=1}^{n} f(u_i)(x_i - x_{i-1})$$

Now let $\dot{P}_u = \{([x_{i-1}, x_i], u_i)\}_{i=1}^{n}$, then the sum on right equals $S(f;\dot{P}_u)$. If we substitute $F(b) - F(a) = S(f;\dot{P}_u)$ into (2), we conclude that

$$\left| F(b) - F(a) - \int_a^b f \right| < \varepsilon.$$

But, since $\varepsilon > 0$ is arbitrary, we infer that equation (1) holds. This completes the proof.

Remark If the function F is differentiable at every point of $[a,b]$, then (by Theorem C) hypothesis (a) is automatically satisfied. If f is not defined for some points $c \in E$, we take $f(c) = 0$. Even if F is differentiable at every points of $[a,b]$, condition (c) is not automatically satisfied, since there exist function F such that F' is not Riemann integrable (This is illustrated in the following Example 5).

Example If $F(x) = \frac{1}{2}x^2$ for all $x \in [a,b]$, then $F'(x) = x$ for all $x \in [a,b]$. Further, $f = F'$ is continuous so it is in $R[a,b]$. Therefore the Fundamental Theorem (with $E = \varnothing$) implies that

$$\int_a^b x\,dx = F(b) - F(a) = \tfrac{1}{2}(b^2 - a^2).$$

Example If $G(x) = \operatorname{Arctan} x$ for $x \in [a,b]$, then $G'(x) = 1/(x^2+1)$ for all $x \in [a,b]$; also G' is continuous, so it is in $R[a,b]$. Therefore the Fundamental Theorem (with $E = \varnothing$) implies that

$$\int_a^b \frac{1}{x^2+1}dx = \operatorname{Arctan} b - \operatorname{Arctan} a.$$

Example If $A(x) = |x|$ for $x \in [-10,10]$, then $A'(x) = -1$ if $x \in [-10,0)$ and $A'(x) = +1$ for $x \in (0,10]$. Then we have $A'(x) = \operatorname{sgn}(x)$ for all $x \in [-10,10]\setminus\{0\}$, where

$$\operatorname{sgn}(x) = \begin{cases} +1 & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x = 0 \\ -1 & \text{for} \quad x < 1 \end{cases}.$$

(sgn is called the **signum function**).

Since the signum function is a step function, it belongs to $R\,[-10,10]$. Therefore the Fundamental Theorem (with $E = \{0\}$) implies that

$$\int_{-10}^{10} \operatorname{sgn}(x)dx = A(10) - A(-10) = 10 - 10 = 0.$$

Example If $H(x) = 2\sqrt{x}$ for $x \in [0,b]$, then H is continuous on $[0,b]$ and $H'(x) = 1/\sqrt{x}$ for $x \in (0,b]$. Since $h = H'$ is not bounded on $(0,b]$, it does not belong to $R[0,b]$ no matter how we define $h(0)$. Therefore, the Fundamental Theorem does not apply.

Example Let $K(x) = x^2\cos(1/x^2)$ for $x \in (0,1]$ and let $K(0) = 0$. It follows, applying the Product Rule and the Chain Rule, that

$$K'(x) = 2x\cos(1/x^2) + (2/x)\sin(1/x^2) \text{ for } x \in (0,1]$$

Further, we have

$$K'(0) = \lim_{x\to 0}\frac{K(x)-K(0)}{x-0} = \lim_{x\to 0}\frac{x^2\cos(\frac{1}{x^2})}{x} = \lim_{x\to 0} x\cdot\cos(\tfrac{1}{x^2}) = 0.$$

Thus K is continuous and differentiable at every point of $[0,1]$. Since the first term in K' is continuous on $[0,1]$, it belongs to R $[0,1]$. However, the second term in K' is not bounded, so it does not belong to R $[0, 1]$. Consequently $K' \notin R$ $[0,1]$, and the Fundamental Theorem does not apply to K'.

The Fundamental Theorem (Second Form)

We now discuss the Fundamental Theorem (Second form) in which we wish to differentiate an integral involving a variable upper limit.

Definition If $f \in R$ $[a,b]$, then the function defined by

$$F(z) = \int_a^z f \quad \text{for } z \in [a,b] \qquad \ldots (3)$$

is called the **indefinite integral** of *f* with **base point** *a*.

We will show that if $f \in R$ $[a,b]$, then its indefinite integral *F* satisfies a Lipschitz condition; hence *F* is continuous on $[a,b]$. We recall a result in the chapter Riemann Integrable Functions.

Additivity Theorem: Let $f:[a,b]\to\mathbb{R}$ and let $c\in(a,b)$. Then $f \in R[a,b]$ if and only if its restrictions to $[a,c]$ and $[c,b]$ are both Riemann integrable. In this case

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Theorem The indefinite integral *F* defined by

$$F(z) = \int_a^z f \quad \text{for } z \in [a,b]$$

is continuous on $[a,b]$. In fact, if $|f(x)| \le M$ for all $x \in [a,b]$, then

$$|F(z) - F(w)| \le M|z-w| \text{ for all } z, w \in [a,b].$$

Proof . The Additivity Theorem implies that if $z, w \in [a,b]$ and $w \le z$, then

$$F(z) = \int_a^z f = \int_a^w f + \int_w^z f = F(w) + \int_w^z f,$$

and hence we have

$$F(z) - F(w) = \int_w^z f$$

Now if $-M \le f(x) \le M$ for all $x \in [a,b]$, then Theorem D implies that

$$\int_w^z -M \le \int_w^z f \le \int_w^z M$$

implies $$-M\int_w^z \le \int_w^z f \le M\int_w^z$$

implies

$$-M(z-w) \le \int_w^z f \le M(z-w)$$

and hence it follows that

$$|F(z) - F(w)| \le \left|\int_w^z f\right| \le M\,|z - w|.$$

This completes the proof.

We will now show that the indefinite integral F is differentiable at any point where f is continuous.

Fundamental Theorem of Calculus (Second Form): Let $f \in R\ [a,b]$ and let f be continuous at a point $c \in [a,b]$. Then the indefinite integral, defined by

$$F(z) = \int_a^z f \quad \text{for } z \in [a, b],$$

is differentiable at c and $F'(c) = f(c)$.

Proof. We will suppose that $c \in [a,b)$ and consider the right-hand derivative of F at c. Since f is continuous at c, given $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that if $c \le x < c + \eta_\varepsilon$, then

$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon. \qquad \ldots(4)$$

Let h satisfy $0 < h < \eta_\varepsilon$. The Additivity Theorem implies that f is integrable on the intervals $[a,c]$, $[a,c+h]$ and $[c,c+h]$ and that

$$F(c+h) - F(c) = \int_c^{c+h} f.$$

Now on the interval $[c,c+h]$ the function f satisfies inequality (4), so that (by Theorem D) we have

$$\int_c^{c+h} (f(c) - £) < \int_c^{c+h} f \le \int_c^{c+h} (f(c) + £).$$

Since $\int_c^{c+h} f = F(c+h) - F(c)$, the above implies

$$(f(c) - \varepsilon)\cdot h \le F(c+h) - F(c) \le (f(c) + \varepsilon)\cdot h$$

If we divide by $h > 0$ and subtract $f(c)$, we obtain

$$-£ \le \frac{F(c+h) - F(c)}{h} - f(c) \le £$$

which implies

$$\left|\frac{F(c+h) - F(c)}{h} - f(c)\right| \le \varepsilon.$$

But, since $\varepsilon > 0$ is arbitrary, we conclude that the right-hand limit is given by

$$\lim_{h \to 0+} \frac{F(c+h) - F(c)}{h} = f(c).$$

It is proved in the same way that the left-hand limit of this difference quotient also equals $f(c)$ when $c \in (a,b]$. This completes the proof.

If f is continuous on all of $[a,b]$, we obtain the following result.

Theorem If f is continuous on $[a,b]$ then the indefinite integral F, defined by

$$F(z) = \int_a^z f \qquad \text{for} \qquad z \in [a,b],$$

is differentiable on $[a,b]$ and $F'(x) = f(x)$ for all $x \in [a,b]$.

Remark The above Theorem can be summarized as follows:

If f is continuous on $[a,b]$, then its indefinite integral is an antiderivative of f.

In general, the indefinite integral need not be an antiderivative (either because the derivative of the indefinite integral does not exist or does not equal $f(x)$). This is illustrated in the following examples.

Example If $f(x) = \operatorname{sgn} x$ on $[-1,1]$, then

If $x \le 0$, then

$$F(x) = \int_{-1}^{x} f(x)\,dx = \int_{-1}^{x} -1 = -1(x+1) = -x-1 = |x| - 1$$

If $x > 0$, then

$$F(x) = \int_{-1}^{x} f(x) = \int_{-1}^{0} f + \int_{0}^{x} f = \int_{-1}^{-1} -1 + \int_{0}^{x} 1 = -1[0-(-1)] + 1[x-0]$$
$$= -1 + x = |x| - 1 .$$

Thus $f \in R\ [-1,1]$ and has the indefinite integral[1] $F(x) = |x| - 1$ with the base point -1. However, since $F'(0)$ does not exist, F is not an antiderivative of f on $[-1,1]$.

Example If h denotes Thomae's function $h:[0,1] \to \mathbb{R}$ defined by defined by $h(x) = 0$ if $x \in [0,1]$ is irrational, $h(0) = 1$ and by $h(x) = 1/n$ if $x \in [0,1]$ is the rational number $x = m/n$ where $m, n \in \mathbb{N}$ have no common integer factors except 1. Then its indefinite integral $H(x) = \int_0^x h$ is identically 0 on[0, 1]. Here, the derivative of this indefinite integral exists at every point and $H'(x) = 0$. But $H'(x) \ne h(x)$ whenever $x \in \mathbb{Q} \cap [0,1]$, so that H is not an antiderivative of h on $[0,1]$.

Substitution Theorem

The next theorem provides the justification for the "***change of variable***" method that is often used to evaluate integrals. This theorem is employed (usually implicitly) in the evaluation by means of procedures that involve the manipulation of "differentials".

Substitution Theorem: Let $J = [\alpha, \beta]$ and let $\varphi : J \to \mathbb{R}$ have a continuous derivative on *J*. If $f : I \to \mathbb{R}$ is continuous on an interval *I* containing $\varphi(J)$, then

$$\int_{\alpha}^{\beta} f(\varphi(t)) \cdot \varphi'(t)\,dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx \qquad \ldots (5)$$

Proof. Define $F(u) = \int_{\varphi(\alpha)}^{u} f(x)\,dx$ for $u \in I$, and $H(t) = F(\varphi(t)) = (F \circ \varphi)(t)$ for $t \in J$. Then $H'(t) = f(\varphi(t))\varphi'(t)$ for $t \in J$ and that

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = F(\varphi(\beta)) = H(\beta) = \int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t)\,dt$$

The hypotheses that *f* and φ' are continuous are restrictive, but are used to ensure the existence of the Riemann integral on the left side of (5).

Example Evaluate the integral

$$\int_1^4 \frac{\sin\sqrt{t}}{\sqrt{t}}\,dt .$$

Here we substitute $\varphi(t) = \sqrt{t}$ for $t \in [1,4]$ so that $\varphi'(t) = 1/(2\sqrt{t})$ is continuous on $[1,4]$. If we let $f(x) = 2\sin x$, then the integrand has the form $(f \circ \varphi) \cdot \varphi'$ and the Substitution Theorem implies that the integral equals $\int_1^2 2\sin x\,dx = -2\cos x\,|_1^2 = 2(\cos 1 - \cos 2)$.

Example Consider the integral $\int_0^4 \frac{\sin\sqrt{t}}{\sqrt{t}}dt$. Since $\varphi(t)=\sqrt{t}$ does not have a continuous derivative on $[0,4]$, the Substitution Theorem is not applicable, at least with this substitution.

Exercises

1. Extend the proof of the Fundamental Theorem to the case of an arbitrary finite set E.
2. If $n\in\mathbb{N}$ and $H_n(x)=x^{n+1}/(n+1)$ for $x\in[a,b]$, show that the Fundamental Theorem implies that $\int_a^b x^n dx=(b^{n+1}-a^{n+1})/(n+1)$. What is the set E here?
3. If $g(x)=x$ for $|x|\geq 1$ and $g(x)=-x$ for $|x|<1$ and if $G(x)=\frac{1}{2}|x^2-1|$, show that $\int_{-2}^3 g(x)dx=G(3)-G(-2)=5/2$.
4. Let $B(x)=-\frac{1}{2}x^2$ for $x<0$ and $B(x)=\frac{1}{2}x^2$ for $x\geq 0$. Show that $\int_a^b |x|dx=B(b)-B(a)$.
5. Let $f:[a,b]\to\mathbb{R}$ and let $C\in\mathbb{R}$.
   (a) If ¤ $:[a,b]\to\mathbb{R}$ is an antiderivative of f on $[a,b]$, show that
   ¤$_C(x)=$¤$(x)+C$ is also an antiderivative of f on $[a,b]$.
   (b) If ¤$_1$ and ¤$_2$ are antiderivative of f on $[a,b]$, show that
   ¤$_1-$¤$_2$ is a constant function on $[a,b]$.
6. If $f\in R\ [a,b]$ and if $c\in[a,b]$, the function defined by $F_c(z)=\int_c^z f$ for $z\in[a,b]$ is called the **indefinite integral** of f with **base point** c. Find a relation between F_a and F_c.
7. Thomae's function is in $R\ [0,1]$ with integral equal to 0. Can the Fundamental Theorem be used to obtain this conclusion? Explain your answer.
8. Let $F(x)$ be defined for $x\geq 0$ by $F(x)=(n-1)x-(n-1)n/2$ for $x\in[n-1,n)$, $n\in\mathbb{N}$. Show that F is continuous and evaluate $F'(x)$ at points where this derivative exists. Use this result to evaluate $\int_a^b [[x]]dx$ for $0\leq a<b$, where $[[x]]$ denotes the greatest integer in x. (The function $x\to[[x]]$ is called the **greatest integer function**, for example, $[[7.2]]=7$, $[[$®$]]=3$, $[[-2.8]]=-3$).
9. Let $f\in R\ [a,b]$ and define $F(x)=\int_a^x f$ for $x\in[a,b]$.
   (a) Evaluate $G(x)=\int_c^x f$ in terms of F, where $c\in[a,b]$.
   (b) Evaluate $H(x)=\int_x^b f$ in terms of F.
   (c) Evaluate $S(x)=\int_x^{\sin x} f$ in terms of F.
10. Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and let $v=[c,d]\to\mathbb{R}$ be differentiable on $[c,d]$ with $v([c,d])\subseteq[a,b]$. If we define $G(x)=\int_a^{v(x)} f$, show that $G'(x)=f(v(x))\cdot v'(x)$ for all $x\in[c,d]$
11. Find $F'(x)$ when F is defined on $[0,1]$ by:

a) $F(x)=\int_0^{x^2}(1+t^3)^{-1}\,dt$ b) $F(x)=\int_{x^2}^{x}\sqrt{1+t^2}\,dt$

12. Let $f:[0,3]\to\mathbb{R}$ be defined by $f(x)=x$ for $0\le x<1$, $f(x)=1$ for $1\le x<2$ and $f(x)=x$ for $2\le x\le 3$. Obtain formulas for $F(x)=\int_0^x f$ and sketch the graphs of f and F. Where is F differentiable? Evaluate $F'(x)$ at all such points.

13. If $f:\mathbb{R}\to\mathbb{R}$ is continuous and $c>0$, define $g:\mathbb{R}\to\mathbb{R}$ by $g(x)=\int_{x-c}^{x+c}f(t)\,dt$. Show that g is differentiable on $\mathbb{R}$ and find $g'(x)$.

14. If $f:[0,1]\to\mathbb{R}$ is continuous and $\int_0^x f=\int_x^1 f$ for all $x\in[0,1]$, show that $f(x)=0$ for all $x\in[0,1]$
.

15. Use the Substitution Theorem to evaluate the following integrals.

a) $\int_0^1 t\sqrt{1+t^2}\,dt$ b) $\int_0^2 t^2(1+t^3)^{-1/2}\,dt$

c) $\int_1^4 \frac{\sqrt{1+\sqrt{t}}}{\sqrt{t}}\,dt$, d) $\int_1^4 \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt$

16. Sometimes the Substitution Theorem cannot be applied but the following result, called the Second Substitution Theorem is useful. In addition to the hypotheses of Substitution Theorem, assume that $¨'(t)\neq 0$ for all $t\in J$, so the function $\cdot:¨(J)\to\mathbb{R}$ inverse to ¨ exists and has derivative $\cdot'(¨(t))=1/¨'(t)$. Then

$$\int_{\sim} f(¨(t))dt=\int_{¨(\sim)}^{¨(\ )} f(x)\cdot'(x)\,dx$$

To prove this, let

$$G(t)=\int_{\sim}^{t} f(¨(s))ds \text{ for } t\in J,$$

so that $G'(t)=f(¨(t))$. Note that $K(x)=G(\cdot(x))$ is differentiable on the interval $¨(J)$ and that

$$K'(x)=G'(\cdot(x))\cdot'(x)=f(¨\circ\cdot(x))\cdot'(x)=f(x)\cdot'(x).$$

Calculate $G(\ )=K(¨(\ ))$ in two ways to obtain the formula.

17. Apply the Second Substitution Theorem to evaluate the following integrals.

a) $\int_1^9 \frac{dt}{2+\sqrt{t}}$ b) $\int_1^3 \frac{dt}{t\sqrt{t+1}}=1\text{n}(3+2\sqrt{2})-1\text{n}3$

c) $\int_1^4 \frac{\sqrt{t}dt}{1+\sqrt{t}}$ d) $\int_1^4 \frac{dt}{\sqrt{t}(t+4)}=\text{Arctan}(1)-\text{Arctan}(1/2)$

18. Explain why Substitution Theorem and/or Second shifting Theorem cannot be applied to evaluate the following integrals, using the indicated substitution.

a) $\int_0^4 \frac{\sqrt{t}dt}{1+\sqrt{t}}$ $¨(t)=\sqrt{t}$ b) $\int_0^4 \frac{\cos\sqrt{t}dt}{\sqrt{t}}$ $¨(t)=\sqrt{t}$

c) $\int_{-1}^1 \sqrt{1+2|t|}dt$ $¨(t)=|t|$ d) $\int_0^1 \frac{dt}{\sqrt{1-t^2}}$ $¨(t)=\text{Arcsin}\,t$

7
POINTWISE AND UNIFORM CONVERGENCE

Let $A \subseteq \mathbb{R}$ be given and suppose that for each $n \in \mathbb{N}$ there is a function $f_n : A \to \mathbb{R}$; we say that (f_n) is a **sequence of functions** on A to $\mathbb{R}$.

Clearly, for each $x \in A$, a sequence of functions gives rise to a sequence of real numbers, namely the sequence

$$(f_n(x)), \qquad \ldots (1)$$

obtained by evaluating each of the functions at the point *x*. For certain values of $x \in A$ the sequence (1) may converge, and for other values of $x \in A$ this sequence may diverge. For each $x \in A$ for which the sequence (1) converges, there is a uniquely determined real number $\lim(f_n(x))$. In general, the value of this limit, when it exists, will depend on the choice of the point $x \in A$. Thus, there arises in this way a function whose domain consists of all numbers $x \in A$ for which the sequence (1) converges. Definition follows:

Let (f_n) be a sequence of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$, let $A_0 \subseteq A$, and let $f : A_0 \to \mathbb{R}$. We say that the **sequence** (f_n) **converges on** A_0 **to** *f* if, for each $x \in A_0$, the **sequence** $(f_n(x))$ converges to $f(x)$ in $\mathbb{R}$. In this case we call *f* the **limit on** A_0 **of the sequence** (f_n). When such a function *f* exists, we say that the sequence (f_n) **is convergent on** A_0, or that (f_n) **converges pointwise on** A_0.

Except for a possible modification of the domain A_0, the limit function is uniquely determined. Ordinarily we choose A_0 to be the largest set possible; that is, we take A_0 to be the set of all $x \in A$ for which the sequence (1) is convergent in $\mathbb{R}$.

If the sequence (f_n) converges on A_0 to *f*, we denote it by

$$f = \lim(f_n) \text{ on } A_0, \quad \text{or} \quad f_n \to f \text{ on } A_0.$$

Sometimes, when f_n and *f* are given by formulas, we write

$$f(x) = \lim f_n(x) \text{ for } x \in A_0, \quad \text{or} \quad f_n(x) \to f(x) \text{ for } x \in A_0$$

We need the following result.

Theorem Let $Y = (y_n)$ be a sequence of real numbers that converge to y and let $x \in \mathbb{R}$. Then the sequence

$$xY = (xy_n)$$

converge to xy. i.e.,

$$\lim(xy_n) = x\lim(y_n).$$

Example Show that $\lim(x/n) = 0$ for $x \in \mathbb{R}$.

For $n \in \mathbb{N}$, let $f_n(x) = x/n$ and let $f(x) = 0$ for $x \in \mathbb{R}$.

For $x \in \mathbb{R}$, it follows that

$$\begin{aligned}\lim(f_n(x)) &= \lim(x/n)\\ &= x\lim(1/n), \text{ using Theorem A}\\ &= x\cdot 0, \text{ as } \lim(1/n) = 0\end{aligned}$$

Hence for all $x \in \mathbb{R}$, $\lim(x/n) = 0$.

Example Discuss the pointwise convergence of the sequence of functions (x^n) .

Let $g_n(x) = x^n$ for $x \in \mathbb{R}, n \in \mathbb{N}$. Clearly, if $x = 1$, then the sequence $(g_n(1))$ is the constant sequence (1) and hence converges to 1. It follows from the result "if $0 < b < 1$, then $\lim(b^n) = 0$" that $\lim(x^n) = 0$ for $0 \le x < 1$ and it is readily seen that this is also true for $-1 < x < 0$. If $x = -1$, then $g_n(-1) = (-1)^n$, and since $((-1)^n)$ is divergent, the sequence $(g_n(-1))$ is divergent. Similarly, if $|x| > 1$, then the sequence (x^n) is not bounded, and so it is not convergent[1] in $\mathbb{R}$.

Hence if

$$g(x) = \begin{cases} 0 & \text{for } -1 < x < 1, \\ 1 & \text{for } \quad x = 1, \end{cases}$$

then the sequence (g_n) converges to g on the set $(-1,1]$.

[1]Reason: If this sequence is convergent, it must be bounded (by Theorem B), which is not the case.

Theorem A convergent sequence of real numbers is bounded

Example Show that $\lim((x^2 + nx)/n) = x$ for $x \in \mathbb{R}$ (Fig. 3).

Let $h_n(x) = (x^2 + nx)/n$ for $x \in \mathbb{R},\ n \in \mathbb{N}$, and let $h(x) = x$ for $x \in \mathbb{R}$. We can write $h_n(x) = \dfrac{x^2}{n} + x.$

Then

$$\lim(h_n(x)) = \lim\left(\frac{x^2}{n} + x\right)$$

$$= \lim\left(\frac{x^2}{n}\right) + \lim(x), \text{ using Theorem A}$$

$$= x^2 \cdot \lim\left(\frac{1}{n}\right) + x, \text{ using Theorem A and noting that for a fixed } x \text{, the limit of the constant sequence } (x) \text{ is } x$$

$$= x^2 \cdot 0 + x$$

$$= x$$

Then $(h_n(x)) \to h(x)$ for all $x \in \mathbb{R}$.

Example Show that $\lim((\frac{1}{n})\sin(nx + n)) = 0$ for $x \in \mathbb{R}$ (Fig. 4).

Let $F_n(x) = (\frac{1}{n})\sin(nx + n)$ for $x \in \mathbb{R}, n \in \mathbb{N}$, and let $F(x) = 0$ for $x \in \mathbb{R}$. Since $|\sin y| \le 1$ for all $y \in \mathbb{R}$ we have

$$|F_n(x) - F(x)| = \left|\frac{1}{n}\sin(nx + n)\right| \le \frac{1}{n} \qquad \dots (2)$$

for all $x \in \mathbb{R}$. Therefore it follows that $\lim(F_n(x)) = 0 = F(x)$ for all $x \in \mathbb{R}$.

It should note that, given any $£ > 0$, if n is sufficiently large, then $|F_n(x) - F(x)| < £$ for all values of x is simultaneously!

Getting an intuition from the remark above, we have the following reformulation of the definition of pointwise convergence

Lemma 1 A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ converges to a function $f : A_0 \to \mathbb{R}$ on A_0 if and only if for each $£ > 0$ and each $x \in A_0$ there is a natural number $K(£, x)$ such that if $n \geq K(£, x)$, then

$$|f_n(x) - f(x)| < £ . \qquad \ldots (3)$$

Example Let $f_n(x) = \dfrac{1}{x+n}$ for $x \in [0, \infty)$, $n = 1, 2, \ldots$ We show that (f_n) converges point wise to *f* where $f(x) = 0$ for all $x \in [0, \infty)$.

For $x = 0$, $f_n(0) = \dfrac{1}{n} \to 1.$

For fixed $x \in (0, \infty)$, it follows that

$$\lim(f_n(x)) = \lim\left(\frac{1}{x+n}\right)$$

$= \dfrac{1}{\lim(x+n)}$, using property of Limit of sequence of real numbers.

$$= \frac{1}{x} \cdot \frac{1}{\lim(1 + (n/x))}$$

$= \dfrac{1}{x} \cdot 0$, as $\lim(1/n) = 0$

$= 0$, as $\lim(1/n) = 0$

Hence (f_n) converges point wise to *f* where $f(x) = 0$ for all $x \in [0, \infty)$.

The above can be done as follows also:

For any $x \in [0, \infty)$, $\left(f_n(x)\right) = \dfrac{1}{x+n} \to 0$ as $n \to \infty$. Now, we let $f(x) = 0$. Let $£ > 0$ be given. We have to find $K(£, x)$ such that if $n \geq K(£, x)$, then

$$|f_n(x) - f(x)| < £ .$$

Now

$$\left|\frac{1}{x+n} - 0\right| < £$$

implies $\dfrac{1}{x+n} < £$

implies $\dfrac{1}{£} < x + n$

implies $\dfrac{1}{£} - x < n.$

Now if we choose

$$k(£,x)=\begin{cases}\text{integral part of } (\frac{1}{£}-x), & \text{when } x<\frac{1}{£}\\ 1, & \text{when } x\geq\frac{1}{£}\end{cases}$$

i.e., $k(£,\ x)$ as the smallest positive integer greater than or equal to $\dfrac{1}{£}-x$, then for $n\geq k(£,\ x)$,

$$\frac{1}{£}-x\leq k(£,\ x)\leq n.$$

Hence, by Lemma, (f_n) converges point wise to f where $f(x)=0$ for all $x\in[0,\infty)$.

Uniform Convergence

Definition A sequence (f_n) of functions on $A\subseteq\mathbb{R}$ to $\mathbb{R}$ **converges uniformly on** $A_0\subseteq A$ to a function $f:A_0\to\mathbb{R}$ if for each $£>0$ there is a natural number $K(£)$ (depending on $£$ but **not** on $x\in A_0$) such that if $n\geq K(£)$, then

$$\left|f_n(x)-f(x)\right|<£ \text{ for all } x\in A_0. \qquad \ldots (4)$$

In this case we say that the sequence (f_n) **is uniformly convergent on A_0**. Sometimes we write

$$f_n \rightrightarrows f \text{ on } A_0, \qquad \text{or } f_n(x)\rightrightarrows f(x) \qquad \text{for } x\in A_0.$$

It is an immediate consequence of the definitions that if the sequence (f_n) is uniformly convergent on A_0 to *f*, then this sequence also converges pointwise on A_0 to *f* in the sense of Definition of pointwise convergence. That the *converse is not always true* is seen by a careful examination of earlier Examples in that section; other examples will be given below.

It is sometimes useful to have the following necessary and sufficient condition for a sequence (f_n) to *fail* to converge uniformly on A_0 to *f*.

The proof of the following result requires only that take the negation of Definition of uniform convergence.

Lemma 2 A sequence (f_n) of functions on $A\subseteq\mathbb{R}$ to $\mathbb{R}$ does not converge uniformly on $A_0\subseteq A$ to a function $f:A_0\to\mathbb{R}$ if and only if for some $£_0>0$ there is a subsequence (f_{n_k}) of (f_n) and a sequence (x_k) in A_0 such that

$$\left|f_{n_k}(x_k)-f(x_k)\right|\geq£_0 \qquad \text{for all } k\in\mathbb{N}.$$

Example Let $f_n(x)=\dfrac{x}{n},\ n\in\mathbb{N}$ and $x\in\mathbb{R}$. We have noted in an earlier example that $\lim(f_n(x))=0$ for all $x\in\mathbb{R}$. Let $f(x)=0$ for all $x\in\mathbb{R}$. Then (f_n) converges to f pointwise. If we let $n_k=k$ and $x_k=k$, then $f_{n_k}(x_k)=\dfrac{k}{k}=1$ so that $\left|f_{n_k}(x_k)-f(x_k)\right|=\left|1-0\right|=1$. Let $£_0=\dfrac{1}{2}$. Then, by the lemma above, the sequence (f_n) does not converge uniformly on $\mathbb{R}$ to *f*.

Example Let $g_n(x)=x^n,\ x\in(-1,\ 1],\ n\in\mathbb{N}$. We have seen in an earlier example that g_n converges to g on the set $(-1,\ 1]$, where

$$g(x)=\begin{cases}0 & \text{for } -1<x<1\\ 1 & \text{for } x=1\end{cases}$$

If $n_k = k$ and $x_k = \left(\frac{1}{2}\right)^{1/k}$, then $g_{n_k}(x_k) = \left(\left(\frac{1}{2}\right)^{\frac{1}{k}}\right)^k = \dfrac{1}{2}$, so that

$$\left|g_{n_k}(x_k) - g(x_k)\right| = \left|\tfrac{1}{2} - 0\right| = \tfrac{1}{2}.$$

Let $£_0 = \dfrac{1}{4}$. Then, by the lemma above, the sequence (g_n) doesnot converge uniformly on $(-1,1]$ to g.

Example Let $h_n(x) = \dfrac{x^2 + nx}{n}$ $x \in \mathbb{R},\ n \in \mathbb{N}$. We have seen in an earlier example that $(h_n) \to h$ for all $x \in \mathbb{R}$, where $h(x) = x$ for all $x \in \mathbb{R}$.

If $n_k = k$ and $x_k = -k$, then $h_{n_k}(x_k) = \dfrac{(-k)^2 + k(-k)}{k} = 0$ and $h(x_k) = -k$ so that $\left|h_{n_k}(x_k) - h(x_k)\right| = k$.

Let $£_0 = \frac{1}{2}k$. Then, by the Lemma above, the sequence (h_n) doesnot converge uniformly on $\mathbb{R}$ to h.

Example Show that the sequence $(f_n(x))$ where

$$f_n(x) = \frac{n}{x+n}, \quad x \geq 0$$

is uniformly convergent in the closed bounded interval $[0, m]$, whatever m may be, but not in the interval $0 \leq x < \infty$.

Here for any given $x \geq 0$,

$$\begin{aligned} f(x) &= \lim_{n\to\infty} f_n(x) \\ &= \lim\left(\frac{n}{x+n}\right) = \lim\left[\frac{1}{(x/n)+1}\right] \\ &= 1. \end{aligned}$$

Then for a given $£ > 0$,

$$\left|f_n(x) - f(x)\right| < £,$$

if $$\left|\frac{n}{x+n} - 1\right| < £$$

i.e., if $$\left|\frac{-x}{x+n}\right| < £$$

i.e., if $$\frac{x}{x+n} < £$$

i.e., if $$n > x\left(\frac{1}{£} - 1\right).$$

Let $K(£, x) =$ the smallest integer greater than $x(1/£ - 1)$. Also, we note that $K(£, x)$ increases as x increases and tends ∞ as $x \to \infty$. Hence it is not possible to choose a positive integer $K(£)$ such that

$$\left|f_n(x) - f(x)\right| < £ \text{ for every } n \geq K(£)$$

and *for every value* of x in $[0, \infty]$. Hence the convergence is nonuniform in $[0, \infty)$.

If we consider any finite interval $[0, m]$ where $m > 0$ is a fixed number, then the maximum value of $x(1/£ - 1)$ on $[0, m]$ is $m(1/£ - 1)$ so that if we take

$$K(£) = \text{any integer greater than } m(1/£ - 1),$$

Then

$$\left| f_n(x) - f(x) \right| < £ \text{ for every } n \geq K(£) \text{ and for every } x \text{ in } [0, m].$$

This shows that the sequence $(f_n(x))$ converges uniformly in the interval $[0, m]$ where m is any fixed positive number.

Example Show that $x = 0$ is a point of nonuniform convergence of the sequence $(f_n(x))$ in $0 \leq x \leq 1$, where

$$f_n(x) = \frac{n^2 x}{1 + n^4 x^2}$$

Solution

Here

$$f(x) = \lim(f_n(x)) = \lim\left(\frac{n^2 x}{1 + n^4 x^2}\right) = \lim\left(\frac{x / n^2}{(1/n^4) + x^2}\right)$$

$$= 0 \text{ for } 0 \leq x \leq 1$$

For a given £, we have

$$\left| f_n(x) - f(x) \right| < £,$$

if $\quad \dfrac{n^2 x}{1 + n^4 x^2} < £$

i.e., if $\quad n^4 x^2 £ - n^2 x + £ > 0$

i.e., if $\quad n^2 > \dfrac{x \pm \sqrt{x^2 - 4x^2 £^2}}{2x^2 £}$

i.e., if $\quad n^2 > \dfrac{1 \pm \sqrt{1 - 4£^2}}{2x£} \qquad ..(1)$

Also (1) shows that if $x \to 0, n \to \infty$ so that it is not possible to choose $K(£) \in \mathbb{N}$ such that

$$\left| f_n(x) - f(x) \right| < £ \text{ for every } n \geq K(£) \text{ and for every } x \in [0, 1].$$

Hence the sequence is nonuniformly convergent in $[0, 1]$.

But if we consider the interval $[m, 1]$ where $0 < m < 1$, the convergence is uniform since in that case it is possible to take

$$K(£) = \text{ an integer just greater than } \left[\frac{1 + \sqrt{1 - 4£^2}}{2x£}\right]^{\frac{1}{2}}.$$

Hence $x = 0$ is the *point of nonuniform convergence* of the given sequence in $[0, 1]$.

Example Test for uniform convergence the sequence $\{e^{-nx}\}$ for $x \geq 0$.

Let $\quad f_n(x) = e^{nx}$.

Then $\quad \lim(f_n(x)) = \lim(e^{-nx}) = \begin{cases} 1 \text{ when } x = 0 \\ 0 \text{ when } x > 0 \end{cases}$

Let $$f(x) = \begin{cases} 1 \text{ when } x = 0 \\ 0 \text{ when } x > 0 \end{cases}$$

Then for each $x \in [0, \infty)$, $\lim\,(f_n(x)) = f(x)$. Hence (f_n) converges point wise to f in $[0, \infty)$.

Now for any $£ > 0$ and $x \in (0, \infty)$,

$$|f_n(x) - f(x)| = e^{-nx} < £$$

if $$e^{nx} > \frac{1}{£}$$

i.e., if $$nx > \log\left(\frac{1}{£}\right)$$

i.e., if $$n > \left(\frac{1}{x}\right)\log\left(\frac{1}{£}\right).$$

Hence choosing

$K(£, x)$ = the smallest integer greater than $(1/x)\log(1/£)$

we get, for all $x \in (0, \infty)$

$$|s_n(x) - S(x)| < £ \quad \text{for all } n \geq K(£, x).$$

Here, obviously, $K(£, x)$ depends on x. We note that $K(£, x)$ increases and tends to ∞ as x tends to 0 i.e., $K(£, x)$ is not a bounded function of x on $(0, \infty)$. Hence it is not possible to find a natural number $K(£)$, depending only on $£$, such that

$$|s_n(x) - S(x)| < £ \text{ whenever } n \geq K(£) \text{ and } x \in [0, \infty)$$

Hence (f_n) is not uniformly convergent on $[0, \infty)$.

If, however, we consider the interval $[a, \infty)$ where a is any fixed real number greater than zero, however small, then $K(£, x)$ is a bounded function on $[a, \infty)$. The maximum value of $K(£, x)$ on $[a, \infty)$ is the integer just greater than $(1/a)\log(1/£)$. Hence if we take

$K(£) =$ the integer just greater than $(1/a)\log(1/£)$,

then

$$|f_n(x) - f(x)| < £ \text{ whenever } n \geq K(£) \text{ and } x \in [a, \infty)$$

Hence the sequence (f_n) is uniformly convergent in $[a, \infty)$ where $a > 0$, but nonuniformly convergent in $[0, \infty)$.

Example Show that if $g_n(x) = \dfrac{x}{nx+1}$ for $x \geq 0$, then (g_n) converges uniformly on $[0, \infty)$.

For any $x \geq 0$, $\lim(g_n(x)) = \lim\left(\dfrac{x}{nx+1}\right) = \lim\left(\dfrac{x/n}{x+1/n}\right) = 0$

Hence (g_n) converges point wise to g where

$$g(x) = 0 \text{ for } x \geq 0 \text{ on } [0, \infty).$$

Now for any $£ > 0$ and $x \in (0, \infty)$,

$$|g_n(x) - g(x)| < £$$

if $$\left|\frac{x}{nx+1}\right| < £$$

i.e., if $\quad nx+1>\frac{x}{£}$

i.e., if $\quad n>\frac{1}{£}-\frac{1}{x}$.

Take $\quad K(£,\,x)=$ the positive integer just greater than $\left(\frac{1}{£}-\frac{1}{x}\right)$.

Then for all $x\in(0,\,\infty)$,

$$\left|g_n(x)-g(x)\right|<£ \quad \text{for all} \quad n\geq k(£,\,x)$$

Here $K(£,\,x)$ = the positive integer just greater than $\left(\frac{1}{£}-\frac{1}{x}\right)$ is a bounded above by $\frac{1}{£}$ and is a function of *x* on $(0,\,\infty)$.

The maximum value of $K(£,\,x)$ on $(0,\,\infty)$ is the integer just greater than $1/£$. Hence if we take

$$K(£)= \text{ integer just greater than } 1/£\,,$$

then

$$\left|g_n(x)-g(x)\right|<£ \;\text{ for all }\; n\geq K(£) \;\text{ and }\; x\in(0,\,\infty)$$

But when $x=0,\;\; g_n(x)=0$ for all $n\in\mathbb{N},$ and hence we have

$$\left|g_n(x)-g(x)\right|<£ \;\text{ for all }\; n\geq K(£) \;\text{ and }\; x\in[0,\,\infty)$$

Hence the sequence (g_n) is uniformly convergent on $[0,\,\infty)$

The Uniform Norm

In discussing uniform convergence, it is often convenient to use the notion of the uniform norm on a set of bounded functions.

If $A\subseteq\mathbb{R}$ and $¨:A\to\mathbb{R}$ is a function, we say that ¨ is **bounded on** A if the set $¨(A)$ is a bounded subset of $\mathbb{R}$. If ¨ is bounded we define the **uniform norm** of ¨ on A by

$$\|¨\|_A=\sup\{|¨(x)|:x\in A\} \qquad \ldots (6)$$

Note that it follows that if $£>0$, then

$$\|¨\|_A\leq £ \quad\Leftrightarrow\quad |¨(x)|\leq £ \qquad \text{for all } x\in A. \quad \ldots (7)$$

Lemma 3 A sequence (f_n) of bounded functions on $A\subseteq\mathbb{R}$ converges uniformly on A to *f* if and only if $\|f_n-f\|_A\to 0$.

Proof $(\Rightarrow)$ If (f_n) converges uniformly on A to f, then by the Definition of uniform convergence, given any $£>0$ there exists $K(£)$ such that if $n\geq K(£)$ and $x\in A$ then

$$|f_n(x)-f(x)|\leq £\,.$$

From the definition of supremum, it follows that

$$\|f_n-f\|_A\leq £ \;\text{ whenever }\; n\geq K(£)\,.$$

Since $£>0$ is arbitrary this implies that $\|f_n-f\|_A\to 0$.

$(\Leftarrow)$ If $\|f_n-f\|_A\to 0$, then given $£>0$ there is a natural number $H(£)$ such that if $n\geq H(£)$ then $\|f_n-f\|_A\leq £$. It follows from (7) that $|f_n(x)-f(x)|\leq £$ for all $n\geq H(£)$ and $x\in A$. Therefore (f_n) converges uniformly on A to *f*. This completes the proof.

We now illustrate the use of Lemma as a tool in examining a sequence of bounded functions for uniform convergence.

Example Show that $\lim(x/n)=0$ for $x\in\mathbb{R}$. Is the convergence uniform on $\mathbb{R}$? Is the convergence uniform on $A=[0,1]$? Explain.

For $n\in\mathbb{N}$, let $f_n(x)=x/n$ and let $f(x)=0$ for $x\in\mathbb{R}$. By Example 1, for all $x\in\mathbb{R}$, $\lim(x/n)=0$, so that the sequence of functions (f_n) converges point wise to f. In particular, for $x=0$, (x/n) converges to 0.

To examine the whether the sequence of functions is uniformly convergent on $\mathbb{R}$, we cannot apply Lemma since the function $f_n(x)-f(x)=\dfrac{x}{n}$ is not bounded on $\mathbb{R}$. But we have verified in an earlier example that this convergence is *not* uniform on $\mathbb{R}$.

Now we examine the uniform convergence on $A=[0,1]$. To see this, we observe that (for a fixed $n\in\mathbb{N}$)

$$\|f_n-f\|_A=\sup\left\{\left|\frac{x}{n}-0\right|:0\le x\le 1\right\}=\frac{1}{n}\sup\{x:0\le x\le 1\}=\frac{1}{n}.$$

so that $\|f_n-f\|_A\to 0$ as $n\to\infty$. Therefore, by Lemma, (f_n) is uniformly convergent on A to f.

We conclude that although the sequence (x/n) *does not converge uniformly on* $\mathbb{R}$ to the zero function, the *convergence is uniform on* A.

Example Let $g_n(x)=x^n$ for $x\in A=[0,1]$ and $n\in\mathbb{N}$, and let $g(x)=0$ for $0\le x<1$ and $g(1)=1$. The functions $g_n(x)-g(x)$ are bounded on A and, we have for any $n\in\mathbb{N}$,

$$\|g_n-g\|_A=\sup\begin{cases}x^n & \text{for } 0\le x<1\\ 0 & \text{for } x=1\end{cases}$$
$$=1$$

Since $\|g_n-g\|_A$ does not converge to 0, we infer, by applying Lemma, that the sequence (g_n) does *not* converge uniformly on A to g.

Example We cannot apply Lemma to the sequence in Example 3 since the function $h_n(x)-h(x)=x^2/n$ is not bounded on $\mathbb{R}$.

Instead, let $A=[0,8]$ and consider

$$\|h_n-h\|_A=\sup\left\{\frac{x^2}{n}:0\le x\le 8\right\}=\frac{64}{n}.$$

Therefore, the sequence (h_n) converges uniformly on A to h.

Example We have seen from (2) that $\|F_n-F\|_{\mathbb{R}}\le\dfrac{1}{n}$. Hence (F_n) converges uniformly on $\mathbb{R}$ to F.

First Derivative Test for Extrema

Let f be continuous on the interval $I=[a,b]$ and let c be an interior point of I. Assume that f is differentiable on (a,c) and (c,b). Then:

(a) If there is a neigbourhood $(c-\text{¢},\ c+\text{¢})\subseteq I$ such that $f'(x)\ge 0$ for $c-\text{¢}<x<c$ and $f'(x)\le 0$ for $c<x<c+\text{¢}$, then f has a relative maximum at c.

(b) If there is a neigbourhood $(c-\text{¢},\ c+\text{¢})\subseteq I$ such that $f'(x)\le 0$ for $c-\text{¢}<x<c$ and $f'(x)\ge 0$ for $c<x<c+\text{¢}$, then f has a relative minimum at c.

Example Let $G(x) = x^n(1-x)$ for $x \in A = [0,1]$. Then the sequence $(G_n(x))$ converges to $G(x) = 0$ for each $x \in A$. To calculate the uniform norm $G_n - G = G_n$ on A, we find the derivative and solve

$$G_n'(x) = x^{n-1}(n-(n+1)x) = 0$$

to obtain the point $x_n = \dfrac{n}{n+1}$. This is an interior point of [0,1], and it is easily verified by using the First derivative Test that G_n attains a maximum on [0,1] at x_n. Therefore, we obtain

$$\|G_n\|_A = G_n(x_n) = (1+\tfrac{1}{n})^{-n} \cdot \frac{1}{n+1}$$

which converges to $(\frac{1}{e}) \cdot 0 = 0$. Thus we see that convergence is uniform on A.

By making use of the uniform norm, we can obtain a necessary and sufficient condition for uniform convergence that is often useful.

Cauchy Criterion for Uniform Convergence of Sequence of Functions: Let (f_n) be a sequence of bounded functions on $A \subseteq \mathbb{R}$. Then this sequence converges uniformly on A to a bounded function f if and only if for each $£ > 0$ there is a number $H(£)$ in $\mathbb{N}$ such that for all $m, n \geq H(£)$, then $\| f_m - f_n \|_A \leq £$.

Proof. ($\Rightarrow$) If $f_n \underset{\to}{\to} f$ on A, then given $£ > 0$ there exists a natural number $K\left(\frac{1}{2}£\right)$ such that if $n \geq K(\frac{1}{2}£)$ then $\| f_n - f \|_A \leq \frac{1}{2}£$. Hence if both $m, n \geq K(\frac{1}{2}£)$, then we conclude that

$$|f_m(x) - f_n(x)| < |f_m(x) - f(x)| + |f_n(x) - f(x)| \leq \tfrac{1}{2}£ + \tfrac{1}{2}£ = £$$

for all $x \in A$. Therefore $\| f_m - f_n \|_A \leq £$, for $m, n \geq K(\frac{1}{2}£) = H(£)$.

($\Leftarrow$) Conversely, suppose that for $£ > 0$ there is $H(£)$ such that if $m, n \geq H(£)$, then $\| f_m - f_n \|_A \leq £$. Therefore, for each $x \in A$ we have

$$|f_m(x) - f_n(x)| \leq \| f_m - f_n \|_A \leq £ \text{ for } m, n \geq H(£). \qquad \ldots (8)$$

It follows that $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$; therefore, being a Cauchy sequence, it is a convergent sequence. We define $f : A \to \mathbb{R}$ by

$$f(x) = \lim(f_n(x)) \qquad \text{for} \quad x \in A.$$

From (8), we have for each $x \in A$

$$-£ \leq f_m(x) - f_n(x) \leq £ \text{ for } m, n \geq H(£).$$

Now fix $m \geq H(£)$ and let $n \to \infty$, then we obtain for $x \in A$,

$$-£ \leq f_m(x) - f(x) \leq £.$$

Hence for each $x \in A$, we have

$$|f_m(x) - f(x)| \leq £ \text{ for } m \geq H(£).$$

Therefore the sequence (f_n) converges uniformly on A to f. This completes the proof.

Exercises

1. Show that $\lim(x/(x+n)) = 0$ for all $x \in \mathbb{R}, x \geq 0$

2. Show that $\lim(nx/(1+n^2x^2)) = 0$ for all $x \in \mathbb{R}$.
3. Evaluate $\lim(nx/(1+nx))$ for $x \in \mathbb{R}, x \geq 0$
4. Evaluate $\lim(x^n/(1+x^n))$ for $x \in \mathbb{R}, x \geq 0$.
5. Evaluate $\lim((\sin nx)/(1+nx))$ for $x \in \mathbb{R}, x \geq 0$.
6. Show that $\lim(\text{Arctan}\, nx) =$ (® / 2) $\text{sgn}\, x$ for $x \in \mathbb{R}$, where sgn is the signum function.
7. Evaluate $\lim(e^{-nx})$ for $x \in \mathbb{R}$, $x \geq 0$.
8. Show that $\lim(xe^{-nx}) = 0$ for $x \in \mathbb{R}, x \geq 0$.
9. Show that $\lim(x^2e^{-nx}) = 0$ and that $\lim(n^2x^2e^{-nx}) = 0$ for $x \in \mathbb{R}, x \geq 0$.
10. Show that $\lim((\cos$ ® $x)^{2n})$ exists for all $x \in \mathbb{R}$. What is its limit?
11. Show that if $a > 0$, then the convergence of the sequence in Exercise 1 is uniform on the interval $[0, a]$, but is not uniform on the interval $[0, \infty)$.
12. Show that if $a > 0$, then the convergence of the sequence in Exercise 2 is uniform on the interval $[0, a]$, but is not uniform on the interval $[0, \infty)$.
13. Show that if $a > 0$, then the convergence of the sequence in Exercise 3 is uniform on the interval $[0, a]$, but is not uniform on the interval $[0, \infty)$.
14. Show that if $0 < b < 1$, then the convergence of the sequence in Exercise 4 is uniform on the interval $[0, b]$, but is not uniform on the interval $[0, 1]$.
15. Show that if $a > 0$, then the convergence of the sequence in Exercise 5 is uniform on the interval $[a, \infty)$, but is not uniform on the interval $[0, \infty)$.
16. Show that if $a > 0$, then the convergence of the sequence in Exercise 6 is uniform on the interval $[a, \infty)$, but is not uniform on the interval $(0, \infty)$.
17. Show that if $a > 0$, then the convergence of the sequence in Exercise 7 is uniform on the interval $[a, \infty)$, but is not uniform on the interval $[0, \infty)$.
18. Show that the convergence of the sequence in Exercise 8 is uniform on $[0, \infty)$.
19. Show that the sequence (x^2e^{-nx}) converges uniformly on $[0, \infty)$.
20. Show that if $a > 0$, then the sequence $(n^2x^2e^{-nx})$ converges uniformly on the interval $[a, \infty)$, but that it does not converge uniformly on the interval $[0, \infty)$.
21. Show that if $(f_n), (g_n)$ converge uniformly on the set A to f, g, respectively, then $(f_n + g_n)$ converges uniformly on A to $f + g$.
22. Show that if $f_n(x) = x + 1/n$ and $f(x) = x$ for $x \in \mathbb{R}$, then (f_n) converges uniformly on $\mathbb{R}$ to f, but the sequence (f_n^2) does not converge uniformly on $\mathbb{R}$. (Thus the product of uniformly convergent sequences of functions may not converge uniformly.)
23. Let $(f_n),(g_n)$ be sequences of bounded functions on A that converge uniformly on A to f, g, respectively. Show that (f_ng_n) converges uniformly on A to fg.
24. Let (f_n) be a sequence of functions that converges uniformly to f on A and that satisfies $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and all $x \in A$. If g is continuous on the interval $[-M, M]$, show that the sequence $(g \circ f_n)$ converges uniformly to $g \circ f$ on A.

8
UNIFORM CONVERGENCE AND CONTINUITY

We first list some examples which shows that, in general, limit of a sequence of continuous functions need not be continuous. Similarly, limit of a sequence of differentiable (resp., Riemann integrable) functions need not be differentiable (resp., Riemann integrable). In this text we discuss the limit of a sequence of continuous functions only.

We will consider results that reveal the importance of uniform convergence that allows interchange of limits which makes the limit of sequence of continuous function also continuous. Let $g_n(x) = x^n$ for $x \in [0,1]$ and $n \in \mathbb{N}$. Then, the sequence (g_n) converges pointwise to the function

$$g(x) = \begin{cases} 0 & \text{for} \quad 0 \le x < 1 \\ 1 & \text{for} \quad x = 1 \end{cases}$$

Although all of the functions g_n are continuous at $x = 1$, the limit function g is not continuous at $x = 1$. Recall that it was shown in Example 9 in the previous chapter that this sequence does not converge uniformly to g on $[0,1]$.

Each of the functions $g_n(x) = x^n$ for $x \in [0,1]$ and $n \in \mathbb{N}$ has a continuous derivative on $[0,1]$. For $x \in [0,\ 1]$, $g_n'(x) = nx^{n-1}$. However, the limit function g does not have a derivative at $x = 1$, since it is not continuous at that point.

Let $f_n : [0,1] \to \mathbb{R}$ be defined for $n \ge 2$ by

$$f_n(x) = \begin{cases} n^2 x & \text{for} \quad 0 \le x \le \frac{1}{n} \\ -n^2(x - \frac{2}{n}) & \text{for} \quad \frac{1}{n} \le x \le \frac{2}{n} \\ 0 & \text{for} \quad \frac{2}{n} \le x \le 1 \end{cases}$$

It is clear that each of the functions f_n is continuous on $[0,1]$; hence it is Riemann integrable.

We note that for $x \in [0,1]$, $f_n(x) \to 0$ as $n \to \infty$. Also, for $n \ge 2$, $\int_0^1 f_n(x)dx = 1$. This can be described as below:

$$\begin{aligned} \int_0^1 f_n(x)dx &= \int_0^{1/n} n^2 x\, dx + \int_{1/n}^{2/n} -n^2(x - \tfrac{2}{n})dx + \int_{2/n}^1 0\, dx \\ &= n^2 \left[\tfrac{x^2}{2} \right]_0^{1/n} - n^2 \left[(\tfrac{x^2}{2} - \tfrac{2}{n} \cdot x) \right]_{1/n}^{2/n} \\ &= \frac{1}{2} + \frac{1}{2} = 1 \text{ for } n \ge 2 \end{aligned}$$

If $x \in [0, 1]$ and $n \to \infty$, we have $\frac{2}{n} \le x \le 1$ so that $f_n(x) \to 0$ as $n \to \infty$. Hence the limit function *f* is such that $f(x) = 0$ for $x \in [0, 1]$. Being a constant function, f is continuous on $[0, 1]$ and hence is Riemann integrable on $[0, 1]$ and $\int_0^1 f(x)\,dx = 0$. Therefore we have

$$\int_0^1 f(x)\,dx = 0 \ne 1 = \lim \int_0^1 f_n(x)\,dx$$

In this example,

$$\int_0^1 \lim f_n(x)\,dx \ne \lim \int_0^1 f_n(x)\,dx,$$

so that interchange of limits is not possible.

Example Consider the sequence (h_n) of functions defined by $h_n(x) = 2nxe^{-nx^2}$ for $x \in [0,1],\ n \in \mathbb{N}$. Since $h_n = H_n'$, where $H_n(x) = -e^{-nx^2}$, the Fundamental Theorem gives

$$\int_0^1 h_n(x)dx = H_n(1) - H_n(0) = 1 - e^{-n}.$$

For $x \in [0,1]$, $h_n(x) = \frac{2nx}{e^{nx^2}} \to 0$ as $n \to \infty$. Hence if we take $h(x) = \lim(h_n(x))$, we have

$$h(x) = \lim(h_n(x)) = 0 \text{ for all } x \in [0,1].$$

Hence

$$\int_0^1 h(x)dx \ne \lim \int_0^1 h_n(x)\,dx.$$

That is interchange of limits is not possible in this case also.

Interchange of Limit and Continuity

Theorem Let (f_n) be a sequence of continuous functions on a set $A \subseteq \mathbb{R}$ and suppose that (f_n) converges uniformly on *A* to a function $f : A \to \mathbb{R}$. Then *f* is continuous on *A*.

Proof. By hypothesis, given $\text{£} > 0$ there exists a natural number $H = H(\tfrac{1}{3}\text{£})$ such that if $n \ge H$ then $|f_n(x) - f(x)| < \tfrac{1}{3}\text{£}$ for all $x \in A$. Let $c \in A$ be arbitrary; we will show that *f* is continuous at *c*. By the Triangle Inequality we have

$$|f(x) - f(c)| \le |f(x) - f_H(x)| + |f_H(x) - f_H(c)| + |f_H(c) - f(c)|$$
$$\le \tfrac{1}{3}\text{£} + |f_H(x) - f_H(c)| + \tfrac{1}{3}\text{£}.$$

Since f_H is continuous at *c*, there exists a number $\text{¢} = \text{¢}(\tfrac{1}{3}\text{£}, c, f_H) > 0$ such that if $|x - c| < \text{¢}$ and $x \in A$, then $|f_H(x) - f_H(c)| < \tfrac{1}{3}\text{£}$. Therefore, if $|x - c| < \text{¢}$ and $x \in A$, then we have $|f(x) - f(c)| < \text{£}$. Since $\text{£} > 0$ is arbitrary, this establishes the continuity of *f* at the arbitrary point $c \in A$.

Hence f is continuous at every point in A. Hence f is continuous on A. This completes the proof.

Although the uniform convergence of the sequence of continuous functions is sufficient to guarantee the continuity of the limit functions, it is not necessary. This is illustrated in the following example.

Example

Let $f_n(x) = \dfrac{nx}{1+n^2x^2}$ for all $x \in [0,1]$ $n = 1,2,3,...$

Then

$$\lim(f_n(x)) = \lim\left(\frac{nx}{1+n^2x^2}\right) = \lim\left(\frac{x/n}{\frac{1}{n^2}+x^2}\right) = 0 \quad \text{for all } x \in [0,1].$$

Let us define $f:[0,1] \to \mathbb{R}$ by

$$f(x) = 0 \text{ for all } x \in [0,1].$$

Then (f_n) converges pointwise to f on [0, 1]. Here clearly each f_n and f are continuous on $[0,1]$. But

$$\|f_n - f\|_{[0,1]} = \sup\{|f_n(x) - f(x)| : x \in [0,1]\}$$

$$\sup\left\{\frac{nx}{1+n^2x^2} : x \in [0,1]\right\} \qquad \ldots (1)$$

To find $\sup\left\{\dfrac{nx}{1+n^2x^2} : x \in [0,1]\right\}$ i.e., the maximum value of $\dfrac{nx}{1+n^2x^2}$ on the set [0, 1], we can use the *Differentiation Method.* By the method, the function $y = \dfrac{nx}{1+n^2x^2}$, attain an extremum when $y' = 0$

i.e., when $\dfrac{(1+n^2x^2)n - nx \cdot 2n^2x}{(1+n^2x^2)^2} = 0$

i.e., when $n + n^3x^2 - 2n^3x^2 = 0$.

i.e., when $x^2 = \dfrac{1}{n^2}$

i.e., when $x = \pm\dfrac{1}{n}$. Hence $y = \dfrac{nx}{1+n^2x^2}$ attains a maximum in [0,1] at $x = \dfrac{1}{n}$ and the maximum value is $\dfrac{n \cdot \frac{1}{n}}{1+n^2 \cdot \frac{1}{n^2}} = \dfrac{1}{2}$. Hence from (1), we have $\|f_n - f\|_{[0,1]} = \dfrac{1}{2}$. Hence $\|f_n - f\|_{[0,1]} \to \dfrac{1}{2}$ as $n \to \infty$.

Since $\|f_n - f\|_{[0,1]}$ does not tend to zero as n tends to infinity, (f_n) is not uniformly convergent to f on $[0,1]$.

Exercises

1. Show that the sequence $((x^n/(1 = x^n))$ does not converge uniformly on $[0,2]$ by showing that the limit function is not continuous on $[0,2]$.
2. Let $f_n:[0,1] \to \mathbb{R}$ be defined for $n \geq 2$ by

$$f_n(x) = \begin{cases} n^2 x & \text{for } 0 \le x \le 1/n \\ -n^2(x - \frac{2}{n}) & \text{for } 1/n \le x \le 2/n \\ 0 & \text{for } 2/n \le x \le 1 \end{cases}$$

Prove that the sequence in (f_n) is an example of a sequence of continuous functions that converges non-uniformly to continuous limit.

3. Construct a sequence of functions on $[0,1]$ each of which is discontinuous at every point of $[0,1]$ and which converges uniformly to a function that is continuous at every point.
4. Suppose (f_n) is a sequence of continuous functions on an interval *I* that converges uniformly on *I* to a function *f*. If $(x_n) \subseteq I$ converges to $x_0 \in I$, show that $\lim(f_n(x_n)) = f(x_0)$.
5. Let $f:\mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let $f_n(x) = f(x + 1/n)$ for $x \in \mathbb{R}$. Show that (f_n) converges uniformly on $\mathbb{R}$ to *f*.
6. Let $f_n(x) = 1/(1+x)^n$ for $x \in [0,1]$. Find the pointwise limit *f* of the sequence (f_n) on $[0,1]$. Does (f_n) converges uniformly to *f* on $[0,1]$?
7. Suppose the sequence (f_n) converges uniformly to *f* on the set *A* and suppose that each f_n is bounded on *A*. (That is, for each *n* there is a constant M_n such that $|f_n(x)| \le M_n$ for all $x \in A$.) Show that the function *f* is bounded on *A*.
8. Let $f_n(x) = nx/(1+nx^2)$ for $x \in A = [0,\infty)$. Show that each f_n is bounded on *A*, but the pointwise limit *f* of the sequence is not bounded on *A*. Does (f_n) converge uniformly to *f* on *A*?

9
SERIES OF FUNCTIONS

If (f_n) is a sequence of functions defined on a subset D of $\mathbb{R}$ with values in $\mathbb{R}$, the sequence of **partial sums** (s_n) of the infinite series $\sum f_n$ is defined for x in D by

$$\begin{aligned} &s_1(x) = f_1(x) \\ &s_2(x) = s_1(x) + f_2(x) \\ &\vdots \qquad \vdots \qquad \vdots \\ &s_{n+1}(x) = s_n(x) + f_{n+1}(x) \\ &\vdots \qquad \vdots \qquad \vdots \end{aligned}$$

In case the sequence (s_n) of functions converges on D to a function f, we say that the infinite series of functions $\sum f_n$ **converges** to f on D. **Notation** We will often write

$$\sum f_n \quad \text{for} \quad \sum_{n=1}^{\infty} f_n$$

to denote either the series or the limit function, when it exists.

If the series $\sum |f_n(x)|$ converges for each x in D, we say that $\sum f_n$ is **absolutely convergent** on D. If the sequence (s_n) of partial sums is uniformly convergent on D to f, we say that $\sum f_n$ is **uniformly convergent on *D***, or that it **converges to *f* uniformly on** D.

One of the main reasons for the interest in uniformly convergent series of functions is the validity of the following results which give conditions justifying the change of order of the summation and other limiting operations.

Theorem 1 If f_n is continuous on $D \subseteq \mathbb{R}$ to $\mathbb{R}$ for each $n \in \mathbb{N}$ and if $\sum f_n$ converges to f uniformly on D, then f is continuous on D.

We note that Theorem 1 is a direct translation of the following Theorem for series.

Theorem A Let (f_n) be a sequence of continuous functions on a set $A \subseteq \mathbb{R}$ and suppose that (f_n) converges uniformly on A to a function $f : A \to \mathbb{R}$. Then f is continuous on A.

Example We now show that the series

$$\sum_{n=1}^{\infty} \frac{x^4}{(1+x^4)^{n-1}} = x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \frac{x^4}{(1+x^4)^3} + \cdots$$

is not uniformly convergent on [0, 1]. Discuss the continuity of the sum function.

Let $f_n(x) = \dfrac{x^4}{(1+x^4)^{n-1}}$ and $s_N(x) = \sum_{n=1}^{N} f_n(x) = \sum_{n=1}^{N} \dfrac{x^4}{(1+x^4)^{n-1}}$.

Hence

$$s_N(x) = \sum_{n=1}^{N} \frac{x^4}{(1+x^4)^{n-1}}$$

$$= \frac{x^4\left[1 - \frac{1}{(1+x^4)^N}\right]}{1 - \frac{1}{1+x^4}} = \frac{(1+x^4)^N - 1}{(1+x^4)^{N-1}}$$

$$= 1 + x^4 - \frac{1}{(1+x^4)^{N-1}} \quad \text{if } x \neq 0. \qquad \ldots(1)$$

When $x = 0$, $f_n(0) = 0$ for all n and hence $s_N(x) = 0$. Hence $s_N(0) = 0 \to 0$ as $N \to \infty$.

For $x \neq 0$, using (1),

$$s_N(x) = 1 + x^4 - \frac{1}{(1+x^4)^{N-1}} \to 1 + x^4 \text{ as } N \to \infty.$$

Hence if $$f(x) = \begin{cases} 1 + x^4, & x \neq 0 \\ 0 & x = 0 \end{cases}$$

then, (s_N) converges point wise to *f* and hence $\sum f_n$ converges pointwise to f on $[0, 1]$.

Each $f_n(x)$ is a continuous function on $[0, 1]$. But its limit $f(x)$ is not continuous at $x = 0$. Hence (with the aid of Theorem 1) $\sum f_n$ does not converge uniformly on $[0, 1]$.

Example We now show that the series

$$\sum \frac{x}{(nx+1)\,[(n-1)\,x+1]}$$

is uniformly convergent on any interval $[a, b], 0 < a < b$, but only point wise on $[0, b]$.

For each $n \in \mathbb{N}$, let

$$f_n(x) = \frac{x}{(nx+1)\,[(n-1)\,x+1]} = \frac{1}{(n-1)\,x+1} - \frac{1}{(nx+1)}.$$

Then the sequence of partial sums of the series $\sum f_n(x)$ is given by

$$s_N(x) = \sum_{n=1}^{N} f_n(x)$$

$$= \sum_{n=1}^{N} \left[\frac{1}{(n-1)x+1} - \frac{1}{nx+1}\right]$$

$$= 1 - \frac{1}{Nx+1}$$ when $x \neq 0$ (as other terms, occur in pairs with same absolute value but with opposite signs and hence, get cancelled)

When $x = 0$, $f_n(0) = 0$ for $n \in \mathbb{N}$

and hence $s_N(x) = 0$ for all $N \in \mathbb{N}$.

Combining the above, we have

$$s_N(x)=\begin{cases}1-\dfrac{1}{Nx+1}, \text{ if } x\neq 0\\ 0 \qquad\quad\;\;, \text{ if } x=0\end{cases}$$

If $x\neq 0$ $\qquad \lim(s_N(x))=\lim\left(1-\dfrac{1}{Nx+1}\right)=1$

and $\qquad \lim s_N(0)=0.$

Therefore (s_N) and hence $\sum f_n$ converges point wise to f where

$$f(x)=\begin{cases}1, \; x\neq 0\\ 0, \; x=0\end{cases}$$

For each $n\in\mathbb{N}$, $f_n(x)$ is continuous on $[0, b]$ but $f(x)$ is not continuous on $[0, b]$. Hence, applying Theorem 1, $\sum f_n$ does not converge uniformly on $[0, b]$.

Now let $J=[a, b]$, where $0<a<b$. Then

$$\|s_N-f\|_J=\sup\{S_N(x)-f(x): x\in[a, b]\}$$

$$=\sup\left\{\left|1-\frac{1}{Nx+1}-1\right|: x\in[a, b]\right\}, \text{ since } x\neq 0$$

$$=\sup\left\{\frac{1}{Nx+1}: x\in[a, b]\right\}$$

$$=\frac{1}{Na+1}\to 0 \text{ as } N\to\infty.$$

Hence (s_N) converges uniformly to f on $[a, b]$. Hence the associated series $\sum f_n$ also converges uniformly to f on $[a, b]$ (where $0<a<b$.

Tests for Uniform Convergence

We now present a few tests that can be used to establish uniform convergence. We first recall Cauchy Criterion for Series of real numbers and then describe Cauchy Criterion for Series of Functions.

Cauchy Criterion for Series: The series

$$\sum_{n=1}^{\infty} x_n = x_1+x_2+\cdots+x_n+\cdots$$

converges if and only if for every $£>0$ there exists an $M(£)\in\mathbb{N}$ such that if $m>n\geq M(£)$, then

$$|x_{n+1}+x_{n+2}+\cdots+x_m|<£.$$

Cauchy Criterion for Series of Functions: Let (f_n) be a sequence of functions on $D\subseteq\mathbb{R}$ to $\mathbb{R}$. The series $\sum f_n$ is uniformly convergent on D if and only if for every $£>0$ there exists an $M(£)$ such that if $m>n\geq M(£)$, then

$$|f_{n+1}(x)+\cdots+f_m(x)|<£ \text{ for all } x\in D.$$

Proof. Let (s_N) be the sequence of Nth partial sums of the series $\sum f_n$. Then

$$s_N(x) = \sum_{i=1}^{N} f_i(x) = f_1(x) + \cdots + f_N(x) \quad \text{for all } x \in D.$$

By definition of uniform convergence of series, the infinite series $\sum f_i$ converges uniformly on D if and only if (s_N) converges uniformly on D. Now, by Cauchy Criterion for Uniform Convergence of a Sequence of Functions, (s_N) converges uniformly on D if and only if for every $\pounds > 0$, there exists a natural number $M(\pounds)$ (depending only on $\pounds$) such that

$$\|s_m - s_n\|_D \le \pounds \quad \text{for all } m > n \ge M(\pounds)$$

$$|s_m(x) - s_n(x)| < \pounds \quad \text{for all } m > n \ge M(\pounds) \text{ and } x \in D$$

i.e., $$\left|\sum_{i=1}^{m} f_i(x) - \sum_{i=1}^{n} f_i(x)\right| < \pounds \quad \text{for all } m > n \ge M(\pounds) \text{ and } x \in D$$

i.e., $$\left|\sum_{i=n+1}^{m} f_i(x)\right| < \pounds \quad \text{for all } m > n \ge M(\pounds) \text{ and } x \in D$$

i.e., $$|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_m(x)| < \pounds$$

for all $m > n \ge M(\pounds)$ and $x \in D$.

Weierstrass *M*-test: Let (M_n) be a sequence of positive real numbers such that

$$|f_n(x)| \le M_n \text{ for } x \in D, n \in \mathbb{N}.$$

If the series $\sum M_n$ is convergent, then $\sum f_n$ is uniformly convergent on D.

Proof. If $m > n$, we have the relation

$$|f_{n+1}(x) + \cdots + f_m(x)| \le M_{n+1} + \cdots + M_m \text{ for } x \in D. \quad \ldots (1)$$

Now by Cauchy Criterion for Series of Real Numbers (Theorem A), the series $\sum M_n$ is convergent implies for every $\pounds > 0$ there exists $M(\pounds) \in \mathbb{N}$ such that if $m > n \ge M(\pounds)$, then

$$|M_{n+1} + \cdots + M_m| < \pounds .$$

Hence from (1), for $m > n \ge M(\pounds)$

$$|f_{n+1}(x) + \cdots + f_m(x)| < \pounds \text{ for } x \in D.$$

Hence by Cauchy Criterion for series of Functions (Theorem 3), the series $\sum f_n$ is uniformly convergent on D.

Example Test for uniform convergence the series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

To discuss the uniform convergence of the given series of functions, we consider the series $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$ neglecting 1. Then the nth term of the remaining series is

$$f_n(x) = \frac{x^n}{n!}.$$

Then $$|f_n(x)| = \left|\frac{x^n}{n!}\right| \le \frac{1}{n!} \quad \text{for } -1 \le x \le 1$$

Noting that $\frac{1}{n!} < \frac{1}{2^n}$ and the series $\sum \frac{1}{2^n}$ is convergent, we have, with the aid of comparison test of series of real numbers, the series $\sum \frac{1}{n!}$ is also convergent.

Thus by Weierstrass's *M*-test, it follows that the given series is uniformly convergent for all values of $x \in [-1, 1]$.

Example We now show that

$$\sum_{1}^{\infty} \frac{1}{n^p + n^q x^2}$$

is uniformly convergent for all values of x if $p > 1$.

Here we have

$$|f_n(x)| = \left|\frac{1}{n^p + n^q x^2}\right| < \frac{1}{n^p} \qquad \text{for all values of } x.$$

It is known that the *p*-series $\sum \frac{1}{n^p}$ is convergent if $p > 1$.

Now take $M_n = \frac{1}{n^p}$. Then by the discussion above, $\sum M_n$ is convergent if $p > 1$. We conclude from Weierstrass's *M*- test that the given series is uniformly convergent for all values of *x*, if $p > 1$.

Example We now show that the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

is uniformly convergent in every interval in $[a, b]$.

Let M be any positive number greater than each of $|a|$ and $|b|$ so that for every $x \in [a, b]$, we have

$$\left|\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right| \le \frac{M^{2n+1}}{(2n+1)!}.$$

Now consider the series

$$M + \frac{M^3}{3!} + \frac{M^5}{5!} + \cdots \qquad \text{..... (1)}$$

Since

$$e^M = 1 + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots,$$

and

$$e^{-M} = 1 - M + \frac{M^2}{2!} - \frac{M^3}{3!} + \cdots,$$

we have

$$\frac{e^M - e^{-M}}{2} = M + \frac{M^3}{3!} + \frac{M^5}{5!} + \cdots \qquad \text{..... (2)}$$

The identity (2) shows that the series (1) is convergent, whatever the positive constant M may be. Hence by Weierstrass's M-test, the given series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

is uniformly convergent in $[a,\ b]$.

Remark. The series of functions given above is the important sine series. In the previous example we have verified the uniform convergence of the sine series

$$\sin x = x - \frac{x^3}{3!} - \frac{x^5}{5!} - \cdots$$

in $[a,\ b]$.

Example We now show that the series

$$\cos x + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2} + \cdots$$

converges uniformly, and also, give the interval of uniform convergence.

Here $\qquad |u_n(x)| = \left|\frac{\cos nx}{n^2}\right| \le \frac{1}{n^2} \qquad$ for all finite values of x.

So taking $M_n = \dfrac{1}{n^2}$ and noting that $\sum M_n = \sum \dfrac{1}{n^2}$ is convergent, with the aid of Weierstrass's M-test, we have the uniform convergence of $\sum^{\infty} \dfrac{\cos nx}{n^2}$ in any finite interval. The interval of uniform convergence is $a \le x \le b$ where a and b are any finite unequal quantities.

Exercises

1. Discuss the convergence and the uniform convergence of the series $\sum f_n$, where $f_n(x)$ is given by
   a) $(x^2 + n^2)^{-1}$
   b) $(nx)^{-2}$ $(x \ne 0)$
   c) $\sin(x/n^2)$
   d) $(x^n + 1)^{-1}$ $(x \ne 0)$
   e) $x^n/(x^n + 1)$ $(x \ge 0)$
   f) $(-1)^n (n + x)^{-1}$ $(x \ge 0)$

Answers/Hints for selected Exercises

1. (a) Take $M_n = 1/n^2$ in the Weierstrass M-test.

   (c) Since $|\sin y| \le |y|$, the series converges for all x. But it is not uniformly convergent on $\mathbb{R}$. If $a > 0$, the series is uniformly convergent for $|x| \le a$.

   (d) If $0 \le x \le 1$, the series is divergent. If $1 < x < \infty$, the series is convergent. It is uniformly convergent on $[a, \infty)$ for $a > 1$. However, it is not uniformly convergent on $(1, \infty)$.

10
IMPROPER INTEGRALS OF FIRST KIND - PART I

Evaluation of the definite integrals of the type $\int_a^b f(x)\,dx$ is required in many problems. So far you may have been seen those definite integrals that have the following two properties:

1. The domain of integration, from a to b, is finite.
2. The range of the integrand is finite (in other words, the integrand is defined at every point in the domain).

Now consider the definite integrals $\int_1^\infty \frac{\ln x}{x^2}dx$ and $\int_0^1 \frac{1}{\sqrt{x}}dx$. Property 1 is violated in the first definite integral while Property 2 is not satisfied at $x=0$ by the second integral. In this chapter we consider the methods for the evaluation of such definite integrals.

An **improper integral** is a definite integral which does not satisfy Property 1 or 2 given above. i.e., integrals with infinite limits of integration and integrals of functions that become infinite at a point within the interval of integration are *improper integrals*. When the limits involved exist, we can evaluate such integrals. We discuss this in this chapter in detail.

Improper Integral of First Kind

The definition or evaluation of the integral

$$\int_a^\infty f(x)dx$$

does not follow from the discussion on Riemann integration since the interval $[a,\infty)$ is not bounded. Such an integral is called an **improper integral of first kind**. The theory of this type of integral resembles to a great extent the theory of infinite series.

We define

$$\int_a^\infty f(x)dx$$

as follows:

If $f \in R\ [a,s]$ for every $s > a$, then

$$\int_a^\infty f(x)dx$$

is defined to be the ordered pair $\langle f, F \rangle$ where

$$F(s) = \int_a^s f(x)\,dx \qquad (a \le s < \infty).$$

Integrals of the type discussed in this section are sometimes called **improper integrals of the first kind.**

Convergence of Improper Integral

We say that $\int_a^\infty f$ is **convergent** to A if $\lim_{s\to\infty} F(s) = A$. In this case we write $\int_a^\infty f = A$. If $\int_a^\infty f$ does not converge, we say that $\int_a^\infty f$ is **divergent.**

Example We now show that the integral $\int_1^\infty \frac{1}{x^2}\, dx$ is convergent.

With the usual notation, we have $F(s) = \int_1^s \frac{1}{x^2}\, dx,$

and then $F(s) = 1 - \frac{1}{s}$ and hence $\lim_{s\to\infty} F(s) = 1.$ Thus

$$\int_1^\infty \frac{1}{x^2}\, dx = 1.$$

Example We now show that the integral $\int_1^\infty \frac{1}{\sqrt{x}}\, dx$ diverges.

The integral

$$\int_1^\infty \frac{1}{\sqrt{x}}\, dx$$

diverges since

$$F(s) = \int_1^s \frac{1}{\sqrt{x}} dx = 2(\sqrt{s} - 1)$$

and $\lim_{s\to\infty} F(s)$ does not exist.

Example We now valuate $\int_1^\infty \frac{\ln x}{x^2} dx,$ if it exists.

Here $F(s) = \int_1^s \frac{\ln x}{x^2}\, dx$

Recall the integration by parts formula,

$$\int_a^b uv' = [uv]_a^b - \int_a^b u'v$$

i.e., $$\int_a^b u(x)dv(x) = [u(x)v(x)]_a^b - \int_a^b du(x)v(x)\,.$$

Taking $u(x) = \ln x,\ dv(x) = \frac{1}{x^2} dx$, we obtain

$$F(s) = \int_1^s \frac{\ln x}{x^2} dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^s - \int_1^s \left(\frac{1}{x}\right)\left(-\frac{1}{x}\right) dx\,.$$

$$= -\frac{\ln s}{s} - \left[\frac{1}{x}\right]_1^s$$

$$= -\frac{\ln s}{s} - \frac{1}{s} + 1.$$

Hence

$$\lim_{s\to\infty} F(s) = \lim_{s\to\infty}\left[-\frac{\ln s}{s} - \frac{1}{s} + 1\right] = -\left[\lim_{s\to\infty}\frac{\ln s}{s}\right] - 0 + 1$$

$$\overset{\mathcal{L}}{=} -\left[\lim_{s\to\infty}\frac{1/s}{1}\right] + 1, \text{ applying L'hospital rule.}$$

$$= 0 + 1 = 1.$$

Hence $\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{s\to\infty} F(s) = 1.$

Example We now Investigate the convergence of $\int_1^\infty \frac{dx}{x}$ and $\int_1^\infty \frac{dx}{x^2}$.

$$\int_1^\infty \frac{dx}{x} = \lim_{s\to\infty}\int_1^s \frac{dx}{x} = \lim_{s\to\infty}(\ln s - \ln 1) = \infty.$$

Hence the integral $\int_1^\infty \frac{dx}{x}$ is divergent.

Now, $\int_1^\infty \frac{dx}{x^2} = \lim_{s\to\infty}\int_1^s \frac{dx}{x^2} = \lim_{s\to\infty}\left(-\frac{1}{s} + 1\right) = 1.$

Hence the integral $\int_1^\infty \frac{dx}{x^2}$ is convergent and its value is 1.

RESULT: If

$$\int_a^\infty f \text{ and } \int_a^\infty g$$

both converge and $c \in R,$ then

(1) $\int_a^\infty (f \pm g)$, converges and

$$\int_a^\infty (f \pm g) = \int_a^\infty f \pm \int_a^\infty g.$$

(2) $\int_a^\infty cf$ converges and

$$\int_a^\infty cf = c\int_a^\infty f.$$

Tests for Convergence and Divergence

In practice, most of the improper integral cannot be evaluated directly. So we turn to the two-step procedure of first establishing the fact of convergence and then approximating the integral numerically. The principal tests for convergence are the **direct comparison** and **limit comparison tests.**

Direct Comparison Test

Let f and g be continuous on $[a,\infty)$ and suppose that $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

1. $\int_a^\infty f(x)dx$ converges if $\int_a^\infty g(x)dx$ converges
2. $\int_a^\infty g(x)dx$ diverges if $\int_a^\infty f(x)dx$ diverges.

Example We now nvestigate the convergence of $\int_1^\infty e^{-x^2}dx$

By definition,

$$\int_1^\infty e^{-x^2}dx = \lim_{b\to\infty}\int_1^b e^{-x^2}dx$$

We cannot evaluate the latter integral directly because it is non-elementary. But we can show that its limit as $b\to\infty$ is finite. We know that $\int_1^b e^{-x^2}dx$ is an increasing function of b. Therefore either it becomes infinite as $b\to\infty$ or it has a finite limit as $b\to\infty$. It does not become infinite: For every value of $x\ge 1$ we have $e^{-x^2}\le e^{-x}$, Also,

$$\int_1^\infty e^{-x}dx = \lim_{b\to\infty}\int_1^b e^{-x}dx = \lim_{b\to\infty}\left[-e^{-x}\right]_1^b = \lim_{b\to\infty}\left[-e^{-b}+e^{-1}\right] < e^{-1} \approx 0.36788.$$

Hence by Direct comparison Test,

$$\int_1^\infty e^{-x^2}dx = \lim_{b\to\infty}\int_1^b e^{-x^2}dx$$

converges to some definite finite value. We do not know exactly what the value is except that it is something less than 0.37.

Example We no investigate the convergence of $\int_1^\infty \frac{\sin^2 x}{x^2}dx$.

We know that $0\le\frac{\sin^2 x}{x^2}\le\frac{1}{x^2}$ on $[1,\infty)$ and $\int_1^\infty \frac{1}{x^2}dx$ converges. Hence by Direct comparison test, $\int_1^\infty \frac{\sin^2 x}{x^2}dx$ converges.

Example We now Investigate the convergence of $\int_1^\infty \frac{1}{\sqrt{x^2-0.1}}dx$.

$\int_1^\infty \frac{1}{\sqrt{x^2-0.1}}dx$ diverges because $\frac{1}{\sqrt{x^2-0.1}}\ge\frac{1}{x}$ on $[1,\infty)$ and $\int_1^\infty \frac{1}{x}dx$ diverges.

RESULT: The improper integral

$$\int_1^\infty \frac{1}{x}dx$$

diverges.

Proof. For any integer N we have

$$\int_1^N \frac{1}{x}dx = \sum_{n=1}^{N-1}\int_n^{n+1}\frac{1}{x}dx \ge \sum_{n=1}^{N-1}\frac{1}{n+1}\int_n^{n+1}1\cdot dx = \sum_{n=1}^{N-1}\frac{1}{n+1} = \sum_{k=2}^{N}\frac{1}{k}. \quad \ldots (4)$$

Again, since as $N\to\infty$ the right side of (4) diverges to infinity, we see that

$$\lim_{s\to\infty}\int_1^s \frac{1}{x}dx$$

does not exist. This proves the theorem.

RESULT: $\int_a^\infty \frac{1}{x^p}dx$, where p is a constant and $a>0$, converges if $p>1$ and diverges if $p\le 1$.

Proof.

$$\int_a^\infty \frac{1}{x^p}dx = \lim_{s\to\infty}\int_a^s \frac{1}{x^p}dx = \lim_{s\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_a^s, \text{ provided } p\ne 1$$

$$= \lim_{s\to\infty}\frac{1}{1-p}\left[s^{1-p}-a^{1-p}\right], \qquad \text{provided } p\ne 1$$

$$= \frac{1}{1-p}\left[\lim_{s\to\infty} s^{1-p}-a^{1-p}\right] \qquad \text{provided } p\ne 1$$

But
$$\lim_{s\to\infty} s^{1-p} = \begin{cases} 0 \text{ if } & p>1, \\ \infty \text{ if } & p<1. \end{cases}$$

Hence $\int_a^\infty \frac{1}{x^p}\,dx$ is convergent when $p>1$ and divergent when $p<1$.

Also, when $p=1$,

$$\int_a^\infty \frac{1}{x^p}dx = \lim_{s\to\infty}\int_a^s \frac{1}{x}\,dx = \lim_{s\to\infty}\left[\ln s-\ln a\right]=\infty.$$

Therefore, $\int_a^\infty \frac{1}{x^p}\,dx$ converges if $p>1$ and diverges if $p\le 1$.

Example Show that $\int_a^\infty e^{-tx}\,dx$ where t is a constant, converges if $t>0$ and diverges if $t\le 0$.

$$\int_a^\infty e^{-tx}\,dx = \lim_{s\to\infty}\int_a^s e^{-tx}\,dx = \lim_{s\to\infty}\left[\frac{e^{-tx}}{-t}\right]_n^s$$

$$= \lim_{s\to\infty}\frac{1}{t}\left[e^{-at}-e^{-st}\right] = \frac{1}{t}\left[e^{-at}-\lim_{s\to\infty}e^{-st}\right]$$

But
$$\lim_{s\to\infty} e^{-ts} = \begin{cases} 0 \text{ if } & t>0, \\ \infty \text{ if } & t<0. \end{cases}$$

Hence $\int_0^\infty e^{-tx}$ converges if $t>0$ and diverges if $t<0$.

When $t=0$, $\int_a^\infty e^{-0x}\,dx = \int_a^\infty dx = \lim_{s\to\infty}\int_a^s dx = \lim_{s\to\infty}(s-a)=\infty$.

Therefore, $\int_a^\infty e^{-tx}$ converges if $t>0$ and diverges if $t\le 0$.

Limit Comparison Test/Quotient Test:

If the positive functions f and g are continuous on $[a,\infty)$ and if

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=L \quad (0<L<\infty)$$

then $\int_a^{\infty} f(x)dx$ and $\int_a^{\infty} g(x)dx$ both converge or both diverge.

Example Using $\int_1^{\infty}\frac{dx}{x^2}$ and the Limit comparison Test, we now discuss the convergence of $\int_1^{\infty}\frac{dx}{1+x^2}$. Also compare the values of each integral.

With $f(x)=1/x^2$ and $g(x)=1/(1+x^2)$, we have

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{1/x^2}{1/(1+x^2)}$$

$$=\lim_{x\to\infty}\frac{1+x^2}{x^2}=\lim_{x\to\infty}\left(\frac{1}{x^2}+1\right)=0+1=1,$$

a positive finite limit. Therefore, $\int_1^{\infty}\frac{dx}{1+x^2}$ converges because $\int_1^{\infty}\frac{dx}{x^2}$ converges.

However, the integrals converge to different values. By an earlier Example, $\int_1^{\infty}\frac{dx}{x^2}=1$.Also,

$$\int_1^{\infty}\frac{dx}{1+x^2}=\lim_{s\to\infty}\int_1^{s}\frac{dx}{1+x^2}=\lim_{s\to\infty}\left[\tan^{-1}x\right]_1^s$$

$$=\lim_{b\to\infty}\left[\tan^{-1}b-\tan^{-1}1\right]=\frac{\text{®}}{2}-\frac{\text{®}}{4}=\frac{\text{®}}{4}.$$

Example We now show that $\int_1^{\infty}\frac{3}{e^x+5}dx$ converges.

We know that $\int_1^{\infty}\frac{1}{e^x}dx$ converges. Now we use Limit Comparison Test with $f(x)=\frac{1}{e^x}$ and $g(x)=\frac{3}{e^x+5}$:

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{1/e^x}{3/(e^x+5)}=\lim_{x\to\infty}\frac{e^x+5}{3e^x}$$

$$=\lim_{x\to\infty}\left(\frac{1}{3}+\frac{5}{3e^x}\right)=\frac{1}{3}+0=\frac{1}{3},$$

a positive finite limit. As

$$0\le\frac{3}{e^x+5}\le\frac{1}{e^x},$$

by Limit Comparison Test, we conclude that $\int_1^{\infty}\frac{3}{e^x+5}dx$ converges.

Example We now test for convergence the integral $\int_1^{\infty}\frac{dx}{x\sqrt{x^2+1}}$

Let $\quad f(x)=\frac{1}{x\sqrt{x^2+1}}$ and $g(x)=\frac{1}{x^2}$.

Then

$$\frac{f(x)}{g(x)}=\frac{x^2}{x\sqrt{x^2+1}}=\frac{1}{\sqrt{1+\frac{1}{x^2}}}\to 1\neq 0, \text{ as } x\to\infty .$$

Since $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}=1$, a finite non zero number, by Quotient Test, $\int_1^\infty f(x)dx$ and $\int_1^\infty g(x)dx$ converge or diverge together. But $\int_1^\infty g(x)\,dx=\int_1^\infty \frac{1}{x^2}\,dx$ converges. Hence $\int_1^\infty f(x)dx=\int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$ converges.

Example We now test for convergence the integral $\int_1^\infty \frac{x^2}{\sqrt{x^5+1}}\,dx$.

Let $f(x)=\frac{x^2}{\sqrt{x^5+1}}$ and $g(x)=\frac{1}{\sqrt{x}}$.

Then

$$\frac{f(x)}{g(x)}=\frac{x^{5/2}}{\sqrt{x^5+1}}=\frac{1}{\sqrt{1+1/x^5}}\to 1 \text{ as } x\to\infty .$$

Since $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}=1$, a finite non zero number, by Quotient Test $\int_1^\infty f(x)\,dx$ and $\int_1^\infty g(x)dx$ converges or diverge together. But $\int_1^\infty g(x)dx=\int_1^\infty \frac{1}{\sqrt{x}}dx=\int_1^\infty \frac{1}{x^{1/2}}dx$ is divergent as $\frac{1}{2}<1$. Hence $\int_1^\infty f(x)dx=\int_1^\infty \frac{x^2}{\sqrt{x^5+1}}$ is divergent.

Example We now test for convergence the integral $\int_0^\infty e^{-x^2}\,dx$

Since 0 is not a point of infinite discontinuity, we have to examine the convergence at ∞ only.

$$\int_0^\infty e^{-x^2}\,dx=\int_0^1 e^{-x^2}\,dx+\int_1^\infty e^{-x^2}dx . \qquad \ldots(1)$$

Since first integral on the right is a proper integral we need only test the convergence of $\int_1^\infty e^{-x^2}\,dx$.

Now , $\quad e^{x^2}>x^2$, $\quad$ for all real x

Hence $\quad e^{-x^2}<\frac{1}{x^2}$ $\quad$ for all $x\geq 1$.

Since $\int_1^\infty \frac{1}{x^2}dx$ converges, by Comparison Test, we have $\int_1^\infty e^{-x^2}\,dx$ converges. Hence using (1), being the sum of two convergent integrals, $\int_0^\infty e^{-x^2}dx$ converges.

Example We now test for convergence the integral $\int_1^\infty \frac{\log x}{x+2}\,dx$.

Let $f(x)=\dfrac{\log x}{x+2}$ and $g(x)=\dfrac{1}{x}$.

Then $\dfrac{f(x)}{g(x)}=\dfrac{x\log x}{x+2}=\dfrac{\log x}{1+2/x}\to\infty$ as $x\to\infty$.

Here $\int_1^\infty g(x)\,dx=\int_1^\infty \frac{1}{x}dx$ diverges. Hence by Quotient test, $\int_1^\infty f(x)\,dx=\int_1^\infty \frac{\log x}{x+2}dx$ diverges.

Definition If $f\in R\,[a,s]$ for every $s>a$ and if $\int_a^\infty |f(x)|\,dx$ converges, we say that $\int_a^\infty f(x)dx$ **converges absolutely**.

RESULT: If

$$F(s)=\int_a^s |f(x)|\,dx$$

and if F is bounded (above) on $[a,\infty)$, then $\lim_{s\to\infty}F(s)$ exists and hence $\int_a^\infty f(x)\,dx$ converges absolutely.

RESULT: If $\int_a^\infty f(x)\,dx$ converges absolutely, if $g\in\mathcal{R}[a,s]$ for every $s>a$, and if

$$|g(x)|\le|f(x)|\quad(a\le x<\infty),$$

then $\int_a^\infty g(x)dx$ converges absolutely.

Proof.

$$G(s)=\int_a^s |g(x)|\,dx\le\int_a^s |f(x)|\,dx\le\int_a^\infty |f(x)|\,dx.$$

Hence G is bounded above on $[a,\infty)$ and the absolute convergence of $\int_a^\infty g(x)dx$ follows from the preceding theorem.

RESULT: If $\int_a^\infty |f(x)|dx$ converges absolutely, then $\int_a^\infty f(x)dx$ converges

Proof.

$|f(x)|$ is either $f(x)$ or $-f(x)$. In either case $f(x)+|f(x)|\ge 0$ and $2|f(x)|\ge f(x)+|f(x)|$. Hence

$$0\le f(x)+|f(x)|\le 2|f(x)|\quad(a\le x<\infty),\qquad\ldots(1)$$

and since (by assumption) $\int_a^\infty 2|f(x)|dx$ converges.

$\int_a^\infty |f(x)|dx$ converges implies $\lim_{s\to\infty}F(s)$ exists where $F(s)=\int_a^s |f(x)|dx$. Being a 'convergent partial integral' F is a bounded (function). Hence from inequalities in (1),

$$\int_a^s \big|f(x)+|f(x)|\big|dx=\int_a^s\big(f(x)+|f(x)|\big)dx \text{ is bounded.}$$

Hence by the theorem above, $\lim_{s\to\infty}\int_a^s \left| f(x)+\left|f(x)\right|\right| dx$ exists and hence $\int_a^\infty f(x)+\left|f(x)\right| dx$ converges absolutely.

It follows that

$$\int_a^\infty \left(f(x)+|f(x)|\right)dx$$

converges absolutely. Since $f(x)+|f(x)|\geq 0$ this means simply that

$$\int_a^\infty \left(f(x)+|f(x)|\right)dx$$

converges. But, since $\int_a^\infty |f(x)|dx$ converges, it follows (by subtraction) that

$$\int_a^\infty \left(f(x)+|f(x)|\right)dx-\int_a^\infty |f(x)|\,dx=\int_a^\infty f(x)\,dx$$

itself converges. This completes the proof.

Example Prove that $\int_1^\infty \frac{\cos x}{x^2+1}\,dx$ converges.

We have $\left|\frac{\cos x}{x^2+1}\right|=\frac{|\cos x|}{x^2+1}\leq\frac{1}{x^2}$ for all $x\geq 1$. since $|\cos x|\leq 1$

Then, since $\int_1^\infty \frac{dx}{x^2}$ converges, by comparison test, it follows that $\int_1^\infty\left|\frac{\cos x}{x^2+1}\right|dx$ converges, i.e., $\int_1^\infty \frac{\cos x}{x^2+1}\,dx$ converges absolutely. Since, every absolutely convergent integral converges, $\int_1^\infty \frac{\cos x}{x^2+1}\,dx$ converges.

If $\int_a^\infty f(x)\,dx$ converges but does not converge absolutely, we say that $\int_a^\infty f(x)\,dx$ **converges conditionally.**

Example Show that $\int_\pi^\infty \frac{\sin x}{x}dx$ is a conditionally convergent improper integral. Also find the value of $\int_\pi^\infty \frac{\sin x}{x}dx$.

To show that $\int_\pi^\infty \frac{\sin x}{x}dx$ converges, we have for any $s>\pi$ (using integration by parts)

$$\int_\pi^s \frac{\sin x}{x}dx=\frac{1}{\pi}-\frac{\cos s}{s}+\int_\pi^s \frac{\cos x}{x^2}dx. \qquad \ldots (2)$$

Now $$\frac{|\cos x|}{x^2}\leq\frac{1}{x^2} \qquad (\pi\leq x<\infty).$$

Since $\int_\pi^\infty \frac{1}{x^2}dx$ converges (absolutely) it follows that $\int_\pi^s \frac{\cos x}{x^2}dx$ converges absolutely and hence converges. Thus as $s\to\infty$ all terms on the right of (2) approach limits. Thus

$$\lim_{s\to\infty}\int_{\pi}^{s}\frac{\sin x}{x}dx$$

exists, which proves that $\int_{\pi}^{\infty}\frac{\sin x}{x}dx$ converges.

Letting $s\to\infty$, (2) gives

$$\int_{\pi}^{\infty}\frac{\sin x}{x}dx=\int_{\pi}^{\infty}\frac{\cos x}{x^2}dx+\frac{1}{\pi}.$$

Note that the second integral is absolutely convergent while the first integral (as we shall now show) is not.

Now we show that $\int_{\pi}^{\infty}\frac{\sin x}{x}dx$ does not converge absolutely. For any $N\in\mathbb{Z}^+$, we have

$$\int_{\pi}^{N\pi}\frac{|\sin x|}{x}dx=\sum_{n=1}^{N-1}\int_{n\pi}^{(n+1)\pi}\frac{|\sin x|}{x}dx\geq\sum_{n=1}^{N-1}\frac{1}{(n+1)\pi}\int_{n\pi}^{(n+1)\pi}|\sin x|dx$$

$$=\frac{1}{\pi}\sum_{n=1}^{N-1}\frac{1}{n+1}\int_{0}^{\pi}|\sin(u+n\pi)|du.$$

Now

$$\sin(u+n\pi)=\sin u\cos n\pi+\cos u\sin n\pi=\sin u\cos n\pi.$$

Since $\cos n\pi=\pm1$ this shows that $|\sin(u+n\pi)|=|\sin u|$. Hence if $0\leq u\leq\pi$, then $|\sin(u+n\pi)|=\sin u$. Thus

$$\int_{\pi}^{N\pi}\frac{|\sin x|}{x}dx\geq\frac{1}{\pi}\sum_{n=1}^{N-1}\frac{1}{n+1}\int_{0}^{\pi}\sin u\,du=\frac{2}{\pi}\sum_{n=1}^{N-1}\frac{1}{n+1}=\frac{2}{\pi}\sum_{k=2}^{N}\frac{1}{k}. \qquad \dots (3)$$

Since the series $\sum_{k=2}^{\infty}\frac{1}{k}$ is divergent, the right side of (3) can be made as large as we please by taking N sufficiently large. This and (3) show that

$$\lim_{s\to\infty}\int_{\pi}^{s}\frac{|\sin x|}{x}dx$$

cannot exist. Hence $\int_{\pi}^{\infty}\frac{\sin x}{x}dx$ does not converge absolutely.

Exercises

In Exercises 1- , discuss the convergence of the following improper integral of the first kind.

1. $\int_{2}^{\infty}\frac{x\,dx}{\sqrt{x^3-1}}$

2. $\int_{1}^{\infty}\frac{1}{x^{\frac{2}{3}}}dx$

3. $\int_{0}^{\infty}\frac{x^2+1}{x^4+1}dx$

4. $\int_{1}^{\infty}\frac{1}{x^{\frac{4}{3}}}dx$

5. $\int_{0}^{\infty}\frac{\sin^2 x}{x^2}dx$

6. $\int_{1}^{\infty}\frac{x}{1+x^2}dx.$

7. $\int_{1}^{\infty}\frac{\log x}{x+e^{-x}}dx$

8. $\int_{1}^{\infty}\frac{43x^2}{1+2x^2+12x^4}dx.$

9. $\int_{1}^{\infty}\frac{x\,dx}{3x^4+5x^2+1}$

10. $\int_{2}^{\infty}\frac{x}{1+x^2}dx.$

11. $\int_1^\infty x\cos x\,dx.$

12. $\int_1^\infty \frac{43x^2\,dx}{1+2x^2+12x^4}dx$

13. $\int_2^\infty \frac{x\,dx}{(\log x)^3}$

14. $\int_1^\infty \frac{1}{(1+x^3)^{\frac{1}{2}}}dx.$

15. $\int_0^\infty \frac{x^{n-1}}{1+x}dx$

16. $\int_1^\infty \frac{1}{(1+x^3)^{\frac{1}{3}}}dx.$

17. $\int_1^\infty \frac{\log x}{x+a}dx$

18. $\int_0^\infty \frac{1-\cos x}{x^2}dx$

19. $\int_2^\infty \frac{x^2-1}{\sqrt{x^6+16}}dx$

20. $\int_0^\infty \frac{x^2+1}{x^4+1}dx$

21. $\int_{-\infty}^\infty \frac{dx}{x^2+4}$

22. $\int_1^\infty \frac{dx}{x\sqrt{3x+2}}$

23. $\int_{-\infty}^\infty \frac{x^2\,dx}{\left(x^2+x+1\right)^{5/2}}$

24. $\int_{-\infty}^\infty \frac{2+\sin x}{x^2+1}dx$

25. Show that $\int_0^\infty \frac{x}{(1+x)^3}dx = \frac{1}{2}\int_0^\infty \frac{1}{(1+x)^2}dx.$

26. Show that $\int_1^\infty \frac{\sin x}{x^2}dx$ is convergent and deduce that $\int_1^\infty \frac{\cos x}{x}dx$ is convergent

27. Show that $$\int_1^\infty \frac{x^{\frac{1}{2}}}{(1+x)^2}dx = \frac{1}{2}+\frac{\pi}{4}.$$

28. Show that the integral $\int_0^\infty \frac{\sin x}{x^p}dx$ where $p>1$ is absolutely convergent.

29. True or false? If f is continuous on $[1,\infty)$ and if $\int_1^\infty f(x)\,dx$ converges, then $\lim_{x\to\infty} f(x)=0.$

30. Show that $\int_0^\infty \frac{x\sin x}{1+x^2}dx$ is convergent.

31. If $\int_1^\infty f(x)\,dx$ converges and if $\lim_{x\to\infty} f(x)=L$, prove that L=0.

32. Give an example of a continuous function f such that

$$f(x)\ge 0 \qquad (1\le x<\infty),$$

and such that $\sum_{n=1}^\infty f(n)$ converges

but $\int_1^\infty f(x)\,dx$ diverges.

33. Give an example of a continuous function f such that

$$f(x)\ge 0 \quad (1\le x<\infty)$$

and such that $\int_1^\infty f(x)\,dx$ converges but $\sum_{n=1}^{\infty} f(n)$ diverges.

34. Show that $\int_0^{\infty} \frac{\sin x}{x^p} dx,\ 0 \le p \le 1$ is convergent but not absolutely convergent.

35. Discuss absolute convergence of $\int_1^{\infty} \frac{\sin bx}{x^2} dx$.

36. If $f(x) \ge 0\ (1 \le x < \infty)$, if f is nonincreasing on $[1, \infty)$, and if

$$\int_1^{\infty} f(x)\,dx \text{ converges,}$$

then show that $\lim_{x \to \infty} xf(x) = 0.$

37. Show that $\int_0^{\infty} \frac{\cos x}{\sqrt{1+x^2}} dx$ is conditionally convergent.

38. Let f be a continuous function on $[a, \infty)$, such that, if

$$F(x) = \int_a^x f(t)\,dt \quad (a \le x < \infty),$$

then f is bounded on (a, ∞). Let g be a function on $[a, \infty)$ such that g' is continuous on $[a, \infty)$, $g'(t) \le 0$ for $a \le t < \infty$, and such that $\lim_{t \to \infty} g(t) = 0$. Prove that

$$\int_a^{\infty} f(t)g(t)\,dt \text{ converges.}$$

39. Use the preceding exercise to show that

$$\int_3^{\infty} \frac{\sin t}{\log t} dt \text{ converges.}$$

40. Show that $\int_1^{\infty} \cos u^2\, du$ is convergent.

41. Evaluate $\int_1^{\infty} \frac{dx}{x^3}$

42. Evaluate $\int_1^{\infty} \frac{dx}{x^{3/2}}$

43. Evaluate $\int_1^{\infty} \frac{dx}{\sqrt{x}}$

44. Evaluate $\int_{-\infty}^{0} e^x\, dx$.

45. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{x^2 + x + 2}$

46. Show that $\int_0^{\infty} e^{-x}\, dx$ is convergent.

11

IMPROPER INTEGRALS OF FIRST KIND - PART II

Integral Test for Series

We have just used the divergence of an infinite series to establish the divergence of an improper integral. It is more usual to use an integral in the investigation of a series. This is known as the **Integral Test for Series.**

RESULT Let f be a nonincreasing function on $[1, \infty)$ such that $f(x) \geq 0 \ (1 \leq x < \infty)$. Then $\sum_{n=1}^{\infty} f(n)$ will converge if $\int_1^{\infty} f(x)\,dx$ converges, and $\sum_{n=1}^{\infty} f(n)$ will diverge if $\int_1^{\infty} f(x)\,dx$ diverges.

Proof. For any $n \in \mathbb{Z}^+$ we have

$$f(n) \geq f(x) \geq f(n+1) \quad (n \leq x \leq n+1)$$

since f is nonincreasing. Integrating from n to $n+1$ we then have

$$\int_n^{n+1} f(n)\,dx \geq \int_n^{n+1} f(x)\,dx \geq \int_n^{n+1} f(n+1)\,dx$$

or
$$f(n)\int_n^{n+1} dx \geq \int_n^{n+1} f(x)\,dx \geq f(n+1)\int_n^{n+1} dx$$

or
$$f(n) \geq \int_n^{n+1} f(x)\,dx \geq f(n+1).$$

Thus, for $N \in \mathbb{Z}^+$ we have

$$\sum_{n=1}^{N-1} f(n) \geq \int_1^N f(x)\,dx \geq \sum_{n=1}^{N-1} f(n+1) = \sum_{k=2}^{N} f(k) \qquad \ldots (5)$$

If $\int_1^{\infty} f(x)\,dx$ converges to A, then, by (5),

$$\sum_{k=2}^{N} f(k) \leq \int_1^N f(x)dx \leq A.$$

The partial sums of $\sum_{k=2}^{\infty} f(k)$ are thus bounded above. i.e., the sequence of Nth partial sums of the series is bounded and hence the sequence of the Nth partial sum is convergent, and hence the series $\sum_{k=2}^{\infty} f(k)$ converges and hence $\sum_{k=1}^{\infty} f(k)$ converges.

If $\int_1^{\infty} f(x)\,dx$ diverges, the divergence of $\sum_{n=1}^{\infty} f(n)$ may be established in similar fashion, using the left-hand inequality in (5). This completes the proof.

Example Using integral test for series establish the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

If $f(x) = 1/x^2 \ (1 \leq x < \infty)$, then

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} f(n).$$

Since
$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x^2}\,dx$$

is convergent, it follows that $\sum_{n=1}^{\infty} f(n)$ converges.

Example Show that $\sum_{n=1}^{\infty}[1/(n\log n)]$ diverges.

If $g(x)=1/(x\log x)$, then g is nonnegative and non-decreasing on $[3,\infty)$. Since $G'(x)=g(x)$ where $G(x)=\log\log x$, then

$$\int_3^s g(x)\,dx=\log\log s-\log\log 3,$$

and hence $\int_3^\infty g(x)$ diverges. It follows from Integral Test for Series that $\sum_{n=1}^{\infty}[1/(n\log n)]$ diverges.

Example Using the Integral Test, show that the ***p*-series**

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{p}+\frac{1}{2^p}+\frac{1}{3^P}+\cdots+\frac{1}{n^p}+\cdots \qquad \ldots(1)$$

(p a real constant) converges if $p>1$, and diverges if $p\le 1$.

***Case* 1** If $p>1$, then $f(x)=\frac{1}{x^p}$ is a positive decreasing function of x for $x>1$. Now,

$$\int_1^\infty \frac{1}{x^p}dx=\int_1^\infty x^{-p}dx=\lim_{s\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^s$$

$$=\frac{1}{1-p}\lim_{s\to\infty}\left(\frac{1}{s^{p-1}}-1\right)=\frac{1}{1-p}(0-1) \text{ , since } s^{p-1}\to\infty \text{ as } s\to\infty \text{ for } p-1>0.$$

$$=\frac{1}{p-1}.$$

Hence $\int_1^\infty \frac{1}{x^p}dx$ converges and hence, by the Integral Test, the given series converges.

***Case* 2** If $p<1$, then $1-p>0$ and

$$\int_1^\infty \frac{1}{x^p}dx=\frac{1}{1-p}\lim_{s\to\infty}(s^{1-p}-1)=\infty, \text{ as } \lim_{s\to\infty}s^{1-p}=\infty \text{ for } 1-p>0.$$

Hence, by the Integral Test, the series diverges for $p<1$.

If $p=1$, we have the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots,$$

which is known to be divergent,

Hence we conclude that the series converges for $p>1$ but diverges for $p\le 1$.

Integrals of the form $\int_{-\infty}^{a} f(x)\,dx$

An integral of the form $\int_{-\infty}^{a} f(x)\,dx$ may be treated by the same methods as those used on integrals of the form $\int_b^\infty f(x)\,dx$. Thus, we say that $\int_{-\infty}^{a} f(x)\,dx$ converges to A if

$$\lim_{s\to\infty}\int_{-s}^{a} f(x)\,dx=A.$$

The change of variable $x = -u$ will change a $\int_{-\infty}^{a}$ problem into a $\int_{b}^{\infty}$ problem.

Example Examine the convergence or divergence of $\int_{-\infty}^{-2} \frac{1}{1-x} dx.$

For any $s > 2$ we have

$$\int_{-s}^{-2} \frac{1}{1-x} dx = \int_{s}^{2} \frac{1}{1+u}(-1)\,du = \int_{2}^{s} \frac{1}{1+u} du.$$

Since $\frac{1}{1+u} \geq \frac{1}{2}\cdot\frac{1}{u}$ for $2 \leq u < \infty,$ it follows, from the fact $\int_{1}^{\infty} \frac{1}{x} dx$ is divergent, that

$$\lim_{s\to\infty} \int_{2}^{s} \frac{1}{1+u} du$$

does not exist. Hence $\lim_{s\to\infty} \int_{-s}^{-2} \frac{1}{1-x} dx$ does not exist, which proves that $\int_{-\infty}^{-2} \frac{1}{1-x} dx$ does not converge. (In this problem the divergence of

$$\int_{-\infty}^{-2} \frac{1}{1-x} dx$$

was deduced from the divergence of $\int_{2}^{\infty} \frac{1}{1+u} du.$)

Example Test for convergence $\int_{-\infty}^{\infty} \frac{x^3 + x^2}{x^6 + 1} dx$

We write

$$\int_{-\infty}^{\infty} \frac{x^3 + x^2}{x^6 + 1} dx = \int_{-\infty}^{0} \frac{x^3 + x^2}{x^6 + 1} dx + \int_{0}^{\infty} \frac{x^3 + x^2}{x^6 + 1} dx \qquad \ldots (1)$$

Now

$$\int_{-\infty}^{0} \frac{x^3 + x^2}{x^6 + 1} dx = \int_{\infty}^{0} \frac{(-y)^3 + (-y)^2}{(-y)^6 + 1}(-dy) \qquad [\text{Putting } x = -y\,]$$

$$= -\int_{0}^{\infty} \frac{y^3 - y^2}{y^6 + 1} dy$$

$$= -\int_{0}^{1} \frac{y^3 - y^2}{y^6 + 1} dy - \int_{1}^{\infty} \frac{y^3 - y^2}{y^6 + 1} dy$$

Taking $f(y) = \frac{y^3 - y^2}{y^6 + 1}$ and $g(y) = \frac{1}{y^3}$, we have

$$\lim_{x\to\infty} \frac{f(y)}{g(y)} = \lim_{x\to\infty} \frac{y^3(y^3 - y^2)}{y^6 + 1} = \lim_{x\to\infty} \frac{1 - 1/y}{1 + 1/y^6} = 1,$$

a non zero finite number. Hence, by Quotient Test, $\int_{1}^{\infty} \frac{y^3 - y^2}{y^6 + 1} dy$ and $\int_{1}^{\infty} \frac{1}{y^3} dy$ converge or diverge together. Since $\int_{1}^{\infty} \frac{1}{y^3} dy$ converges, $\int_{1}^{\infty} \frac{y^3 - y^2}{y^6 + 1} dy$ converges.

Since $\int_{0}^{1} \frac{y^3 - y^2}{y^6 + 1} dy$ is a proper integral, it follows that $\int_{-\infty}^{0} \frac{x^3 + x^2}{x^6 + 1} dx$ converges.

Similarly, we can show that $\int_{-\infty}^{0} \frac{x^3+x^2}{x^6+1}\,dx$ converges.

Hence $\int_{-\infty}^{\infty} \frac{x^3+x^2}{x^6+1}$, (being a sum of two convergent improper integrals) is convergent.

Example Test the convergence of $\int_{-\infty}^{0} \frac{dx}{(1-3x)^2}$.

$$\int_{-\infty}^{0} \frac{dx}{(1-3x)^2} = \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{(1-3x)^2}\,dx$$

$$= \lim_{a\to-\infty} \left[\frac{1}{3(1-3x)}\right]_a^0 = \lim_{a\to-\infty} \left[\frac{1}{3} - \frac{1}{3(1-3a)}\right]$$

$$= \frac{1}{3} - \frac{1}{3} \lim_{a\to-\infty} \frac{1}{1-3a} = \frac{1}{3} - \frac{1}{3}\times 0 = \frac{1}{3}.$$

Hence, the given integral is convergent and has value $\frac{1}{3}$.

Example Test the convergence of $\int_{-\infty}^{0} \cosh x\, dx$.

$$\int_{-\infty}^{0} \cosh x\,\, dx = \lim_{a\to-\infty} \int_{a}^{b} \cosh x\, dx = \lim_{a\to-\infty} \left[\sinh x\right]_a^0$$

$$= \lim_{a\to-\infty} \left[\sinh 0 - \sinh a\right] = \lim_{a\to-\infty} \left[0 - \frac{e^a - e^{-a}}{2}\right] = \infty.$$

Hence, the given integral is divergent.

Example Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{1+x^2}\,dx + \lim_{b\to\infty} \int_{0}^{b} \frac{1}{1+x^2}\,dx$$

$$= \lim_{a\to-\infty} \left[\tan^{-1} x\right]_a^0 + \lim_{b\to\infty} \left[\tan^{-1} x\right]_0^b$$

$$= \lim_{a\to-\infty} \left(\tan^{-1} 0 - \tan^{-1} a\right) + \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} 0\right)$$

$$= \tan^{-1} 0 - \left(-\frac{®}{2}\right) + \frac{®}{2} - \tan^{-1} 0, \text{ since}$$

$$\lim_{a\to-\infty} \tan^{-1} a = -\frac{®}{2} \text{ and } \lim_{b\to\infty} \tan^{-1} b = -\frac{®}{2}$$

$$= \frac{®}{2} + \frac{®}{2} = ®$$

Hence, the given integral is convergent and has value ®.

Example The cross sections of the solid horn in Fig.1 perpendicular to the x-axis are circular disks with diameters reaching from the x-axis to the curve $y = e^x$, $-\infty < x \le \ln 2$. Find the volume of the horn.

Solution

For a typical cross section, radius is $\frac{1}{2}y$ and area is

$$A(x) = \circledR\,(\text{radius})^2 = \circledR\left(\frac{1}{2}y\right)^2 = \frac{\circledR}{4}e^{2x}.$$

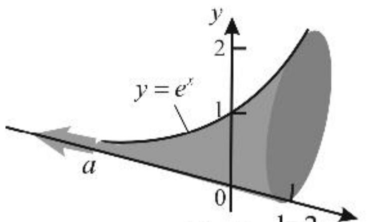


Fig.1

Now the volume of the horn in the Fig.1 is $\int_{-\infty}^{\ln 2} A(x)\,dx$. i.e., the volume of the horn is the limit as $a \to -\infty$ of the volume V_a of the portion from *a* to ln 2. By the method of slicing, the volume V_a of this portion is

$$V_a = \int_a^{\ln 2} A(x)dx = \int_a^{\ln 2} \frac{\circledR}{4}e^{2x}dx = \left[\frac{\circledR}{8}e^{2x}\right]_b^{\ln 2}$$

$$= \frac{\circledR}{8}(e^{\ln 4} - e^{2a}) = \frac{\circledR}{8}(4 - e^{2a}).$$

As $a \to -\infty,\ e^{2a} \to 0$ and $V_a \to (\circledR/8)(4-0) = \circledR/2$. Hence

$$\int_{-\infty}^{\ln 2} A(x)\,dx = \lim_{a\to-\infty} V_a = \frac{\pi}{2}.$$

i.e., the volume of the given horn is $\circledR/2$.

Example Test for convergence $\int_{-\infty}^{-1} \frac{e^x}{x}dx$.

Let $x = -y$. Then,

$$\int_{-\infty}^{-1} \frac{e^x}{x}dx = \int_{\infty}^{1} \frac{e^{-y}}{-y}(-dy) = -\int_1^{\infty} \frac{e^{-y}}{y}dy.$$

Now, $\frac{e^{-y}}{y} \le e^{-y}$ for all $y \ge 1$ and, by Example above with $a = 1$ and $t = 1$, we have $\int_1^{\infty} e^{-y}\,dy$ is convergent. Hence by comparison test, $\int_1^{\infty} \frac{e^{-y}}{y}\,dy$ converges. Therefore

$$\int_{-\infty}^{-1} \frac{e^x}{x}dx = -\int_1^{\infty} \frac{e^{-y}}{y}dy$$

also converges.

Exercises

In Exercises 1- , discuss the convergence of the following improper integral of the first kind.

1. $\int_2^\infty \frac{x\,dx}{\sqrt{x^3-1}}$

2. $\int_1^\infty \frac{1}{x^{\frac{2}{3}}}dx$

3. $\int_0^\infty \frac{x^2+1}{x^4+1}dx$

4. $\int_1^\infty \frac{1}{x^{\frac{4}{3}}}dx$

5. $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$

6. $\int_1^\infty \frac{x}{1+x^2}dx.$

7. $\int_1^\infty \frac{\log x}{x+e^{-x}}dx$

8. $\int_1^\infty \frac{43x^2}{1+2x^2+12x^4}dx.$

9. $\int_1^\infty \frac{x\,dx}{3x^4+5x^2+1}$

10. $\int_2^\infty \frac{x}{1+x^2}dx.$

11. $\int_1^\infty x\cos x\,dx.$

12. $\int_1^\infty \frac{43x^2\,dx}{1+2x^2+12x^4}dx$

13. $\int_2^\infty \frac{x\,dx}{(\log x)^3}$`

14. $\int_1^\infty \frac{1}{(1+x^3)^{\frac{1}{2}}}dx.$

15. $\int_0^\infty \frac{x^{n-1}}{1+x}dx$

16. $\int_1^\infty \frac{1}{(1+x^3)^{\frac{1}{3}}}dx.$

17. $\int_1^\infty \frac{\log x}{x+a}dx$

18. $\int_0^\infty \frac{1-\cos x}{x^2}dx$

19. $\int_2^\infty \frac{x^2-1}{\sqrt{x^6+16}}dx$

20. $\int_0^\infty \frac{x^2+1}{x^4+1}dx$

21. $\int_{-\infty}^\infty \frac{dx}{x^2+4}$

22. $\int_1^\infty \frac{dx}{x\sqrt{3x+2}}$

23. $\int_{-\infty}^\infty \frac{x^2\,dx}{\left(x^2+x+1\right)^{5/2}}$

24. $\int_{-\infty}^\infty \frac{2+\sin x}{x^2+1}dx$

25. Show that $\int_0^\infty \frac{x}{(1+x)^3}dx = \frac{1}{2}\int_0^\infty \frac{1}{(1+x)^2}dx.$

26. Show that $\int_1^\infty \frac{\sin x}{x^2}dx$ is convergent and deduce that $\int_1^\infty \frac{\cos x}{x}dx$ is convergent

27. Show that $\int_1^\infty \frac{x^{\frac{1}{2}}}{(1+x)^2}dx = \frac{1}{2}+\frac{\pi}{4}.$

28. Show that the integral $\int_0^\infty \frac{\sin x}{x^p}dx$ where $p>1$ is absolutely convergent.

29. True or false? If f is continuous on $[1,\infty)$ and if $\int_1^\infty f(x)\,dx$ converges, then $\lim_{x\to\infty} f(x)=0.$

30. Show that $\int_0^\infty \frac{x\sin x}{1+x^2}dx$ is convergent.

31. If $\int_1^\infty f(x)\,dx$ converges and if $\lim_{x\to\infty} f(x)=L,$ prove that L=0.

32. Give an example of a continuous function f such that

$$f(x) \geq 0 \qquad (1 \leq x < \infty),$$

and such that $\sum_{n=1}^{\infty} f(n)$ converges

but $\int_1^{\infty} f(x)\,dx$ diverges.

33. Give an example of a continuous function *f* such that

$$f(x) \geq 0 \quad (1 \leq x < \infty)$$

and such that $\int_1^{\infty} f(x)\,dx$ converges

but $\sum_{n=1}^{\infty} f(n)$ diverges.

34. Show that $\int_0^{\infty} \frac{\sin x}{x^p} dx,\ 0 \leq p \leq 1$ is convergent but not absolutely convergent.

35. Discuss absolute convergence of $\int_1^{\infty} \frac{\sin bx}{x^2} dx$.

36. If $f(x) \geq 0\ (1 \leq x < \infty)$, if f is nonincreasing on $[1,\infty)$, and if

$$\int_1^{\infty} f(x)\,dx \text{ converges,}$$

then show that $\lim_{x \to \infty} xf(x) = 0$.

37. Show that $\int_0^{\infty} \frac{\cos x}{\sqrt{1+x^2}} dx$ is conditionally convergent.

38. Let *f* be a continuous function on $[a,\infty)$, such that, if

$$F(x) = \int_a^x f(t)\,dt \quad (a \leq x < \infty),$$

then *f* is bounded on (a,∞). Let g be a function on $[a,\infty)$ such that g' is continuous on $[a,\infty)$, $g'(t) \leq 0$ for $a \leq t < \infty$, and such that $\lim_{t \to \infty} g(t) = 0$. Prove that

$$\int_a^{\infty} f(t)g(t)\,dt \text{ converges.}$$

39. Use the preceding exercise to show that

$$\int_3^{\infty} \frac{\sin t}{\log t} dt \text{ converges.}$$

40. Show that $\int_1^{\infty} \cos u^2\,du$ is convergent.

41. Evaluate $\int_1^{\infty} \frac{dx}{x^3}$

42. Evaluate $\int_1^{\infty} \frac{dx}{x^{3/2}}$

43. Evaluate $\int_1^{\infty} \frac{dx}{\sqrt{x}}$

44. Evaluate $\int_{-\infty}^{0} e^x\,dx$.

45. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{x^2+x+2}$

46. Show that $\int_0^{\infty} e^{-x}\,dx$ is convergent.

12

IMPROPER INTEGRALS OF SECOND AND THIRD KINDS – PART I

Improper Integrals of Second Kind

The definition of

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

does not follow from the discussion on Riemann integration because the function *f* defined by

$$f(x)=\frac{1}{\sqrt{x}} \qquad (0<x\le 1)$$

is not bounded. Note, however, that *f* is bounded (and continuous) on $[\varepsilon,1]$ for every $\varepsilon>0$. This suggests treating

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx \text{ as the } \lim_{\varepsilon\to 0+}\int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx$$

which equals to $\qquad \lim_{£\to 0^+}\left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_£^1 = 2$.

Definition If $f\in\mathcal{R}\,[a+\varepsilon,b]$ for all ε such that $0<\varepsilon<b-a$, but $f\notin\mathcal{R}[a,b]$, we define $\int_a^b f(x)\,dx$ as the ordered pair $\langle f,F\rangle$ where

$$F(\varepsilon)=\int_{a+\varepsilon}^b f(x)\,dx \quad (0<\varepsilon<b-a).$$

We say that $\int_a^b f$ **converges** to A if $\lim_{£\to 0+} F(£)=A$. We say that $\int_a^b f$ **diverges** if $\int_a^b f$ does not converge. The integral $\int_a^b f$ is called an **improper integral of the second kind.**

By the discussion just above the Definition,

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

converges.

Example Examine the convergence of $\int_0^1 \frac{dx}{x^2}$.

The integrand $\frac{1}{x^2}$ is unbounded at $x=0$. Hence the given is an improper integral of second kind.

Hence $\int_0^1 \frac{1}{x^2}\,dx=\lim_{£\to 0+}\int_£^1 \frac{1}{x^2}\,dx=\lim_{£\to 0+}\left[-\frac{1}{x}\right]_£^1=\lim_{£\to 0}\left(\frac{1}{£}-1\right)=\infty$.

Hence $\int_0^1 \frac{dx}{x^2}$ is divergent.

Example Find the value of $\int_{-1}^7 \frac{dx}{\sqrt[3]{x+1}}$, if it converges.

The integrand is unbounded at $x=-1$. Hence

$$\int_{-1}^{7}\frac{1}{\sqrt[3]{x+1}}dx=\lim_{£\to 0+}\int_{-1+£}^{7}(x+1)^{-1/3}\,dx=\lim_{£\to 0+}\left[\frac{(x+1)^{2/3}}{2/3}\right]_{-1+£}^{7}$$

$$=\lim_{£\to 0+}\left[\frac{3}{2}\cdot 8^{2/3}-\frac{3}{2}\cdot £^{2/3}\right]=\lim_{£\to 0+}\left[6-\frac{3}{2}£^{2/3}\right]=6\,.$$

Hence $\int_{-1}^{7}\frac{dx}{\sqrt[3]{x+1}}$ converges and has value 6.

Example Show that $\int_a^b\frac{dx}{(x-a)^p}$ converges if $p<1$ and diverges if $p\ge 1$.

The given integral is a proper integral if $p\le 0$ and hence converges. So let $p>0$. Then the integrand $\frac{1}{(x-a)^p}$ is unbounded at $x=a$.

Hence, $\int_a^b\frac{1}{(x-a)^p}dx=\lim_{£\to 0+}\int_{a+£}^{b}\frac{1}{(x-a)^p}dx$

$$=\lim_{£\to 0+}\left[\frac{(x-a)^{-p+1}}{-p+1}\right]_{a+£}^{b},\ \text{if}\ p\neq 1$$

$$=\lim_{£\to 0+}\frac{1}{1-p}\left[(b-a)^{-p+1}-£^{-p+1}\right],\text{if}\ p\neq 1$$

$$=\frac{1}{1-p}(b-a)^{-p+1}-\frac{1}{1-p}\lim_{£\to 0+}£^{-p+1}$$

But $\lim_{£\to 0+}£^{-p+1}=0,$ if $p<1$

$=\infty,$ if $p>1$.

Hence, $\int_a^b\frac{1}{(x-a)^p}\,dx$ is convergent if $p<1$ and divergent if $p>1$.

When $p=1$,

$\int_a^b\frac{1}{(x-a)^p}\,dx=\int_a^b\frac{1}{x-a}dx$, improper integral of second kind

$$=\lim_{£\to 0+}\int_{a+£}^{b}\frac{1}{x-a}\,dx$$

$$=\lim_{£\to 0+}\left[\ln(x-a)\right]_{a+£}^{b}$$

$$=\lim_{£\to 0+}\left[\ln(b-a)-\ln £\right]$$

$$=\ln(b-a)-\lim_{£\to 0+}\ln £\,.$$

Since the limit on the right does not exist $\int_a^b\frac{1}{(x-a)^p}dx$ is not convergent at $p=1$.

Remark As in the above example we can show that $\int_a^b\frac{1}{(b-x)^p}\,dx$ converges if $p<1$ and diverges if $p\ge 1$.

Both improper integrals

$$\int_a^b \frac{dx}{(x-a)^p} \quad \text{and} \quad \int_a^b \frac{dx}{(b-x)^p}$$

are widely used as comparison integrals in testing convergence of improper integrals.

Convergence Tests

Direct Comparison Test: Let f and g be two positive functions, both are bounded at $x=a$ and such that

$$f(x) \le g(x) \text{ for } a < x \le b.$$

Then

(i) $\int_a^b f(x)\,dx$ converges if $\int_a^b g(x)\,dx$ converges.

and (ii) $\int_a^b g(x)\,dx$ diverges if $\int_a^b f(x)\,dx$ diverges.

Limit Comparison Test/ Quotient Test: If $f(x) \ge 0, g(x) \ge 0$ for $a < x \le b$, $f(x)$ and $g(x)$ are unbounded at $x=a$ and if $\lim_{x\to a}\frac{f(x)}{g(x)} = A$, then

(a) If $A \ne 0$ or ∞ i.e., if A is a non zero finite number, then the two integrals $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ converge or diverge together;

(b) If $A=0$ and $\int_a^b g(x)\,dx$ converges, then $\int_a^b f(x)\,dx$ converges;

and

(c) If $A=\infty$ and $\int_a^b g(x)\,dx$ diverges, then $\int_a^b f(x)\,dx$ diverges.

Example Test for convergence the improper integral $\int_1^5 \frac{dx}{\sqrt{x^4-1}}$.

We have $\frac{1}{\sqrt{x^4-1}} < \frac{1}{\sqrt{x-1}}$ for all $x>1$.

Also, $\int_1^5 \frac{1}{\sqrt{x-1}}\,dx = \int_1^5 \frac{1}{\sqrt{x^4-1}}\,dx$ converges.

Hence by comparison test, $\int_1^5 \frac{1}{\sqrt{x^4-1}}\,dx$ converges

Example Test for convergence the improper integral $\int_3^6 \frac{\log x}{(x-3)^4}\,dx$

We have $\frac{\log x}{(x-3)^4} > \frac{1}{(x-3)^4}$ for all $x>3$.

Also, $\int_3^6 \frac{1}{(x-3)^4}$ diverges since $4>1$.

Hence, by comparison test, $\int_3^6 \frac{\log x}{(x-3)^4}\,dx$ also diverges.

Example Investigate the convergence of $\int_2^3 \frac{dx}{x^2(x^3-8)^{2/3}}$.

Let $f(x) = \frac{1}{x^2(x^3-8)^{2/3}}$ and $g(x) = \frac{1}{(x-2)^{2/3}}$.

Then,

$$\frac{f(x)}{g(x)} = \frac{(x-2)^{2/3}}{x^2(x^3-2^3)^{2/3}} = \frac{1}{x^2(x^2+2x+4)^{2/3}}, \text{ since } \quad x^3-2^3 = (x-2)(x^2+2x+4)$$

Hence

$$\lim_{x\to 2+} \frac{f(x)}{g(x)} = \lim_{x\to 2+} \frac{1}{x^2(x^2+2x+4)^{2/3}} = \frac{1}{8\sqrt[3]{18}},$$

a non zero finite number. Hence by Quotient Test $\int_2^3 f(x)\,dx$ and $\int_2^3 g(x)\,dx$ converges or diverges together. Now, since $\frac{2}{3} > 1$, $\int_2^3 g(x)\,dx = \int_2^3 \frac{1}{(x-2)^{2/3}}\,dx$ converges. Hence,

$$\int_2^3 f(x)\,dx = \int_2^3 \frac{dx}{x^2(x^3-8)^{2/3}}$$

also converges.

Example Investigate the convergence of $\int_0^{\text{®}} \frac{\sin x}{x^3}\,dx$

Let $f(x) = \frac{\sin x}{x^3}$.

Since $\sin x \geq 0$ in $[0, \text{®}]$, $f(x)$ is non negative in $[0, \text{®}]$.

Take $g(x) = \frac{1}{x^2}$.

Then

$$\frac{f(x)}{g(x)} = \frac{x^2 \sin x}{x^3} = \frac{\sin x}{x}.$$

Hence $\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{\sin x}{x} = 1$, a non zero finite number. Hence by Quotient Test $\int_0^{\text{®}} f(x)\,dx$ and $\int_0^{\text{®}} g(x)\,dx$ converge or diverge together. Now, $\int_0^{\text{®}} g(x)\,dx = \int_0^{\text{®}} \frac{1}{x^2}\,dx$ diverges since $2 > 1$. Hence, $\int_0^{\text{®}} \frac{\sin x}{x^3}\,dx$ diverges.

Example Investigate the convergence of $\int_1^5 \frac{dx}{\sqrt{(5-x)(x-1)}}$

The given integrand is unbounded at two points $x = 1$ and $x = 5$.
We write

$$\int_1^5 \frac{dx}{\sqrt{(5-x)(x-1)}} = \int_1^3 \frac{1}{\sqrt{(5-x)(x-1)}}\, dx + \int_3^5 \frac{1}{\sqrt{(5-x)(x-1)}}\, dx$$

We first consider the improper integral $\int_1^3 \frac{1}{\sqrt{(5-x)(x-1)}}\, dx.$

Let $\quad f(x) = \frac{1}{\sqrt{(5-x)(x-1)}} \quad$ and $\quad g(x) = \frac{1}{\sqrt{x-1}}.$

Then

$$\lim_{x\to 1+} \frac{f(x)}{g(x)} = \lim_{x\to 1+} \frac{\sqrt{x-1}}{\sqrt{(5-x)(x-1)}} = \lim_{x\to 1+} \frac{1}{\sqrt{5-x}} = \frac{1}{2},$$

a finite non zero number, and hence by the Quotient Test $\int_1^3 f(x)\, dx$ and $\int_1^3 g(x)\, dx$ converge or diverge together. Now, since $\frac{1}{2} < 1,$ we have $\int_1^3 g(x)\, dx = \int_1^3 \frac{1}{(x-1)^{1/2}}\, dx$ converges and hence $\int_1^3 f(x)\, dx = \int_1^3 \frac{1}{\sqrt{(5-x)(x-1)}}\, dx$ also converges.

Now consider the improper integral $\int_3^5 \frac{1}{\sqrt{(5-x)\,(x-1)}}\, dx$.

Let $f(x) = \frac{1}{\sqrt{(5-x)(x-1)}}$ and $g(x) = \frac{1}{(5-x)^{1/2}}.$

Then

$$\lim_{x\to 5+} \frac{f(x)}{g(x)} = \lim_{x\to 5+} \frac{\sqrt{5-x}}{\sqrt{(5-x)(x-1)}} = \lim_{x\to 5+} \frac{1}{\sqrt{x-1}} = \frac{1}{2},$$

a finite no zero number. Hence $\int_3^5 f(x)\, dx$ and $\int_3^5 g(x)\, dx$ converge or diverge together. Since $\frac{1}{2} < 1,$ $\int_3^5 g(x)\, dx = \int_3^5 \frac{1}{(5-x)^{1/2}}\, dx$ converges, and hence $\int_3^5 \frac{dx}{\sqrt{(5-x)(x-1)}}$ converges. Combining all the above, we have

$$\int_1^3 \frac{1}{\sqrt{(5-x)(x-1)}}\, dx + \int_3^5 \frac{1}{\sqrt{(5-x)(x-1)}} = \int_1^5 \frac{dx}{\sqrt{(5-x)(x-1)}} \text{ converges.}$$

Absolute and Conditional Convergence

Properties such as absolute convergence and conditional convergence for improper integrals of the second kind are defined in the same way as for improper integrals of the first kind, and results on these properties carry over without difficulty to improper integrals of the second kind.

$\int_a^b f(x)\, dx$ is said to be **absolutely convergent** if $\int_a^b |f(x)|\, dx$ converges. If $\int_a^b f(x)\, dx$ converges but $\int_a^b |f(x)|\, dx$ diverges, then $\int_a^b f(x)\, dx$ is said to be **conditionally convergent.**

RESULT: If $\int_a^b |f(x)|dx$ converges i.e., if $\int_a^b f(x)\,dx$ converges absolutely, then $\int_a^b f(x)\,dx$ converges.

RESULT: If $\int_a^b f(x)dx$ is an absolutely convergent improper integral, and if $|g(x)| \le |f(x)| \, (a < x \le b)$, then $\int_a^b g(x)dx$ converges absolutely.

Example Show that

$$\int_0^1 \frac{\sin(1/x)}{x^p}\,dx \quad (p > 0)$$

converges absolutely for $p > 1$.

We have $\left|\frac{\sin(1/x)}{x^p}\right| = \frac{|\sin(1/x)|}{x^p} < \frac{1}{x^p}$ for all $0 \le x \le 1$ and $p > 0$.

Since $\int_0^1 \frac{1}{x^p}\,dx$ converges if and only if $p < 1$, by Comparison Test it follows that $\int_0^1 \left|\frac{\sin(1/x)}{x^p}\right| dx$ converges if $p < 1$. Therefore $\int_0^1 \frac{\sin(1/x)}{x^p}$ converges absolutely if $p < 1$.

Example The improper integral $\int_0^1 \frac{\sin x}{\sqrt{x}}\,dx$ converges absolutely,

since $\frac{|\sin x|}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$ $(0 < x \le 1)$ and $\int_0^1 \frac{1}{\sqrt{x}}$ converges (absolutely).

Conversion of Second Kind Integral to First Kind

It is often useful to convert an improper integral of the second kind by a change of variable into an improper integral of the first kind. This is illustrated in the next result.

RESULT The improper integral

$$\int_0^1 \frac{1}{x}\,dx$$

diverges.

Proof. For $0 < \varepsilon < 1$ let

$$F(\varepsilon) = \int_\varepsilon^1 \frac{1}{x}\,dx.$$

If $\varphi(u) = \frac{1}{u} \left(1 \le u \le \frac{1}{\varepsilon}\right)$, then $\varphi'(u) = -\frac{1}{u^2}du \left(1 \le u \le \frac{1}{\varepsilon}\right)$. Hence, by the Substitution Theorem which provides justification for the change of variable method, we have

$$F(\varepsilon) = \int_{\frac{1}{\varepsilon}}^1 u\left(\frac{-1}{u^2}\right)du = \int_1^{\frac{1}{\varepsilon}} \frac{1}{u}\,du.$$

As the first kind improper integral $\int_1^\infty \frac{1}{x}\,dx$ diverges, we have $\lim_{\varepsilon \to 0+} \int_1^{\frac{1}{\varepsilon}} \frac{1}{u}\,du$ does not exist. Hence $\lim_{\varepsilon \to 0+} F(\varepsilon)$ does not exist, and the theorem follows.

Integrand Unbounded at Right End Point

So far in this chapter we have treated only integrals $\int_a^b f$ where f is "bad" near a. Corresponding theory holds in the case where f is "bad" near b. Thus, if

$$f \in R\ [a, b-\varepsilon]$$

for all ε such that $0 < \varepsilon < b - a,$ and if

$$\lim_{\varepsilon \to 0+} \int_a^{b-\varepsilon} f(x)\,dx$$

exists, we again say that $\int_a^b f(x)\,dx$ is a convergent improper integral.

Example Investigate the convergence of $\int_0^1 \frac{1}{1-x}dx.$

The integrand $f(x) = 1/(1-x)$ is continuous on $[0,1)$ but becomes infinite as $x \to 1^-$.

$$\int_0^{1-£} \frac{1}{1-x}dx = \left[-\ln|1-x|\right]_0^{1-£}$$
$$= -\ln[1-(1-£)] + 0$$
$$= -\ln £.$$

Hence $\lim_{£ \to 0+} \int_0^{1-£} \frac{1}{1-x}dx = \lim_{£ \to 0+}\left[-\ln(£) + 0\right]$
$$= \infty.$$

The above limit is infinite, so the integral

$$\int_0^1 \frac{1}{1-x}dx = \lim_{£ \to 0+} \int_0^{1-£} \frac{1}{1-x}\,dx$$

diverges.

Example Examine the convergence of $\int_0^1 \frac{dx}{\sqrt{1-x}}.$

The integrand $\frac{1}{\sqrt{1-x}}$ is unbounded at $x = 1$. Hence

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{£ \to 0+} \int_0^{1-£} \frac{1}{\sqrt{1-x}}\,dx = \lim_{£ \to 0+}\left[-2\sqrt{1-x}\,\right]_0^{1-£}$$
$$= \lim_{£ \to 0+}\left[2 - 2\sqrt{£}\,\right] = 2\,.$$

Hence $\int_0^1 \frac{dx}{\sqrt{1-x}}$ is convergent and has value 2.

Example Examine the convergence of $\int_0^2 \frac{dx}{2x - x^2}$

The integrand $\frac{1}{2x-x^2}$ is unbounded at both the end points $x = 0$. and $x = 2$. Hence

$$\int_0^2 \frac{dx}{2x-x^2} = \int_0^1 \frac{1}{2x-x^2}dx + \int_1^2 \frac{1}{2x-x^2}dx$$

$$= \lim_{£_1 \to 0+} \int_{£_1}^{1} \frac{1}{2x - x^2} dx + \lim_{£_2 \to 0+} \int_{1}^{2-£_2} \frac{1}{2x - x^2} dx$$

$$= \lim_{£_1 \to 0+} \left[\frac{1}{2} \ln \frac{x}{2-x} \right]_{£_1}^{1} + \lim_{£_2 \to 0+} \left[\frac{1}{2} \ln \frac{x}{2-x} \right]_{1}^{2-£_2},$$ where $\ln$ is the natural logarithm function

$$= \lim_{£_1 \to 0+} \frac{1}{2} \left[\ln 1 - \ln \frac{£_1}{2-£_1} \right] + \lim_{£_2 \to 0+} \frac{1}{2} \left[\ln \frac{2-£_2}{£_2} - \ln 1 \right]$$

$$= -\frac{1}{2} \lim_{£ \to 0+} \ln \left[\frac{£_1}{2-£_1} \right] + \frac{1}{2} \lim_{£ \to 0+} \ln \frac{2-£}{£_2}.$$

Since the limits on the right do not exist $\int_0^2 \frac{dx}{2x - x^2}$ is not convergent.

RESULT: The improper integral

$$\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$$

is convergent.

Proof. The integral is improper since $\frac{1}{\sqrt{1-x^2}}$ is not bounded on $[0,1]$. We first show that the improper integral

$$\int_0^1 \frac{1}{\sqrt{1-x}} dx$$

is convergent (and hence absolutely convergent, since $\sqrt{1-x} \geq 0$). If $0 < \varepsilon < 1$, we have

$$\int_0^{1-\varepsilon} \frac{1}{\sqrt{1-x}} dx = 2 - 2\sqrt{\varepsilon}$$

and so $$\lim_{\varepsilon \to 0+} \int_0^{1-\varepsilon} \frac{1}{\sqrt{1-x}} dx = 2$$

Hence $\int_0^1 \frac{1}{\sqrt{1-x}} dx$ converges absolutely.

But, for $0 \leq x < 1$,

$$\frac{1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1+x}} . \frac{1}{\sqrt{1-x}} \leq \frac{1}{\sqrt{1-x}} .$$

Hence $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$ is absolutely convergent, and hence the improper integral

$$\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$$

is convergent.

Exercises

Discuss the convergence of the following improper integrals of the second kind

1. $\int_0^{®} \frac{\sin x}{x^p} dx$

2. $\int_0^1 \frac{1}{x^{\frac{2}{3}}} dx.$

3. $\int_0^{®/2} \frac{dx}{(\cos x)^{1/n}}, n > 1$.

4. $\int_0^1 \frac{1}{x^{\frac{4}{3}}} dx$.

5. $\int_0^1 \frac{x^n}{1-x} dx$

6. $\int_0^1 \frac{dx}{(x+1)\sqrt{1-x^2}}$.

7. $\int_0^2 \frac{x}{(16-x^4)^{\frac{1}{3}}} dx$.

8. $\int_2^3 \frac{x^2+1}{x^2-4} dx$.

9. $\int_{a-1}^{a+1} \frac{1}{(x-a)^{\frac{1}{3}}} dx$.

10. $\int_0^1 \frac{\cos x}{x^2} dx$.

11. $\int_0^1 \frac{\sin x}{x^{\frac{3}{2}}} dx$.

12. $\int_0^1 \frac{\log(1/x)}{\sqrt{x}} dx$.

13. $\int_0^\infty \frac{t^{-\frac{1}{2}}}{1+t} dt$

14. $\int_0^3 \frac{x^2}{(3-x)^2} dx$.

15. Evaluate $\int_{-1}^1 \frac{dx}{x^2}$

16. Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

17. Prove that if $s < 1$ then

$$\int_a^b (x-a)^{-s}\, dx = \frac{(b-a)^{1-s}}{1-s}$$

Prove that if $s \geq 1$ then the integral diverges.

18. Show that $\int_0^{®/2} \log \sin x\, dx$ is convergent and hence evaluate it.

19. Prove that $\int_0^\infty \frac{x^{s-1}}{1+x}$

is convergent if and only if $0 < s < 1$.

20. Show that $\int_0^{®/2} \sin x \log(\sin x)\, dx$ converges and that its value is $\log\left(\frac{2}{e}\right) = \log 2 - 1$.

21. Show that $\int_0^1 \frac{\log x}{\sqrt{x}} dx$ is convergent, but $\int_1^2 \frac{\sqrt{x}}{\log x} dx$ is divergent.

22. Let $$F(x) = \int_0^x \frac{\sin t}{t^{\frac{3}{2}}} dt \qquad (0 < x < \infty).$$

Prove that the maximum value of $F(x)$ is attained when $x = \pi$.

23. Prove that $\int_0^{®} \frac{1}{x} \sin\frac{1}{x}\, dx$ converges conditionally.

24. Show that $\int_0^1 \frac{\sin x}{\sqrt{x}} dx$ converges absolutely.

25. Test for convergence $\int_0^\infty \frac{dx}{\sqrt[3]{x^4 + x^2}}$

26. Test for convergence $\int_0^\infty \frac{\sin x}{\sqrt{x}} dx$

27. (Objective Type Questions) Classify the following according to the type of improper integral

(a) $\int_0^\infty \frac{dx}{\sqrt[3]{x^4 + x^2}}$; (b) $\int_0^\infty \frac{dx}{1 + \tan x}$

(c) $\int_5^{100} \frac{x\,dx}{(x-3)^8}$ (d) $\int_{-\infty}^\infty \frac{x^2\,dx}{5x^4 + x^2 + 1}$

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IMPROPER INTEGRALS OF SECOND AND THIRD KINDS – PART II

The Integrand Becomes Infinite at an Interior Point

If f becomes infinite at an interior point d of $[a, b]$, then

$$\int_a^b f(x)dx = \int_a^d f(x)dx + \int_d^b f(x)dx. \qquad \dots (5)$$

The integral from a to b **converges** if the integrals from a to d and d to b both converge. Otherwise, the integral from a to b **diverges**.

Example Investigate the convergence of $\int_0^3 \frac{dx}{(x-1)^{2/3}}$

The integrand $f(x) = 1/(x-1)^{2/3}$ becomes infinite at the interior point $x = 1$ but is continuous on $[0, 1)$ and $(1, 3]$. The convergence of the integral over $[0, 3]$ depends on the integrals from 0 to 1 and 1 to 3. On $[0, 1]$ we have

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{c \to 1^-} \int_0^c \frac{dx}{(x-1)^{2/3}} = \lim_{c \to 1^-} \left[3(x-1)^{1/3} \right]_0^c$$

$$= \lim_{c \to 1^-} [3(c-1)^{1/3} - 3(0-1)^{1/3}] = 3.$$

On $[1, 3]$ we have

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c \to 1^+} \int_c^3 \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{c \to 1^+} [3(3-1)^{1/3} - 3(c-1)^{1/3}] = 3\sqrt[3]{2}.$$

Both limits are finite, so the integral of f from 0 to 3 converges and its value is $3 + 3\sqrt[3]{2}$. i.e., $\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}.$

Improper Integral of Third Kind

Improper Integrals of the third kind can be expressed in terms of improper integrals of the first and second kinds.

The integral

$$\int_0^\infty \frac{1}{x^2 + \sqrt{x}} dx$$

does not fall into any one of the categories we have so far described since it is an integral over $(0, \infty)$ *and* $\frac{1}{\left(x^2 + \sqrt{x}\right)}$ is not bounded for x near 0. However, we shall agree to call $\int_0^\infty \frac{1}{x^2 + \sqrt{x}} dx$ a *convergent* improper integral since we can break it up into

$$J_1 = \int_0^1 \frac{1}{x^2 + \sqrt{x}} dx \text{ and } \qquad J_2 = \int_1^\infty \frac{1}{x^2 + \sqrt{x}} dx .$$

Now J_1 is a convergent integral of the second kind [since $\frac{1}{x^2 + \sqrt{x}} \le \frac{1}{\sqrt{x}}$ for $0 < x \le 1$ and since $\int_0^1 \frac{1}{\sqrt{x}} dx$ is convergent] and J_2 is a convergent improper integral of the first kind [since $\frac{1}{x^2 + \sqrt{x}} \le \frac{1}{x^2}$ for $1 \le x < \infty$ and since $\int_1^\infty \frac{1}{x^2}$ is convergent].

In general if an integral J can be broken up in this way into two or more improper integrals $J_1, \dots, J_n$ of the first or second kinds, and if *each* $J_k (k = 1, \dots, n)$ is convergent, we shall say that J is a convergent improper integral. However, if one or more of the J_k is divergent, we shall say that *J* is a improper integral.

Example $\int_0^\infty \frac{1}{x^2}dx$ is a divergent improper integral since

$$\int_0^1 \frac{1}{x^2}dx \text{ and} \int_1^\infty \frac{1}{x^2}dx$$

are improper integrals (of the second and first kinds, respectively) one of which is divergent.

Example

$$\int_{-\infty}^\infty \frac{1+x}{1+x^2}dx$$

is a divergent improper integral since both

$$\int_0^\infty \frac{1+x}{1+x^2}dx \text{ and } \int_{-\infty}^0 \frac{1+x}{1+x^2}dx$$

are divergent improper integrals of the second kind. (Note $\frac{1+x}{1+x^2} \geq \frac{1}{x}$ for $1 \leq x \leq \infty$.) and $\int_1^\infty \frac{1}{x}dx$ is divergent.

As we have just observed, the integral

$$\int_{-\infty}^\infty \frac{1+x}{1+x^2}dx$$

diverges since $\int_0^s \frac{1+x}{1+x^2}dx$ … (1)

does not approach a limit as $s \to \infty$. Similarly,

$$\int_{-s}^0 \frac{1+x}{1+x^2}dx \qquad \ldots (2)$$

does not approach a limit as $s \to \infty$.

However, the sum of(1) and (2) does approach a limit as $s \to \infty$. For the sum of (1) and (2) is

$$\int_{-s}^0 \frac{1+x}{1+x^2}dx + \int_0^s \frac{1+x}{1+x^2}dx = \int_0^s \frac{1-u}{1+u^2}du + \int_0^s \frac{1+u}{1+u^2}du$$

$$= 2\int_0^s \frac{1}{1+u^2}du,$$

and $\lim_{s\to\infty} 2\int_0^s \frac{1}{1+u^2}du$

does exist. The sum of (1) and (2) may be also be written $\int_{-s}^s \frac{1+x}{1+x^2}dx$. Hence, we have shown that

$$\lim_{s\to\infty} \int_{-s}^s \frac{1+x}{1+x^2}dx$$

exists, even though $\int_{-\infty}^\infty \frac{1+x}{1+x^2}dx$ diverges. We call

$$\lim_{s\to\infty} \int_{-s}^s \frac{1+x}{1+x^2}dx$$

the **Cauchy Principle Value** of $\int_{-\infty}^\infty \frac{1+x}{1+x^2}dx.$

Definition The **Cauchy principal value (C.P.V.)** of $\int_{-\infty}^{\infty} f(x)\,dx$ is defined to be

$$\lim_{s\to\infty}\int_{-s}^{s} f(x)\,dx,$$

if this limits exists. We denote Cauchy Principle Value by C.P.V.

i.e., C.P.V $\int_{-\infty}^{\infty} f(x)\;dx = \lim_{s\to\infty}\int_{-s}^{s} f(x)\;dx$.

Theorem If $\int_{-\infty}^{\infty} f(x)\;dx$ converges to A, then

$$\text{C.P.V. } \int_{-\infty}^{\infty} f(x)\;dx = A.$$

Proof.

If $\int_{-\infty}^{\infty} f(x)\;dx$ converges to A, then

$$A = \int_{-\infty}^{\infty} f(x)\;dx = \lim_{a\to\infty}\int_{-a}^{0} f(x)\;dx + \lim_{b\to\infty}\int_{0}^{b} f(x)\;dx \qquad \dots (1)$$

Also

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)\;dx = \lim_{R\to\infty}\left[\int_{-R}^{0} f(x)\;dx + \int_{0}^{R} f(x)\;dx\right] \qquad \dots (2)$$

The last two limits in (2) are the same as the limits on the right in equation (1). Hence left side of (1) and (2) are the same. i.e.,

$$A = \int_{-\infty}^{\infty} f(x)\;dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\;dx.$$

Hence

$$\text{C.P.V. } \int_{-\infty}^{\infty} f(x)\;dx = A.$$

The converse is not, in general, true, because Cauchy Principal Value of $\int_{-\infty}^{\infty} f(x)\;dx$ may exist even if the integral diverges. The following is an example.

Example $\int_{-\infty}^{\infty} x\;dx$ diverges but

$$\text{C.P.V. } \int_{-\infty}^{\infty} x\;dx = \lim_{s\to\infty}\int_{-s}^{s} x\,dx = \lim_{s\to\infty}\left[\frac{x^2}{2}\right]_{-s}^{s} = \lim_{s\to\infty} 0 = 0.$$

Cauchy Principal Value – Unboundedness at Interior Point

If $f(x)$ is unbounded at an interior point $x = x_0$ of the interval $a \le x \le b$, then by definition,

$$\int_{a}^{b} f(x)\;dx = \lim_{£_1\to 0+}\int_{a}^{x_0-£} f(x)\;dx + \lim_{£_2\to 0}\int_{x_0+£_2}^{b} f(x)\;dx \qquad \dots (1)$$

It may happen that the limits on the right of (1) do not exist when $£_1$ and $£_2$ approach zero independently. In such case it is possible that by choosing $£_1 = £_2 = £$ in (1), i.e., writing

$$\int_{a}^{b} f(x)\;dx = \lim_{£\to 0+}\left\{\int_{a}^{x_0-£} f(x)dx + \int_{x_0+£}^{b} f(x)\;dx\right\} \qquad \dots (2)$$

the limit des exist. If the limit on the right of (2) does exist, we call this limiting value the **Cauchy Principal Value** of $\int_{a}^{b} f(x)\;dx$ and is usually denoted by C.P.V. $\int_{a}^{b} f(x)\;dx$.

Thus

$$\text{C.P.V.} \int_a^b f(x)\,dx = \lim_{\pounds\to 0+}\left\{\int_a^{x_0-\pounds} f(x)\,dx + \int_{x_0+\pounds}^b f(x)\,dx\right\}$$

Example Determine whether $\int_{-1}^{5}\frac{dx}{(x-1)^3}$ converges (a) in the usual sense (b) in the Cauchy Principal Value sense.

(a) By definition,

$$\int_{-1}^{5}\frac{1}{(x-1)^3}\,dx = \lim_{\pounds_1\to 0+}\int_{-1}^{1-\pounds_1}\frac{1}{(x-1)^3}\,dx + \lim_{\pounds_2\to 0}\int_{1+\pounds_2}^{5}\frac{1}{(x-1)^3}\,dx$$

$$= \lim_{\pounds_1\to 0+}\left[\frac{-1}{2(x-1)^2}\right]_{-1}^{1-\pounds_1} + \lim_{\pounds_2\to 0+}\left[\frac{-1}{2(x-1)^2}\right]_{1+\pounds_2}^{5}$$

$$= \lim_{\pounds_1\to 0+}\left(\frac{1}{8}-\frac{1}{2\pounds_1^2}\right) + \lim_{\pounds_2\to 0+}\left(\frac{1}{2\pounds_2^2}-\frac{1}{32}\right)$$

Since the limits on right do not exist, the given integral doesnot converge in the usual sense.

(b) C.P.V. $\int_{-1}^{5}\frac{dx}{(x-1)^3} = \lim_{\pounds\to 0+}\left\{\int_{-1}^{1-\pounds}\frac{1}{(x-1)^3}dx + \int_{1+\pounds}^{5}\frac{1}{(x-1)^3}dx\right\}$

$$= \lim_{\pounds\to 0+}\left\{\frac{1}{8}-\frac{1}{2\pounds^2}+\frac{1}{2\pounds^2}-\frac{1}{32}\right\} = \frac{3}{32}.$$

Hence the given integral converges in the C.P.V. sense.

Exercises

Discuss the convergence of the following improper integrals of the second kind

1. $\int_0^{®}\frac{\sin x}{x^p}dx$
2. $\int_0^1 \frac{1}{x^{\frac{2}{3}}}dx.$
3. $\int_0^{®/2}\frac{dx}{(\cos x)^{1/n}}, n>1.$
4. $\int_0^1 \frac{1}{x^{\frac{4}{3}}}dx.$
5. $\int_0^1 \frac{x^n}{1-x}\,dx$
6. $\int_0^1 \frac{dx}{(x+1)\sqrt{1-x^2}}.$
7. $\int_0^2 \frac{x}{(16-x^4)^{\frac{1}{3}}}dx.$
8. $\int_2^3 \frac{x^2+1}{x^2-4}dx.$
9. $\int_{a-1}^{a+1}\frac{1}{(x-a)^{\frac{1}{3}}}dx.$
10. $\int_0^1 \frac{\cos x}{x^2}dx.$
11. $\int_0^1 \frac{\sin x}{x^{\frac{3}{2}}}dx.$
12. $\int_0^1 \frac{\log(1/x)}{\sqrt{x}}\,dx.$
13. $\int_0^{\infty}\frac{t^{-\frac{1}{2}}}{1+t}dt$
14. $\int_0^3 \frac{x^2}{(3-x)^2}dx.$

15. Evaluate $\int_{-1}^{1} \frac{dx}{x^2}$

16. Evaluate $\int_{0}^{1} \frac{dx}{\sqrt{1-x^2}}$

17. Prove that if $s < 1$ then

$$\int_a^b (x-a)^{-s}\, dx = \frac{(b-a)^{1-s}}{1-s}$$

Prove that if $s \geq 1$ then the integral diverges.

18. Show that $\int_0^{\text{®}/2} \log \sin x\, dx$ is convergent and hence evaluate it.

19. Prove that $$\int_0^{\infty} \frac{x^{s-1}}{1+x}$$
is convergent if and only if $0 < s < 1$.

20. Show that $\int_0^{\text{®}/2} \sin x \log(\sin x)\, dx$ converges and that its value is $\log\left(\frac{2}{e}\right) = \log 2 - 1$.

21. Show that $\int_0^1 \frac{\log x}{\sqrt{x}}\, dx$ is convergent, but $\int_1^2 \frac{\sqrt{x}}{\log x}\, dx$ is divergent.

22. Let $$F(x) = \int_0^x \frac{\sin t}{t^{\frac{3}{2}}}\, dt \qquad (0 < x < \infty).$$
Prove that the maximum value of $F(x)$ is attained when $x = \pi$.

23. Prove that $\int_0^{\text{®}} \frac{1}{x} \sin \frac{1}{x}\, dx$ converges conditionally.

24. Show that $\int_0^1 \frac{\sin x}{\sqrt{x}}\, dx$ converges absolutely.

25. Test for convergence $\int_0^{\infty} \frac{dx}{\sqrt[3]{x^4 + x^2}}$

26. Test for convergence $\int_0^{\infty} \frac{\sin x}{\sqrt{x}}\, dx$

27. (Objective Type Questions) Classify the following according to the type of improper integral

(a) $\int_0^{\infty} \frac{dx}{\sqrt[3]{x^4 + x^2}}$; (b) $\int_0^{\infty} \frac{dx}{1 + \tan x}$

(c) $\int_5^{100} \frac{x\, dx}{(x-3)^8}$ (d) $\int_{-\infty}^{\infty} \frac{x^2\, dx}{5x^4 + x^2 + 1}$

In Exercises 1-3, which of the following integrals does the Cauchy principle value exist? Do any of the integrals converge?

28. $\int_{-\infty}^{\infty} \sin t\, dt.$

29. $\int_{-\infty}^{\infty} |\sin t|\, dt.$

30. $\int_{-\infty}^{\infty} \frac{1}{1+t^2}\, dt$

31. Prove that $\int_0^s \frac{dx}{4-x}$ diverges in the usual sense but converges in the Cauchy principal value sense.

32. Show that C.P.V. $\int_{-1}^{1} \frac{1}{x}\, dx = 0$ and C.P.V $\int_{-1}^{1} \frac{1}{|x|}\, dx$ does not exist.

33. If f is continuous on $(-\infty, \infty)$ and if $\int_{-\infty}^{\infty} f(x)\, dx$ converges to A, prove that

C.P.V. $\int_{-\infty}^{\infty} f(x)\, dx = A.$.

34. If f is continuous on [0,1], prove that

$$\int_0^1 \frac{f(x)}{\sqrt{1-x^2}}\, dx$$

is convergent. Then prove that

$$\int_0^1 \frac{f(x)}{\sqrt{1-x^2}}\, dx = \int_0^{\frac{\pi}{2}} f(\sin u)\, du.$$

Is the integral on the right improper?

35. If f is well behaved on $[a,b]$ except near the point $c \varepsilon (a,b)$, we define the Cauchy principal value of $\int_a^b f$ as

$$\lim_{\varepsilon \to 0+} \left(\int_a^{c-\varepsilon} f + \int_{c+\varepsilon}^{b} f \right).$$

(a) Show that C.P.V. $\int_{-1}^{1} \frac{1}{x} dx = 0.$

(b) Show that C.P.V. $\int_{-1}^{1} \frac{1}{|x|} dx$ does not exist.

14
BETA AND GAMMA FUNCTIONS

The Beta Function

If m, n are positive, then the definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is called the **Beta function**, (or **Beta Integral**) denoted by (m, n). That is,

$$(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \text{ for } m>0, n>0. \qquad \ldots(1)$$

The beta integral is some times called **Eulerian Integral of the first kind.**

Convergence of Beta Function

- For $m \geq 1$ and $n \geq 1$, the beta function (m, n) given by (1) is a proper integral and hence is convergent.
- If $m<1$ and $n<1$, then (m, n) is an *improper integral of the second kind.* The convergence is verified as follows:

We have

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}\,dx + \int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}\,dx$$

Let $I_1 = \int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}\,dx$ and $I_2 = \int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}\,dx$.

Then

$$\int_0^1 x^{m-1}(1-x)^{n-1} = I_1 + I_2$$

Convergence of I_1

We take $f(x) = x^{m-1}(1-x)^{n-1}$ and $g(x) = \dfrac{1}{x^{1-m}}$.

Then, $\dfrac{f(x)}{g(x)} = (1-x)^{n-1} \to 1$ as $x \to 0$.

i.e., $\lim\limits_{x\to 0+} \dfrac{f(x)}{g(x)} = 1$, a non zero finite number.

Hence, by Limit Comparison Test (Quotie nt test) $\int_0^{\frac{1}{2}} f(x)\,dx$ and $\int_0^{\frac{1}{2}} g(x)\,dx$ converge or diverge together.

But $\int_0^{\frac{1}{2}} g(x)\,dx = \int_0^{\frac{1}{2}} \dfrac{1}{x^{1-m}}\,dx$ converges if and only if $1-m<1$ i.e., $m>0$. Hence $\int_0^{\frac{1}{2}} f(x)\,dx = \int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}\,dx = I_1$ converges if and only if $m>0$.

Convergence of I_2

We take $f(x) = x^{m-1}(1-x)^{n-1}$ and $g(x) = \dfrac{1}{(1-x)^{1-n}}$.

Then, $\lim\limits_{x\to 1+} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to 1+} x^{m-1} = 1$, a non zero finite number.

Hence, by Limit Comparison Test, both the integrals $\int_{\frac{1}{2}}^{1} f(x)\,dx$ and $\int_{\frac{1}{2}}^{1} g(x)\,dx$ converges or diverge together. But $\int_{\frac{1}{2}}^{1} g(x)\,dx = \int_{\frac{1}{2}}^{1} \frac{1}{(1-x)^{1-n}}\,dx$ converges if and only if $1-n<1$, i.e., if and only if $n>0$. Hence $\int_{\frac{i}{2}}^{1} f(x)\,dx = \int_{\frac{i}{2}}^{1} x^{m-1}(1-x)^{n-1}\,dx = I_2$ converges if and only if $n>0$.

Therefore if $m>0$ and $n>0$, both I_1 and I_2 converges and hence $\beta(m,n)$ converges.

Example Express $\int_0^2 (8-x^3)^{-1/3}\,dx$ in terms of a Beta function.

Let $\quad I = \int_0^2 (8-x^3)^{-1/3}\,dx$

Put $x^3 = 8z$. Then $x = 2z^{1/3}$ and $dx = \frac{2}{3} z^{2/3}\,dz.$

Also, when $x=0$, $z=0$; when $x=2, z=1$

Hence (1) becomes

$$I = \int_0^1 (8-8z)^{1/3} \cdot \frac{2}{3} z^{-2/3}\,dz = 8^{-1/3} \cdot \frac{2}{3} \int_0^1 (1-z)^{-1/3} z^{-2/3} dz$$

$$= \frac{1}{3}\int_0^1 (1-x)^{-1/3} x^{-2/3}\,dx \text{, using the fact that}$$

$$\int_a^b f(z)dz = \int_a^b f(x)\,dx$$

$$= \frac{1}{3}\int_0^1 x^{-2/3}(1-x)^{-1/3}\,dx$$

$$= \frac{1}{3}\left(\frac{-2}{3}+1, \frac{-1}{3}+1\right) = \frac{1}{3}\left(\frac{1}{3}, \frac{2}{3}\right).$$

Example Evaluate $\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx,\ m>0,\ n>0$.

Put $\quad x = a+(b-a)z \quad$ so that $\quad dx = (b-a)\,dz.$

Hence $\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx$

$$= \int [(b-a)z]^{m-1}\,[b-a-(b-a)z]^{n-1}\,(b-a)\,dz$$

$$= (b-a)^{m+n-1}\int_0^1 z^{m-1}(1-z)^{n-1}\,dz$$

$$= (b-a)^{m+n-1}\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = (b-a)^{m+n-1} \cdot \ (m,n)$$

Properties of Beta Function

1. The beta function is symmetric in m and n.

i.e., $\quad (m, n) = \ (n, m).$

Proof . We have

$$(m,\ n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \int_0^1 (1-x)^{m-1}(1-(1-x))^{n-1}\,dx,$$ obtained by taking $f(x) = x^{m-1}(1-x)^{n-1}$ and $a = 1$ in the result $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

$$= \beta(n, m)$$

2. When n is a positive integer,

$$(m,\ n) = \frac{(n-1)!}{m(m+1)(m+2)\ \cdots\ (m+n-1)}$$

Proof.

Case 1) If $n = 1$,

$$\beta(m,1) = \int_0^1 x^{m-1}(1-x)^0\,dx = \left[\frac{x^m}{m}\right]_0^1 = \frac{1}{m}.$$

Case 2) If n is an integer and $n > 1$, we have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \left[(1-x)^{n-1}\frac{x^m}{m}\right]_0^1 - \int_0^1 (n-1)(1-x)^{n-2}(-1)\frac{x^m}{m}\,dx,$$

by integrating by parts

$$= 0 + \frac{n-1}{m}\int_0^1 x^m\ (1-x)^{n-2}\,dx$$

$$= \frac{n-1}{m}\beta(m+1, n-1)$$

By repeated application, we have

$$\beta(m,n) = \frac{n-1}{m}.\frac{n-2}{m+1}.\frac{n-3}{m+2}.\ \ldots\ .\frac{1}{m+n-2}\beta(m+n-1,1)$$

$$= \frac{n-1}{m}.\frac{n-2}{m+1}.\frac{n-3}{m+2}.\ \ldots\ .\frac{1}{m+n-2}.\frac{1}{m+n-1},$$

since by Case 1, $(m',1) = \dfrac{1}{m'}$

$$= \frac{(n-1)!}{m(m+1)(m+2)\ \cdots\ (m+n-1)} \qquad \text{... (1)}$$

3. When m is a positive integer, proceeding as in Property 2, we obtain

$$(m,\ n) = \frac{(m-1)!}{n(n+1)(n+2)\ \cdots\ (n+m-1)}$$

4. If m and n are positive integers, then

$$(m,\ n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}. \qquad \ldots (2)$$

This can be obtained by multiplying both numerator and denominator of Eq. (1) by $(m-1)!$.

5. $$(1,\ n) = \frac{1}{n}.$$

This can be obtained by putting $m=1$ in Eq.(2).

6. If m and n are positive integers, then

$$\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx = (b-a)^{m+n-1}\ \ (m,n) \qquad \ldots (3)$$

Proof. In the given integral, we put $x-a=(b-a)y$ so that $dx=(b-a)dy$. When $x = a$, $y = 0$; and when $x = b$, $y = 1$. Hence,

$$\text{The given integral } = \int_0^1 \left[(b-a)y\right]^{m-1}\left[(b-a)(1-y)\right]^{n-1}(b-a)\,dy$$

$$= (b-a)^{m+n-1}\int_0^1 y^{m-1}(1-y)^{n-1}dy$$

$$= (b-a)^{m+n-1}\ \ (m,n)$$

7. $$\int_0^b x^{m-1}(b-x)^{n-1}\,dx = b^{m+n-1}\ \ (m,n)$$

This can be obtained by putting $a=0$ in Eq.(3).

8. $$(m,n) = 2\int_0^{®/2} \sin^{2m-1}¯\ \cos^{2n-1}¯\ d¯\ .$$

Proof. Put $x=\sin^2\theta$, so that $1-x=\cos^2\theta$ and $dx = 2\sin\theta\cos\theta\,d\theta$. When $x = 0$, ¯ = 0 and when $x = 1$, ¯ = ®/2.

$$\therefore \qquad \beta(m,n) = \int_0^{\frac{\pi}{2}} \sin^{2(m-1)}\theta\ \cos^{2(n-1)}\theta\ \ 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\ \ d\theta$$

9. $\beta(m,n)=\int_0^{\infty}\frac{y^{n-1}}{(1+y)^{m+n}}\,dy$.

Proof. Consider the expression $(m,\,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$. Put $x=\frac{1}{1+y}$, so that $1-x=1-\frac{1}{1+y}=\frac{y}{1+y}$ and $dx=-\frac{1}{(1+y)^2}$. When x = 1, y = 0 and when $x=0,\ ^{-}=\infty.$ Hence

$$\beta(m,n)=\int_{\infty}^{0}\frac{1}{(1+y)^{m-1}}\left(\frac{y}{1+y}\right)^{n-1}\left(-\frac{1}{(1+y)^2}\right)dy$$

$$=\int_0^{\infty}\frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

10. $(m,n)=\int_0^{\infty}\frac{x^{m-1}}{(1+x)^{m+n}}\,dx,$ where $m>0,\ n>0$

Proof. In the expression for $\beta(m,n)$, put $z=\frac{x}{1+x}$, so that $x=z+xz$ or $x=\frac{z}{1-z}$ and $dx=\frac{(1-z)\cdot 1-z(-1)}{(1-z)^2}dz=\frac{dz}{(1-z)^2}$. When x = 0, z = 0 and when $x=\infty,$ $z=\lim_{x\to\infty}\frac{x}{1+x}=1.$ Hence

$$\int_0^{\infty}\frac{x^{m-1}}{(1+x)^{m+n}}\,dx=\int_0^1\left(\frac{z}{1-z}\right)^{m-1}(1-z)^{m+n}\frac{1}{(1-z)^2}dz, \text{ since}$$

$$1+x=1+\frac{z}{1-z}=\frac{1}{1-z}.$$

$$=\int_0^1\frac{z^{m-1}}{(1-z)^{m-1}}\frac{(1-z)^{m+n}}{(1-z)^2}dz$$

$$=\int_0^1 z^{m-1}(1-z)^{n-1}dz=\ (m,\,n).$$

Example We now show that

$$\int_0^{\frac{\pi}{2}}\sin^p x\cos^q x\,dx=\frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right),\qquad p>-1,\ q>-1$$

By an earlier example, we have

$$(m,n)=2\int_0^{\frac{®}{2}}\sin^{2m-1}{}^{-}\ \cos^{2n-1}{}^{-}\ d^{-}. \qquad \ldots(4)$$

In (4), put $m=\frac{p+1}{2},\ n=\frac{q+1}{2}$ or $p=2m-1,\ q=2n-1,$ and we obtain

$$\left(\frac{p+1}{2},\frac{q+1}{2}\right) \; =2\int_0^{\frac{\circledR}{2}} \sin^p x \cos^q x \, dx.$$

The above formula can be used to evaluate integrals. For example,

$$\int_0^{\circledR/2} \sin^7 x \cos^5 x \, dx = \frac{1}{2} \; \left(\frac{7+1}{2},\frac{5+1}{2}\right) = \frac{1}{2} \; \left(4,\, 3\right) = \frac{1}{2}\frac{3!\,2!}{6!} = \frac{1}{120}.$$

Example Express in terms of Beta function, the integral

$$\int_0^1 x^m (1-x^2)^n dx \; (m>1, n>-1).$$

Put $\qquad x^2 = z$ so that $2x\; dx = dz.$

Also when $\qquad x=0,$ we have $z=0$ and when $x=1, \quad z=1.$

Hence

$$\int_0^1 x^m (1-x^2) dx = \int_0^1 x^{m-1} (1-x^2)^n \; x\,dx$$

$$= \int_0^1 z^{(m-1)/2} (1-z)^n \frac{1}{2} dz$$

$$= \frac{1}{2}\int_0^1 z^{(m-1)/2} (1-z)^n dz$$

$$= \frac{1}{2}\beta\left(\frac{m-1}{2}+1, n+1\right) = \frac{1}{2}B\left(\frac{m+1}{2}, n+1\right)$$

Example Express $\int_0^1 \frac{x^2}{\sqrt{1-x^5}} dx$ in terms of beta function.

We have $\quad \int_0^1 \frac{x^2}{\sqrt{1-x^5}} \; dx = \int_0^1 x^2 (1-x^5)^{\frac{1}{2}} \, dx.$

Here put $x^5 = z$ so that $5x^4 dx = dz.$

Also when $x=0$, $z=0$ and when $x=1, \; z=1.$ Therefore, we have

$$\int_0^1 \frac{x^2}{\sqrt{1-x^5}} dx = \int_0^1 x^{-2} (1-x^5)^{\frac{-1}{2}} x^4 \qquad dx = \int_0^1 z^{\frac{-2}{5}} (1-z)^{\frac{-1}{2}} \frac{1}{5} dz$$

$$= \frac{1}{5}\int_0^1 z^{\frac{-2}{5}} (1-z)^{\frac{-1}{2}} \; dz = \frac{1}{5} B\left(-\tfrac{2}{5}+1, -\tfrac{1}{2}+1\right) = \frac{1}{5} B\left(\tfrac{3}{5}, \tfrac{1}{2}\right).$$

Example Express in terms of Beta Functions, the integer

$$\int_a^b (x-a)^{m-1} (b-x)^{n-1} \, dx.$$

Here put $x = a + (b-a)z,$ so that $dx = (b-a)dz.$

Also when $x=a, \; z=0$ and when $x=b, \; z=1$ .Therefore, we get

$$\int_a^b (x-a)^{m-1} (b-x)^{n-1} \qquad = \int_0^1 \left[(b-a)z\right]^{m-1} \left[b-a-(b-a)z\right]^{n-1} (b-a)dz$$

$$= \int_0^1 (b-a)^{m-1} z^{m-1} \left(b-a\right)^{n-1} \left(1-z\right)^{n-1} \left(b-a\right)dz$$

$$= (b-a)^{m+n-1} \int_0^1 z^{m-1} (1-z)^{n-1} dz$$

$$= (b-a)^{m+n-1} B(m,n).$$

Example If $p > 0, q > 0,$ prove that

(i) $\frac{\beta(p,q+1)}{q} = \frac{\beta(p+1,q)}{p} = \frac{\beta(p,q)}{p+q};$

(ii) $\beta(p,q) = \beta(p+1,q) + \beta(p,q+1).$

We have

$$\beta(p,q+1) = \int_0^1 x^{p-1}(1-x)^q\,dx = \int_0^1 (1-x)^q x^{p-1} dx\,,$$

$$= \left[(1-x)^q \frac{x^p}{p}\right]_0^1 - \int_0^1 q(1-x)^{q-1}(-1)\frac{x^p}{p}\,dx,$$

by integrating by parts

$$= 0 + \frac{q}{p}\int_0^1 x^p(1-x)^{q-1}\,dx = \frac{q}{p}\beta(p+1,q). \qquad \text{... (1)}$$

$$\therefore \quad \frac{\beta(p,q+1)}{q} = \frac{\beta(p+1,q)}{p} \qquad \text{... (2)}$$

Also from (2), we obtain

$$\frac{\beta(p,q+1)}{q} = \frac{1}{p}\beta(p+1,q) = \frac{1}{p}\int_0^1 x^p(1-x)^{q-1}\,dx \frac{1}{p}\int_0^1 x^{p-1}x(1-x)^{q-1}\,dx$$

$$= \frac{1}{p}\int_0^1 x^{p-1}\left[1-(1-x)\right](1-x)^{q-1}\,dx$$

$$= \frac{1}{p}\int_0^1 x^{p-1}(1-x)^{q-1}\,dx - \frac{1}{p}\int_0^1 x^{p-1}(1-x)^q\,dx$$

$$= \frac{1}{p}\beta(p,q) - \frac{1}{p}\beta(p,q+1$$

i.e., $\frac{\beta(p,q+1)}{q} + \frac{\beta(p,q+1)}{p} = \frac{\beta(p,q)}{p}.$

Hence $\frac{(p+q)\beta(p,q+1)}{pq} = \frac{\beta(p,q)}{p}.$

Therefore, $\frac{\beta(p,q+1)}{q} = \frac{\beta(p,q)}{p+q}.$ ... (3)

From (2) and (3), we get

$$\frac{\beta(p,q+1)}{q} = \frac{\beta(p+1,q)}{p} = \frac{\beta(p,q)}{p+q} \qquad \text{... (4)}$$

(ii) From (4), we get

$$\beta(p,q+1) = \frac{q}{p+q}\beta(p,q,), \qquad \text{... (5)}$$

and

$$\beta(p+1,q) = \frac{p}{p+q}\beta(p,q,). \qquad \text{...(6)}$$

Adding (5) and (6), we have

$$\beta(p,q+1) + \beta(p+1,q) = \frac{q}{p+q}\beta(p,q,) + \frac{q}{p+q}\beta(p,q)$$

$$= \frac{p+q}{p+q}\beta(p,q) = \beta(p,q)$$

The Gamma Function

If $n > 0$, the definite integral $\int_0^\infty e^{-x} x^{n-1} dx$ is called the **Gamma Function** and is denoted by $\Gamma(n)$.

Thus

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, \qquad x > 0, \quad n > 0.$$

It is also called **Eulerian integral of the second kind.** Also $\Gamma(1) = 1$.

Convergence of Gamma Function

The Gamma function $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$ is an improper integral of the first kind if $n \geq 1$ and is an improper integral of the third kind $n < 1$. Now we can write

$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \; dx = \int_0^1 e^{-x} x^{n-1} \; dx + \int_1^\infty e^{-x} x^{n-1} \; dx = I_1 + I_2$, where

$$I_1 = \int_0^1 e^{-x} x^{n-1} dx \text{ and } I_2 = \int_1^\infty e^{-x} x^{n-1} dx .$$

Case1: $n \geq 1$

When $n \geq 1$, I_1 is a proper integral and I_2 is an improper integral of the first kind.

Here $\quad f(x) = e^{-x} x^{n-1}$

Take $\quad g(x) = \frac{1}{x^2}$.

Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} x^{n+1} e^{-x} = 0$.

Since $\int_1^\infty g(x)\, dx = \int_1^\infty \frac{1}{x^2} dx$ converges, by Limit Comparison Test, we have

$\int_1^\infty f(x)\, dx = \int_1^\infty e^{-x} x^{n-1} \; dx = I_2$ is convergent.

Hence, $I_1 + I_2 = \Gamma(n)$ converges if $n \geq 1$

Case 2: $n < 1$

When $n < 1$, I_1 is an improper integral of the second kind and I_2 is an improper integral of the first kind.

Here $\qquad f(x) = e^{-x} x^{n-1}$

Take $\qquad g(x) = \frac{1}{x^{1-n}}$.

Then $\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} e^{-x} = 1$, a non zero finite number.

Since $\int_0^1 g(x)\,dx = \int_0^1 \frac{1}{x^{1-n}}\,dx$ converges if $1-n<1$, i.e., if $n>0$, by Limit Comparison Test, it follows that $\int_0^1 f(x)\,dx = \int_0^1 x^{n-1}e^{-x}\quad dx = I_1$ converges if $n>0$. As in Case 1 we can show that $I_2 = \int_1^{\infty} e^{-x}x^{n-1}dx$ converges for all $n>1$

Hence, $I_1 + I_2 = \Gamma(n)$ converges for all $0<n<1$.

Combining the results obtained in Case 1 and Case 2, we see that $\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}dx$ converges for all $n>0$.

Recurrence Formula For Gamma Function

$$\Gamma(n) = (n-1)\Gamma(n-1) \qquad \text{for} \quad n>1.$$

We have

$$\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}dx$$

$$= \int_0^{\infty} x^{n-1}\, e^{-x}dx$$

$$= \lim_{t\to\infty}\int_0^{t} x^{n-1}\, e^{-x}dx$$

$$= \lim_{t\to\infty}\left[x^{n-1}\frac{e^{-x}}{-1}\right]_0^t - \lim_{t\to\infty}\int_0^t (n-1)\,x^{n-2}\,\frac{e^{-x}}{-1}dx\,,$$

integrating by parts

$$= \lim_{t\to\infty}\left[\frac{x^{n-1}}{-e^{x}}\right]_0^t - (n-1)\lim_{t\to\infty}\int_0^t x^{n-2}\,\frac{e^{-x}}{-1}dx$$

$$= \lim_{t\to\infty}\frac{t^{n-1}}{-e^{t}} - (n-1)\int_0^{\infty} x^{n-2}\,\frac{e^{-x}}{-1}dx$$

$= 0 + (n-1)\Gamma(n-1)$, since, by applying L'Hospital Rule repeatedly, $\lim_{t\to\infty}\frac{t^{n-1}}{-e^{t}} = 0$

$= (n-1)\Gamma(n-1).$

Theorem When n is a positive integer $\Gamma(n) = (n-1)!$

We have $\quad \Gamma(n) = (n-1)\Gamma(n-1)$

When n is a positive integer, by repeated application of the above formula

$$\begin{aligned}\Gamma(n) &= (n-1)\Gamma(n-1)\\ &= (n-1)(n-2)\Gamma(n-2)\\ &= (n-1)(n-2)(n-3)\Gamma(n-3)\\ &= (n-1)(n-2)(n-3)\ldots 1\Gamma(1).\\ &= (n-1)(n-2)(n-3)\cdots 1\cdot\Gamma(1).\end{aligned}$$

But $\quad \Gamma(1) = \int_0^{\infty} e^{-x}x^0 \quad dx = \int_0^{\infty} e^{-x}\quad dx = \lim_{x\to\infty}\int_0^t e^{-x}\,dx$

$$= \lim_{t\to\infty}\left[-e^{-x}\right]_0^t = \lim_{t\to\infty}\left(1-e^{-t}\right) = 1-0 = 1.$$

Hence

$$\Gamma(n) = (n-1)(n-2)(n-3)\cdots 1\cdot 1 = (n-1)!,$$

when n is a positive integer.

Remark $\Gamma(n+a) = (n+a-1)(n+a-2)\cdots a\cdot\Gamma(a)$,
when n is a positive integer.

Example Show that $\int_0^\infty e^{-ay} y^{n-1} dy = \dfrac{\Gamma(n)}{a^n}$.

Putting $x = ay$ in $\Gamma(n)$, we have

$$\Gamma(n) = \int_0^\infty e^{-ay}(ay)^{n-1} a\,dy = a^n \int_0^\infty e^{-ay} y^{n-1}\,dy$$

Hence
$$\int_0^\infty e^{-ay} y^{n-1} dy = \frac{\Gamma(n)}{a^n}$$

Example Show that

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}} dx = \frac{1}{(a+b)^m a^n} \quad (m, n)$$

$$I = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}} dx = \int_0^1 \left(\frac{x}{a+bx}\right)^{m-1}\left(\frac{1-x}{a+bx}\right)^{n-1}\frac{1}{(a+bx)^2} dx$$

Put $\dfrac{x}{a+bx} = \dfrac{y}{a+b}$; so that $\dfrac{1}{(a+bx)^2} dx = dy$

Also
$$\frac{1-x}{a+bx} = \frac{1}{a}\frac{a-ax}{a+bx} = \frac{1}{a}\left[\frac{a+bx-bx-ax}{a+bx}\right]$$
$$= \frac{1}{a}\left[1-\frac{x(a+b)}{a+bx}\right] = \frac{1-y}{a}.$$

Also when $x = 0$, $y = 0$; and when $x = 1$, $y = 1$. Hence

$$I = \int_0^1 \left(\frac{y}{a+b}\right)^{m-1}\left(\frac{1-y}{a}\right)^{n-1}\frac{1}{a(a+b)} dy$$

$$= \frac{1}{(a+b)^m a^n}\int_0^1 y^{m-1}(1-y)^{n-1} dy = \frac{\beta(m,n)}{(a+b)^m a^n}.$$

Relation Between Beta and Gamma Functions

Theorem $(m,\, n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Proof. Using the relation $\dfrac{\Gamma(n)}{a^n} = \int_0^\infty e^{-a\,x} x^{n-1} dx$, we obtain (by taking z in place of a)

$$\Gamma(m) = z^m \int_0^\infty e^{-z\,x} x^{m-1} dx = \int_0^\infty z^m e^{-z\,x} x^{m-1} dx$$

Multiplying both sides by $e^{-z} \cdot z^{n-1}$, we obtain

$$\Gamma(m) e^{-z}\, z^{n-1} = \int_0^\infty e^{-z(1+x)} z^{m+n-1} x^{m-1} dx .$$

Integrating both sides from 0 to ∞,

$$\Gamma(m)\int_0^\infty e^{-z}\, z^{n-1} dz = \int_0^\infty\int_0^\infty e^{-z(1+x)} z^{m+n-1} x^{m-1} dx\, dz$$

That is,

$$\Gamma(m)\Gamma(n) = \int_0^\infty\int_0^\infty e^{-z(1+x)} z^{m+n-1}\, dz\, x^{m-1} dx \ ,$$

by changing the order of integration.

$$\Gamma(m)\Gamma(n) = \int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}} x^{m-1} dx = \Gamma(m+n)\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$= \Gamma(m+n)\beta(m,\, n)$$

Hence $\qquad (m,\, n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

The above can be proved as follows also:

$$\beta(m,n) = \int_0^\infty x^{m-1}(1-x)^{n-1}\, dx = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta \qquad \ldots (1)$$

by putting $x = \sin^2\theta$.

$$\Gamma(m) = \int_0^\infty e^{-x} x^{m-1} \quad dx = 2\int_0^\infty e^{-t^2} t^{2m-1}\, dt, \text{ by putting } x = t^2$$

$$= 2\int_0^\infty e^{-x^2} \quad x^{2m-1} dx \text{, by changing the variable } t \text{ to } x \qquad \ldots (2)$$

Similarly, we have

$$\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1}\, dy. \qquad \ldots (3)$$

Hence

$$\Gamma(m)\Gamma(n) = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dx\, dy. \qquad \ldots (4)$$

In (4) put $x = r\cos\theta, y = r\sin\theta$ .Then $dxdy = r\,dr\,d\theta$. Since x and y vary from 0 to ∞, x and y may be taken as the coordinates of any point in the first quadrant. Also by putting $x = r\cos\theta, y = r\sin\theta$, we are transforming Cartesian coordinates into polar coordinates. Again in polar coordinates, the first quadrant can be covered by varying r from 0 to ∞ and θ from 0 to $\frac{\pi}{2}$. So

$$\begin{aligned}\Gamma(m)\Gamma(n) &= 4\int_0^\infty \int_0^{\frac{\pi}{2}} e^{-r^2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1} r\,dr\,d\theta \\ &= 4\int_0^\infty \int_0^{\frac{\pi}{2}} e^{-r^2} r^{2m+2n} \sin^{2n-1}\theta \ \cos^{2m-1}\theta \ dr\,d\theta \\ &= \left(2\int_0^\infty e^{-r^2} r^{2m+2n-1} dr\right)\left(2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta \cos^{2m-1}\theta\, dr\, d\theta\right) \qquad \ldots (5)\end{aligned}$$

Now

$$\begin{aligned}2\int_0^\infty e^{-r^2} r^{2m+2n-1}\, dr &= \int_0^\infty e^{-1} t^{m+n-1}\, dt, \quad \text{by putting } r^2 = t \\ &= \Gamma(m+n)\end{aligned}$$

Also, we have

$$2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta \cos^{2m-1}\theta\, d\theta = \beta(n,m) = \beta(m,n).$$

Hence (5) becomes

$$\Gamma(m)\cdot\Gamma(n) = \Gamma(m+n)\cdot\beta(m,n).$$

Hence $$\beta(m,n) = \frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}.$$

Corollary $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof. We know that $(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ for $m > 0, n > 0$

Taking $m = ½$ and $n = ½$, we get

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} \ .$$

Since $\Gamma(1) = 1$, the above gives

$$\Gamma\left(\frac{1}{2}\right)^2 = \beta\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\frac{\pi}{2}} \sin^0\theta \ \cos^0\theta \ d\theta = 2\int_0^{\frac{\pi}{2}} d\theta = \pi.$$

Hence $$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Theorem $\Gamma(n)\Gamma(1-n) = \dfrac{n}{\sin(\pi n)}$, where $0 < n < 1$.

Proof. As

$$(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx, \quad \text{for } m>0,\, n>0$$

and $$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \quad (m,n) \qquad \text{for } m>0,\, n>0,$$

we have $$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx.$$

Putting $m+n=1,\ m=1-n,$ we obtain

$$\frac{\Gamma(1-n)\Gamma(n)}{\Gamma(1)} = \int_0^\infty \frac{x^{n-1}}{1+x}\,dx \qquad 0<n<1$$

Also $$\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin \pi x}.$$

Hence $$\Gamma(n)\Gamma(1-n) = \frac{n}{\sin(\pi n)},$$ where $0 < n < 1$.

Example Evaluate $\displaystyle\int_0^1 x^m \left(\log\frac{1}{x}\right)^n dx.$

Put $x = e^{-t}$, then $dx = -e^{-t}dt$. Hence

$$\int_0^1 x^m \left(\log\frac{1}{x}\right)^n dx = \int_0^\infty \left(e^{-t}\right)^m t^n \left(-e^{-t}\right)dt = \int_0^\infty e^{-(m+1)t} t^n\, dt$$

Now put $(m+1)t = y$, then $dt = \dfrac{dy}{m+1}$, so that

$$\int_0^1 x^m \left(\log\frac{1}{x}\right)^n dx = \int_0^\infty \frac{e^{-y}y^n}{(m+1)^n}\frac{dy}{m+1} = \frac{1}{(m+1)^{n+1}}\Gamma(n+1).$$

Example Evaluate $\displaystyle\int_0^\infty e^{-x^2}\,dx$

Put $x^2 = t$, then $2x\,dx = dt$ or $dx = \dfrac{dt}{2x} = \dfrac{dt}{2\sqrt{t}}$. Hence

$$\int_0^\infty e^{-x^2}\,dx = \int_0^\infty e^{-t}\,\frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int_0^\infty t^{\frac{1}{2}-1} e^{-t}\, dt = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Example

$$\int_0^{\pi/2} \sin^7\theta \cos^5\theta \, d\theta = \frac{1}{2}\beta\left(\frac{7+1}{2}, \frac{5+1}{2}\right)$$

$$= \frac{1}{2}\beta(4,3) = \frac{\Gamma(4)\Gamma(3)}{2\Gamma(7)} = \frac{1}{2}\frac{3!\,2!}{6!} = \frac{1}{120}$$

Example Express the following in terms of Gamma functions:

$$\int_0^1 x^p (1-x^q)^n \, dx, \text{ where } p>0,\ q>0;\ n>0.$$

Put $x^q = z$ or $x = z^{\frac{1}{q}}$. Then $dx = \frac{1}{q} z^{\left(\frac{1}{q}\right)-1} dz$. Also when $x=0,\ z=0$ and when $x=1,\ z=1$. Hence

$$\int_0^1 x^p (1-x^q)^n \, dx = \int_0^1 z^{\frac{p}{q}} (1-z)^n \frac{1}{q} z^{\left(\frac{1}{q}\right)-1} dz$$

$$= \frac{1}{q}\int_0^1 z^{\frac{p+1}{q}} (1-z)^{(n+1)-1} dz$$

$$= \frac{1}{q}\beta\left(\frac{p+1}{q},\ n+1\right) = \frac{1}{q}\frac{\Gamma\left(\frac{p+1}{q}\right)\Gamma(n+1)}{\Gamma\left(\frac{p+1}{q}+n+1\right)}$$

Example Express the following in terms of Gamma functions:

$$\int_o^a x^{p-1}(a-x)^{q-1}\, dx \quad \text{where } p>0,\ q>0$$

Here put $x = az$ so that $dx = a\, dz$

When $x=0,\ z=0$ and when $x=a,\ z=1$

$$\therefore \int_0^a x^{p-1}(a-x)^{a-1}\, dx = \int_0^1 (az)^{p-1}(a-az)^{q-1}\, a\, dz$$

$$= \int_0^1 a^{p-1} z^{p-1} a^{q-1} (1-z)^{q-1}\, a\, dz$$

$$= a^{p+q-1}\int_0^1 z^{p-1}(1-z)^{q-1}\, dz$$

$$= a^{p+q-1}\beta(p,q) = a^{p+q-1}\frac{\Gamma(p).\Gamma(q)}{\Gamma(p+q)}$$

Example Show that $\Gamma(n) = \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx.$

Put $\log\frac{1}{x} = y.$ Then $x = e^{-y}$ and $dx = -e^{-y}\, dy.$

Also when $x=0\ \ y=\infty$ and when $x=1,\ y=0.$

$$\therefore \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx = \int_0^\infty y^{n-1}\left(-e^{-y}dy\right) = -\int_\infty^0 e^{-y} y^{n-1}\, dy$$

$$= \int_0^\infty e^{-y} y^{n-1}\, dy = \Gamma(n).$$

Example Prove the **duplication formula**:

$$\sqrt{\pi}\,\Gamma(2m) = 2^{2m-1}\Gamma(m)\Gamma\left(m+\frac{1}{2}\right).$$

We know that $\quad \beta(m,n) = \dfrac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}.$ … (1)

$$\therefore \qquad \beta(m,m) = \frac{\left[\Gamma(m)\right]^2}{\Gamma(2m)}.$$

Also $\quad \beta(m,m) = \int_0^1 x^{m-1}(1-x)^{m-1}\, dx.$

In the above integral put $x = \sin^2\theta$ so that $dx = 2\,\sin\theta\cos\theta\, d\theta$.

When $x = 0, \theta = 0$ and when $x = 1\ \theta = \frac{\pi}{2}$ therefore, we have

$$(m,m) = \int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta \cos^{2m-2}\theta . 2\sin\theta\cos\theta\ d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2m-1}\theta\ d\theta\ \ 2\int_0^{\frac{\pi}{2}}\left(\frac{2\sin\theta\cos\theta}{2}\right)^{2m-1} d\theta$$

$$= 2.\int_0^{\frac{\pi}{2}}\left(\frac{\sin 2\theta}{2}\right)^{2m-1} d\theta = 2\int_0^{\frac{\pi}{2}} \frac{1}{2^{2m-1}}\sin^{2m-1} 2\theta\ d\theta$$

$$= \frac{1}{2^{2m-2}}\int_0^{\frac{\pi}{2}} \sin^{2m-1} 2\theta\ d\theta = \frac{1}{2^{2m-2}}\int_0^{\pi} \sin^{2m-1}\phi . \frac{1}{2}\ d\phi$$

by putting $2\theta = \phi$

$$= \frac{1}{2^{2m-1}}\int_0^{\frac{\pi}{2}} \sin^{2m-1}\phi\ d\phi$$

$$= \frac{1}{2^{2m-1}} \times 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\phi\ d\phi\,, \qquad \text{using the result}$$

$$\int_0^{2a} f(x)\ dx = 2\int_0^a f(x)\, dx \text{ if } f(2a-x) = f(x).$$

$$= \frac{1}{2^{2m-2}}\int_0^{\frac{\pi}{2}} \sin^{2m-1}\phi\cos^0\phi\ d\phi$$

$$= \frac{1}{2^{2m-2}} . \frac{\Gamma\left(\frac{2m-1+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{2m-1+1}{2}+\frac{0+1}{2}\right)}$$

$$= \frac{1}{2^{2m-1}} . \frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$= \frac{1}{2^{2m-1}} \cdot \frac{\Gamma(m)\sqrt{\text{®}}}{\Gamma\left(m+\frac{1}{2}\right)} \qquad \ldots (2)$$

From (1) and (2), we conclude that

$$\frac{[\Gamma(m)]^2}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \cdot \frac{\Gamma(m)\sqrt{\text{®}}}{\Gamma\left(m+\frac{1}{2}\right)}$$

Hence $\qquad \sqrt{\text{®}} \cdot \Gamma(2m) = 2^{2m-1}\Gamma(m)\cdot\Gamma(m+\frac{1}{2}).$

Example Prove that $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\text{®}}}{2}$.

Let $\qquad I = \int_0^\infty e^{-x^2}\,dx$

Put $\ x^2 = z.$ Then $\ 2x\,dx = dz\ $ and hence $\ dx = \frac{dz}{2x} = \frac{1}{2\sqrt{z}}dz.$

Also, when $x=0,\ z=0$; when $x\to\infty,\ z\to\infty$. Thus

$$I = \int_0^\infty e^{-z}\frac{dz}{2\sqrt{z}} = \frac{1}{2}\int_0^\infty \frac{e^{-z}}{\sqrt{z}}dz$$

$$= \frac{1}{2}\int_0^\infty z^{\frac{-1}{2}}e^{-z}\,dz$$

$$= \tfrac{1}{2}\Gamma(-\tfrac{1}{2}+1) = \tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \tfrac{1}{2}\sqrt{\text{®}}$$

Example Prove that $\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\text{®}}$

$\int_{-\infty}^\infty e^{-x^2}\,dx = 2\int_0^\infty e^{-x^2}$, using the result "if $f(x)$ is an even function of x, then

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx\text{"}$$

$$= 2\cdot\frac{\sqrt{\text{®}}}{2} = \sqrt{\text{®}}.$$

Example Show that $\int_0^\infty \sqrt{x}e^{-x^3}dx = \frac{1}{3}\sqrt{\text{®}}.$

Put $x^3 = z$ so that $dx = \frac{dz}{3x^2}$. Also, when $x=0,\ z=0$; when $x\to\infty, z\to\infty$. Hence

$$\int_0^\infty \sqrt{x}e^{-x^3}dx = \int_0^\infty \sqrt{x}\,e^{-z}\frac{dz}{3x^2}$$

$$= \frac{1}{3}\int_0^\infty x^{-3/2}e^{-z}dz = \frac{1}{3}\int_0^\infty (z^{1/3})^{-3/2}\,e^{-z}\,dz$$

$$= \frac{1}{3}\int_0^\infty z^{\frac{-1}{2}}\,e^{-z} = \frac{1}{3}\Gamma(\tfrac{1}{2}) = \frac{1}{3}\sqrt{\text{®}}, \quad \text{since } \Gamma(\tfrac{1}{2}) = \sqrt{\text{®}}.$$

In an integral of the type $\int e^{-f(x)}dx$, we may put $f(x) = z$. The above example illustrated this.

Example Show that $\dfrac{(m+2,n-2)}{(m,n)}=\dfrac{m(m+1)}{(n-1)(n-2)}$

$$\frac{(m+2,n-2)}{(m,n)}=\frac{\dfrac{\Gamma(m+2)\,\Gamma(n-2)}{\Gamma(m+2+n-2)}}{\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

$$=\frac{\Gamma(m+2)\,\Gamma(n-2)}{\Gamma(m)\,\Gamma(n)}=\frac{(m+1)\,m\Gamma(m)\Gamma(n-2)}{\Gamma(m)(n-1)(n-2)\Gamma(n-2)},$$

since $\Gamma(n)=(n-1)\,\Gamma(n-1)]$

$$=\frac{m(m+1)}{(n-1)(n-2)}.$$

Example Evaluate

(i) $\int_0^\infty e^{-4x}\,x^{3/2}\,dx$ (ii) $\int_0^1 x^4\,(1-x)^3\;dx$

(iii) $\int_0^2 \dfrac{x^2}{\sqrt{2-x}}dx$ (iv) $\int_0^1 x^5(1-x^3)^3\;dx$

(i) Putting $4x=y$, we have $dx=\dfrac{1}{4}dy$. Also, when $x=0, y=0$; when $x\to\infty, y\to\infty$. Hence

$$\int_0^\infty e^{-4x}x^{3/2}dx=\frac{1}{32}\int_0^\infty e^{-y}\;y^{5/2-1}dy$$

$$=\frac{1}{32}\Gamma\left(\frac{5}{2}\right)=\frac{1}{32}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)=\frac{3}{128}\sqrt{\text{®}}\;.$$

(ii) $\int_0^1 x^4\,(1-x)^3\;dx=\quad(5,4)=\dfrac{\Gamma(5)\Gamma(4)}{\Gamma(9)}=\dfrac{4!3!}{8!}=\dfrac{1}{280}$.

(iii) Putting $x=2y$ so that $dx=2dy$

Also, when $x=0, y=0$; when $x=2, y=1$. Hence

$$\int_0^2\frac{x^2\cdot dx}{\sqrt{2-x}}=4\sqrt{2}\int_0^1 y^2(1-y)^{-1/2}\,dy=4\sqrt{2}\quad(3,\tfrac{1}{2})$$

$$=4\sqrt{2}\,\frac{\Gamma(3)\Gamma(\frac{1}{2})}{\Gamma\left(\dfrac{7}{2}\right)}=\frac{4\sqrt{2}\cdot 2!\Gamma(\frac{1}{2})}{\dfrac{5}{2}\cdot\dfrac{3}{2}\cdot\dfrac{1}{2}\cdot\Gamma\left(\dfrac{1}{2}\right)}=\frac{64\sqrt{2}}{15}.$$

(iv) Put $x^3=y$. Proceeding as above, we get

$$\frac{1}{3}\quad(2,4)=\frac{1}{3}\frac{1!\cdot 3!}{5!}=\frac{1}{60}$$

Example Evaluate

(i) $\int_0^\infty \dfrac{x^8(1-x^6)}{(1+x)^{24}}dx$ (ii) $\int_0^\infty \dfrac{x^4(1+x^5)}{(1+x)^{15}}dx$

(i) $I=\int_0^\infty \dfrac{x^8(1-x^6)}{(1+x)^{24}}dx=\int_0^\infty \dfrac{x^8}{(1+x)^{24}}\,dx-\int_0^\infty \dfrac{x^{14}}{(1+x)^{24}}dx$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}} dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}} dx$$

$$= \quad (9,15) - \quad (15-9) = 0 \text{, since } \quad (m,n) = \quad (n,m)$$

$$= \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)} = \frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{2})}{2\Gamma(1)} = \frac{1}{2}\Gamma(\tfrac{1}{4})\Gamma(\tfrac{3}{4})$$

Now by *Duplication Formula,* we have

$$2^{2p-1}\ \Gamma(p)\ \Gamma(p+\tfrac{1}{2}) = \sqrt{\text{®}}\ \Gamma(p).$$

Taking $p = \frac{1}{4}$,

$$2^{-1/2}\Gamma(\tfrac{1}{4})\ \Gamma(\tfrac{3}{4}) = \sqrt{\text{®}}\ \Gamma(\tfrac{1}{2})$$

$$\frac{1}{\sqrt{2}}\Gamma(\tfrac{1}{4})\ \Gamma(\tfrac{3}{4}) = \sqrt{\text{®}}\ \sqrt{\text{®}}\ , \qquad \text{since } \Gamma(\tfrac{1}{2}) = \sqrt{\text{®}}$$

implies

$$\Gamma(\tfrac{1}{4})\ \Gamma(\tfrac{3}{4}) = \sqrt{2}\,\text{®} \qquad \dots (2)$$

Using (2) in (1), we get

$$\int_0^{\text{®}/2} \sqrt{\tan x}\ dx = \frac{1}{2}(\sqrt{2}\text{®}) = \frac{\text{®}}{\sqrt{2}}$$

Exercises

In Exercise 1-4, express in terms of Beta functions:

1. $\int_0^2 x^3(8-x^3)^{-1/3}\, dx$
2. $\int_0^1 \frac{x^2}{1-x^5}\ dx$
3. $\int_0^1 \frac{x^2}{\sqrt{1-x^5}}\, dx$
4. $\int_0^1 x^{l-1}(1-x^2)^{m-1}\, dx$

In Exercise 5-20, evaluate the given expressions:

5. $\quad (3,5)$,
6. $\left(\frac{1}{2},\frac{1}{2}\right)$.
7. $(\frac{1}{2},\frac{3}{2})$
8. $\left(\frac{3}{2},2\right)$
9. $\int_0^{\frac{\text{®}}{2}} \sin^{10} d\bar{}$
10. $\int_0^{\frac{\text{®}}{2}} \sin^7 d\bar{}\ \cos^{5\bar{}}\ d\bar{}$
11. $\int_0^{\frac{\text{®}}{2}} \sin^{4\bar{}} \sin^{6\bar{}}\ d\bar{}$

12. $\int_0^{\frac{®}{2}} \sqrt{\tan^{-}}\ d^{-}$

13. $\int_0^{\frac{®}{2}} \frac{dx}{\sqrt{\cos x}}$

14. $\int_0^{\frac{®}{2}} \frac{\sqrt[3]{\sin^8 x}}{\sqrt{\cos x}} dx$

15. $\int_0^1 x^5 (1-x^3)^{10}\, dx.$

16. $\int_0^1 (1-x^2)^{\frac{-1}{2}} dx$

17. $\int_0^1 x^7 (1-x)^8\, dx$

18. $\int_0^1 x^2 (1-x)^3\, dx$

19. $\int_0^2 (4-x^2)^{\frac{3}{2}}\, dx$

20. $\int_0^1 \sqrt{\frac{(1-x)}{x}}\, dx$

21. Show that $\int_0^a \left(8-x^3\right)^{\frac{-1}{3}} = \frac{1}{3} \ \left(\frac{1}{3}, \frac{1}{3}\right).$

22. Prove that $\int_0^a \left(a-x\right)^{m-1} x^{n-1} dx = a^{m+n-1} \ (m,n).$

23. Prove that $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} dx = \ (m,n).$

24. Prove that $\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} = 2 \ (m,n).$

25. Prove that $\int_0^\infty \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}} dx = \frac{1}{(a+b)^m a^n} \ (m,n).$

26. Prove that $\int_0^p x^m \left(p^q - x^q\right)^n dx = \frac{p^{qn+m+1}}{q} \ \left(n+1, \frac{m+1}{q}\right)$

27. Show that if p, q are positive, then

$$\beta(p+1,\ q) + \beta(p,\ q+1) = \beta(p, q).$$

28. Evaluate $\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx.$

29. Prove that $\int_0^a (a-x)^{m-1}\ x^{n-1}\ dx = a^{m+n-1} \ (m,n).$

30. Using the property $(m,n) = \ (n,m)$, evaluate $\int_0^1 x^3 (1-x)^{4/3}\ dx$

31. Show that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}} dx = \frac{(m,n)}{a^n (1+a)^n}$

32. Show that

(i) $\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} dx = 2 \ (m,n),\ m>0, n>0$

(ii) $\int_0^\infty \frac{x^{m-1} - x^{n-1}}{(1+x)^{m+n}} dx = 0, \quad m > 0, n > 0$.

33. Express $\int_0^1 x^m (1-x^n)^p \, dx$ in terms of Beta function and hence evaluate $\int_0^1 x^5 (1-x^3)^3 \, dx$

34. Prove that $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}} = \frac{1}{a^n b^m} \quad (m,n)$

where $m > 0,\ n > 0,\ a > 0,\ b > 0$

35. (i) Express $\int_0^1 x^m (1-x^n)^p \, dx$ in terms of the Beta function and hence evaluate $\int_0^1 x^5 (1-x^3)^{10} \, dx$.

(ii) Show that $\int_0^1 x^{m-1} \left(1-x^2\right)^{n-1} \, dx = \frac{1}{2} \ \left(\frac{1}{2}m, n\right) = \frac{\Gamma\left(\frac{m}{2}\right)\Gamma(n)}{2\Gamma\left(\frac{1}{2}m+n\right)}$

(iii) Prove that $\frac{(p,q+1)}{q} = \frac{(p+1,q)}{p} = \frac{(p,q)}{p+q}, \ p > 0, q > 0$

(iv) Prove that $\frac{(m+1,n)}{(m,n)} = \frac{m}{m+n}, \ m > 0,\ n > 0$.

(v) Prove that $(m,n) = 2^{1-2m} \quad (m, \frac{1}{2})$.

(vi) Show that $\int_0^{\frac{®}{2}} \sin^3 ¯ \cos^5 ¯ \ d¯ \ = \frac{1}{2} \ (2,3)$ and hence evaluate it.

In Exercises 36-45, evaluate:

36. $\frac{\Gamma(7)}{2\Gamma(4)\Gamma(3)}$

37. $\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)$

38. $\frac{\Gamma(3)\Gamma(\frac{3}{2})}{\Gamma(\frac{9}{2})}$

39. $\int_0^\infty x^3 e^{-x} dx = 6$

40. $\int_0^\infty \sqrt{x} e^{-x^2} \, dx$

41. $\int_0^\infty x^4 e^{-x^4} \, dx$

42. $\int_0^\infty e^{-x^4} \, dx$

43. $\int_0^\infty e^{-x^3} \, dx$

44. $\int_0^1 \left[x \log(\frac{1}{x}) \right]^{\frac{1}{3}} dx$

45. $\int_0^\infty x^3 e^{-x^3} \, dx$

46. Show that $\int_0^\infty \sqrt{x} e^{-x^3} \, dx = \frac{1}{3}\sqrt{®}$

47. Show that $\int_0^\infty 4x^4 \, e^{-x^4} \, dx = \Gamma\left(\frac{5}{4}\right)$

48. Show that $\int_0^{\infty} x^6 e^{-2x}\, dx = \frac{45}{8}\text{®}$

49. Show that $\int_0^1 x^2 (1-x)^3\, dx = \frac{1}{60}$

50. Show that $\int_0^3 \frac{dx}{\sqrt{3x - x^2}} = \text{®}$

51. Show that $\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{\sqrt{\text{®}}\,\Gamma\left(\frac{1}{3}\right)}{3\Gamma\left(\frac{5}{6}\right)}$

52. Prove that $\Gamma(n+1) = n\Gamma(n)$, where $n > 0$.

53. Show that $\int_0^1 y^{q-1} \left(\log \frac{1}{y}\right)^{p-1} dy = \frac{\Gamma(p)}{q^p}$, where $p > 0,\ q > 0$.

54. Show that $\int_0^{\frac{\text{®}}{2}} \sqrt{\sin \text{¯}}\, d\text{¯} \times \int_0^{\frac{\text{®}}{2}} \frac{1}{\sqrt{\sin \text{¯}}} d\text{¯} = \text{®}$.

55. Show that the area bounded by the normal curve $y = \frac{1}{\text{±}\sqrt{2\text{®}}} e^{-(x^2/2\text{±}^2)}$ and x-axis is unity.

56. Prove that $\int_0^1 \left[\frac{x^3}{(1-x^3)}\right]^{\frac{1}{2}} dx = \frac{2^{\frac{4}{3}}\text{®}^2}{3^{\frac{1}{2}}\left[\Gamma\left(\frac{1}{3}\right)\right]^3}$

57. Show that $\int_0^1 \sqrt{1-x^4}\ dx = \frac{1}{6\sqrt{2\text{®}}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$

58. Show that $\int_0^1 \frac{x^2 dx}{(1-x^4)^{\frac{1}{2}}} \times \int_0^1 \frac{dx}{(1-x^4)^{\frac{1}{2}}} = \frac{\text{®}}{4}$.

59. Show that if $a > 1, \int_0^{\infty} \frac{x^a}{a^x} dx = \frac{\Gamma(a+1)}{(\log a)^{a+1}}$

60. Prove that $\int_0^{\infty} \frac{t^2 dt}{1+t^4} = \frac{\text{®}}{2\sqrt{2}}$

61. Prove that $\int_0^1 \frac{dx}{\sqrt{1-x^n}}. = \frac{\sqrt{\text{®}}\,\Gamma\left(\frac{1}{n}\right)}{n\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$

62. Show that $\Gamma(a) = 2\int_0^{\infty} e^{-t^2} t^{2a-1}\ dt = \int_0^1 \left(\log \frac{1}{u}\right)^{a-1} du$.

63. Prove that $n > -1, \int_0^{\infty} x^n e^{-a^2 x^2} dx = \frac{1}{2a^{n+1}}\Gamma\left(\frac{n+1}{2}\right)$.

64. Show that $\Gamma(n) = \int_0^1 (-1)^{n-1} (\log y)^{n-1} dy$.

65. Show that $\int_0^{\infty} \frac{x^{n-1} - x^{m-1}}{(1+x)^{m+n}} dx = 0. \qquad m > 0,\ n > 0.$

66. (i) Prove that $\int_0^{\infty} x^n e^{-a^2x^2}dx = \frac{1}{2a^{n+1}}\Gamma\left(\frac{n+1}{2}\right)$. Hence show that $\int_{-\infty}^{\infty} e^{-a^2x^2}dx = \frac{\sqrt{\pi}}{a}$

(ii) Prove that $\int_0^{\infty} e^{-a^2x^2}dx = \frac{\sqrt{\pi}}{2a},\ a>0$

67. (a) Show that $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{n}\right)}{n\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$

(b) Prove that $\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}}{4}\cdot\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$.

68. Show that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}dx = \frac{\Gamma(m)\,\Gamma(n)}{a^n(1+a)^m\,\Gamma(m+n)}$

69. (i) Show that $\int_0^{\pi/2} \sin^n\theta\, d\theta = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)},\ n>-1$

(ii) Show that $\int_0^{\pi/2}\sqrt{\sin\theta}\,d\theta \times \int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin\theta}} = \pi$

(iii) Show that $\int_0^{\pi/2}\sin^p x\,dx \times \int_0^{\pi/2}\sin^{p+1}x\,dx = \frac{\pi}{2(p+1)}$.

70. Show that $\int_0^{\infty}\frac{x^a}{a^x}dx = \frac{\Gamma(a+1)}{(\log a)^{n+1}}$, if $a>1$.

71. Show that $\int_{-1}^{\infty}\frac{x+1}{(x+2)^6}dx = \frac{1}{20}$

72. Show that $\Gamma\left(n+\frac{1}{2}\right) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}\sqrt{\pi}$.

73. Show that $\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n} = \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(n+1)}$

74. Show that $\int_0^{\frac{\pi}{2}}\sin^m\theta\cos^n\theta\,d\theta\ \frac{1}{2}\ \left(\frac{m+1}{2},\frac{n+1}{2}\right) = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$

75. Show that $\int_0^1\frac{(1-x^4)^{3/4}}{(1+x^4)^2}dx = \frac{1}{4(2^{\frac{1}{2}})}\ \left(\frac{7}{4},\frac{1}{4}\right)$.